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ON LIOUVILLE THEOREM AND HÖLDER CONTINUITY OF WEAK SOLUTIONS TO SOME QUASILINEAR ELLIPTIC SYSTEMS OF HIGHER ORDER

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Summary. The aim of this paper is to show that the Liouville-type property is a sufficient and necessary condition for the regularity of weak solutions of quasilinear elliptic systems of higher orders.

Keywords: regularity of weak solutions, quasilinear elliptic systems

AMS classification: 35J60, 35D10

INTRODUCTION

In this paper we shall deal with quasilinear elliptic systems. More precisely we shall consider the following problem.

Let $\Omega$ be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$, $n \geq 2$. Let us denote $\sigma(n, k) = \binom{n+k-1}{k}$, $\varphi(n, k) = \binom{n+k}{k}$, $n, k \in \mathbb{N}$. We shall study weak solutions $u \in H^m(\Omega) \cap H^{m-1, \infty}(\Omega)$ to the system

$$
\sum_{i=1}^{N} \sum_{|\alpha| \leq m_i, |\beta| = m_j} (-1)^{|\alpha|} D^\alpha (A_{ij}^\alpha(x, \delta(u)) D^\beta u^j) = \sum_{|\alpha| \leq m_i} (-1)^{|\alpha|} D^\alpha g^i,
$$

(0.1)

$i = 1, \ldots, N$, in $\Omega$.

By a weak solution of (0.1) we mean a function $u \in H^m(\Omega) (H^m(\Omega) = H^{m_1}(\Omega) \times \cdots \times H^{m_N}(\Omega), H^{m_i}(\Omega) —$ Sobolev space, $m_i \geq 1$ for $i = 1, \ldots, N$, $u = (u^1, \ldots, u^N) —$ see
such that
\[
\sum_{i,j=1}^{N} \sum_{|\alpha| \leq m_i} \int A_{ij}^{\alpha\beta}(x, \delta(u)) D^\alpha u^i D^\beta \varphi^i dx = \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i} \int g_{\alpha}^i D^\alpha \varphi^i dx,
\]
(0.2)

\[\varphi \in [D(\Omega)]^N.\]
\[\delta(u) = \{D^\alpha u^i : |\alpha| \leq m_i - 1, i = 1, \ldots, N\}.
\]

We shall assