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A CERTAIN TYPE OF PARTIAL DIFFERENTIAL EQUATIONS
ON TORI

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Summary. The existence of classical solutions for some partial differential equations on tori is shown.

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1. INTRODUCTION

The purpose of this paper is to show the existence of $C^2$-smooth solutions for the singularly perturbed equation

$u_{yy} + \varepsilon u_{xx} = \varepsilon f(u, y, x),$

where $u$ is $2\pi$-periodic in $x$ and $y$, $f \in C^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is $2\pi$-periodic in $x$ and $y$, $\varepsilon > 0$ is a small parameter. We shall show that (1) possesses a solution provided $f$ is globally Lipschitz in $u$ uniformly for $y, x$ with a Lipschitz constant $K < 1$ and a certain ordinary differential equation has a $2\pi$-periodic solution. We conclude this paper with a discussion of the equations (1) when $f$ is independent on $y$. We also show a geometric interpretation of this special case.

Singularly perturbed equations on tori have been studied by several authors [2], [3], [4]. Usually they have used the approach via the Nash-Moser implicit function theorem. We will use only the Banach fixed point theorem.
2. MAIN RESULTS

**Theorem 2.1.** If there is a constant $K$, $0 < K < 1$ such that

\[(+) \quad |f(u_1, \cdot, \cdot) - f(u_2, \cdot, \cdot)| \leq K |u_1 - u_2|\]

for all $u_1, u_2 \in \mathbb{R}$, then (1) has a solution $u_\varepsilon$ for each small $\varepsilon > 0$ having the form

\[u_\varepsilon(x, y) = \bar{v}(x) + O(\varepsilon)\]

where $\bar{v}$ is a stable (see ($-$) in the proof of this theorem) $2\pi$-periodic solution of the equation

\[(2) \quad v'' = \frac{1}{2\pi} \int_0^{2\pi} f(v, s, x) \, ds.\]

**Proof.** First of all, we investigate the equation (2). Let

\[H = \left\{ v: \mathbb{R} \to \mathbb{R}, v \text{ is } 2\pi\text{-periodic}, 2\pi \left\| v \right\|^2 = \int_0^{2\pi} v^2(s) \, ds < \infty \right\}.\]

It is well-known that $H$ is a Hilbert space with the basis

\[\{\sin nt, \cos mt\}_{n \geq 1, m \geq 0}.\]

**Lemma 2.2.** The equation

\[v'' = g, \quad g \in H, \quad \int_0^{2\pi} g(s) \, ds = 0\]

has a unique solution $v(g)$ in $H$ such that $\int_0^{2\pi} v = 0$ and $\left\| v \right\| \leq \left\| g \right\|$.

**Proof of Lemma 2.2.** If $g = \sum_{i=1}^{\infty} a_i \cdot \sin it + b_i \cdot \cos it$ then

\[v = -\sum_{i=1}^{\infty} (a_i \cdot \sin it + b_i \cdot \cos it)/i^2.\]
We put \( S(g) = v(g) \), \( F(g) = \frac{1}{2\pi} \int_0^{2\pi} f(g, s, x) ds \) and \( P g = \frac{1}{2\pi} \int_0^{2\pi} g(s) ds \). Then (2) has the form

\[
(3) \quad s = S(I - P) \cdot F(s + t), \quad 0 = PF(s + t),
\]

where \( s \in \text{Ker} \, P \), \( t \in \text{Im} \, P \cong \mathbb{R} \). Since \( f \) has the property (+) we have

\[
\|S(I - P)(F(s_1 + t) - F(s_2 + t))\| \leq K \cdot \|s_1 - s_2\|
\]

for all \( s_1, s_2 \in \text{Ker} \, P \). Using the Banach fixed point theorem we can solve the first equation of (3) for each \( t \). We insert this solution \( s(t) \) into the second equation of (3) obtaining

\[
(4) \quad 0 = PF(s(t) + t).
\]

We see that each solution of (4) determines a unique solution of (2). If a zero of (4) is simple then we say that the solution of (2) determined by this zero is the stable solution of (2) (−).

Without loss of generality we can assume that \( \bar{v} \equiv 0 \), i.e. \( t = 0 \), \( s(0) = 0 \). We denote

\[
X = \left\{ u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, u \text{ is } 2\pi \text{-periodic in } x, y, \quad \|u\| = \frac{1}{2\pi} \sqrt{\int_0^{2\pi} \int_0^{2\pi} u^2(x, y) \, dx \, dy} < \infty \right\}.
\]

\( X \) is a Hilbert space with the basis

\[
\{ \sin mt \cdot \sin it, \sin mt \cdot \cos it, \cos mt \cdot \cos it, \cos mt \cdot \sin it \}
\]

**Lemma 2.3.** The equation

\[
w_{yy} + \varepsilon w_{xx} = g, \quad g \in X, \quad \int_0^{2\pi} g(x, \cdot) \, dx = 0
\]

has a unique solution \( w_g \in X \) satisfying \( \int_0^{2\pi} w_g(x, \cdot) \, dx = 0 \). Moreover,

\[
\|w_g\| \leq \|g\| \cdot \frac{1}{\varepsilon}.
\]

**Proof.** The proof is the same as that of Lemma 2.2.
We put
\[ T_\varepsilon(g) = w_\varepsilon, \quad \tilde{R}g = \frac{1}{2\pi} \int_0^{2\pi} g(x,y) \, dx, \quad G(g) = f(g,\cdot,\cdot). \]

Then (1) has the form

(5) \[
\begin{align*}
    w &= \varepsilon \cdot T_\varepsilon \cdot (I - \tilde{R}) \cdot G(w + v + t), \\
    v &= \varepsilon \cdot S \cdot (I - P) \cdot \tilde{R} \cdot G(w + v + t), \\
    0 &= P \cdot \tilde{R} \cdot G(w + v + t),
\end{align*}
\]

where \( w \in \text{Ker} \tilde{R}, \ v \in \text{Im} \tilde{R} \cap \text{Ker} P, \ t \in \text{Im} P \cong R. \) We note that \( v \) is independent on \( x \) since \( \text{Im} \tilde{R} \subseteq H. \) By (+), Lemma 2.2, Lemma 2.3 we see that the mapping

\[ (w, v) \rightarrow (\varepsilon \cdot T_\varepsilon \cdot (I - \tilde{R}) \cdot G(w + v + t), \varepsilon \cdot S \cdot (I - P) \tilde{R} \cdot G(w + v + t)) \]

defined on \( \text{Ker} \tilde{R} \times \text{Ker} P \) with the norm \( \| \cdot \| + \| \cdot \| \) is Lipschitz with a Lipschitz constant \( K_1, \ K < K_1 < 1 \) for \( \varepsilon > 0 \) small, \( t \in \mathbb{R}. \)

Thus the first two equations of (5) have unique solutions \( w_\varepsilon(t), v_\varepsilon(t) \) for each \( t \in \mathbb{R} \), \( \varepsilon > 0 \) small, and \( \| w_\varepsilon(t) \|, \| v_\varepsilon(t) \| \) are bounded on each bounded subset of \( \mathbb{R}. \) Using these estimates and the Sobolev imbedding theorem we see that \( w_\varepsilon(t), v_\varepsilon(t) \in C^3 \) and \( |w_\varepsilon(t)|_{C^3}, |v_\varepsilon(t)|_{C^3}, \) are uniformly bounded for \( \varepsilon > 0 \) small, \( |t| \leq 1. \) We take a sequence \( \varepsilon_i \rightarrow 0, \varepsilon_i > 0, t_i \rightarrow t, |t_i| \leq 1, \varepsilon_i \) small. Then by the Arzela-Ascoli theorem, \( \{ w_\varepsilon(t_i), v_\varepsilon(t_i) \}_{i=0}^\infty \) has a subsequence tending to \( (\bar{w}, \bar{v}) \) in \( C^2. \)

On the other hand, (5) implies

\[ v_{yy} = \frac{\varepsilon}{2\pi} \int_0^{2\pi} f(w + v + t, y, x) \, dx, \]

\[ w_{yy} + \varepsilon w_{xx} = \varepsilon \left( f(w + v + t, y, x) - \frac{1}{2\pi} \int_0^{2\pi} f(w + v + t, y, x) \, dx \right). \]

It follows that \( \bar{v} \equiv 0, \bar{w} \) is independent on \( y, \bar{w} = \bar{w}(x) \) satisfies

\[ \bar{w}'' = \frac{1}{2\pi} \int_0^{2\pi} f(\bar{w} + t, y, x) \, dy - \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\bar{w} + t, y, x) \, dy \, dx. \]

However, this equation is precisely the first equation of (3) and thus \( \bar{w} \approx s(t). \) This implies

\[ \lim_{\varepsilon \to 0_+} w_\varepsilon(t) = s(t), \quad \lim_{\varepsilon \to 0_+} v_\varepsilon(t) = 0. \]
in the space $C^2$. Hence for $\varepsilon > 0$ small the last equation of (5) is $C^1$-close to the equation

$$0 = P \cdot \tilde{R} \cdot G(\bar{w} + t) = P \cdot F(s(t) + t)$$

on the interval $\left(-\frac{1}{2}, \frac{1}{2}\right) \subset (-1,1)$. But we know that $P \cdot F(s(0) + 0) = 0$ and this root is simple. Thus the equation

$$0 = P \cdot \tilde{R} \cdot G(w_\varepsilon(t) + v_\varepsilon(t) + t)$$

has a solution on $\left(-\frac{1}{2}, \frac{1}{2}\right)$ for $\varepsilon > 0$ small tending to 0 as $\varepsilon \to 0$. This completes the proof. \qed

It is clear that we can repeat the above proof if $f$ depends smoothly also on $\varepsilon$, i.e. $f = f(u, y, x, \varepsilon)$.

Remark 2.4. Since a small smooth perturbation of a function having a simple root also has a simple root, it is not difficult to see that each stable solution $\bar{v}$ of (2) has the following property: Each $2\pi$-periodic (smooth) perturbation of (2) possesses a $2\pi$-periodic solution near $\bar{v}$.

Finally, let $f(u, y, x, 0) = g(u)$ and $g(c) = 0$, $g'(c) \neq 0$, $|g'(.)| < 1$. Then the equation (3) has the form

$$s'' = g(s + t) - \frac{1}{2\pi} \int_0^{2\pi} g(s(u) + t) \, du,$$

$$0 = \frac{1}{2\pi} \int_0^{2\pi} g(s(u) + t) \, du.$$  

We see that the first equation has a unique solution $s \equiv 0$ for each $t \in \mathbb{R}$ and thus (4) has the form

$$0 = g(t).$$

Since $g(c) = 0$, $g'(c) \neq 0$, the trivial solution $u \equiv c$ of $u'' = g(u)$ is stable.

3. A SPECIAL CASE

In this section we assume that $f$ is independent on $y$, i.e. we investigate the equation

$$\frac{1}{\varepsilon} u_{yy} + u_{xx} = f(u, x)$$

on the torus $S^1 \times S^1$. 

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We suppose that there is a $K > 0$ satisfying
\[
\left| \frac{\partial f}{\partial u}(\cdot, \cdot) \right| < K.
\]
The operator $A_\varepsilon : \text{Dom}(A_\varepsilon) \subset X \to X$,
\[
A_\varepsilon u = \frac{1}{\varepsilon} u_{yy} + u_{xx},
\]
has the invariant subspace
\[
H_1 = \text{span}\{\sin mx, \cos mx\}.
\]
Further,
\[
H_1 \oplus H_2 = X,
\]
\[
H_2 = \text{span}\{\sin my \cdot \cos jx, \sin my \cdot \sin jx, \cos my \cdot \cos jx, \cos my \cdot \sin jx\}_{m \geq 1}.
\]
Hence the spectrum of $A_\varepsilon/H_2$ is
\[
\left\{-\frac{1}{\varepsilon} m^2 - j^2 \right\}_{m \geq 1} = \sigma(A_\varepsilon/H_2).
\]
On the other hand, if $F(u) = f(u, \cdot)$ then
\[
F(H_1) \subset H_1, \quad \|F(u_1) - F(u_2)\| \leq K \cdot \|u_1 - u_2\|.
\]
Summing up we obtain $\sigma(A_\varepsilon/H_2) \cap (-K, K) = \emptyset$ for $\varepsilon > 0$ small.

Thus applying Theorem 2 from [1] we obtain

**Theorem 3.5.** For $\varepsilon > 0$ small each $2\pi$-periodic solution of (6) is independent on $y$.

Finally, Theorem 3.5 has the following simple geometric interpretation: Consider the equation
\[
u_{yy} + u_{xx} = f(u, z)
\]
on the torus $M_\varepsilon = S^1 \times \{z \in \mathbb{R}^2, |z| = \varepsilon\} (x \in S^1)$. Then by using a suitable scaling of variables (7) can be transformed into (6). Hence for $\varepsilon > 0$ small the equation (7) has only $C^2$-solutions on $M_\varepsilon$ which are independent on $y$. Of course, provided they exist.
Remark 3.6. Similarly we can study the following problem: Let us consider the system of equations
\[ E^p_{x} u_p + \varepsilon E^p_{y} u_p = \varepsilon f_p(x, y, u_1, \ldots, u_m), \quad p = 1, \ldots, m \]
where \((x, y) \in T^m \times T^m, u_p = u_p(x, y) \in \mathbb{R}, \varepsilon\) is a small nonnegative parameter, 
\(E^p_x = E^p, E^p_y = E^p, E^p\) is a strongly elliptic operator on the \(m\)-dimensional torus 
\(T^m = S^1 \times \ldots \times S^1\), i.e.
\[ E^p u = \sum_{i,j} \frac{\partial}{\partial z_i} (a^p_{i,j}(z) \frac{\partial}{\partial z_j} u), \]
where \(a^p_{i,j}\) are \(2\pi\)-periodic in all coordinates of \(z\) and the matrices \(\{a^p_{i,j}(.)\}\) are symmetric positive definite. Further, \(f_p\) are \(2\pi\)-periodic in \((x, y)\) and globally Lipschitz in \(u = (u_1, \ldots, u_m)\) with a Lipschitz constant \(K_p\) i.e.
\[ |f_p(\cdot, \cdot, u^1_1, \ldots, u^1_m) - f_p(\cdot, \cdot, u^2_1, \ldots, u^2_m)| \leq K_p \sqrt{(u^1_1 - u^2_1)^2 + \ldots + (u^1_m - u^2_m)^2}. \]
Let \(A_p\) be the first nonzero eigenvalue of \(E^p\). We assume
\[ \sum_{p=1}^{m} (K_p/A_p)^2 < 1. \]
Then following the above procedure we obtain: The above mentioned equation has a solution \(u\) in the form \(u_p = v_p + O(\varepsilon), p = 1, \ldots, m\), for each \(\varepsilon \geq 0\) small where \(v = (v_1, \ldots, v_m)\) is a stable solution of
\[ E^p_{y} v_p = \frac{1}{(2\pi)^m} \int_0^{2\pi} \cdots \int_0^{2\pi} f_p(x, y, v_1, \ldots, v_m) \, dx. \]
The stability of \(v\) means that under a small perturbation of the right hand side of this equation there always exists a unique solution near \(v\).

References
Souhrn

URČITÝ TYP PARCIÁLNÝCH DIFERENCIÁLNÝCH ROVNÍC NA TÓROCH

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V práci sa študujú špeciálne parciálne diferenciálne rovnice na tóroch, príčom sa dokazuje existencia ich klasického riešenia.

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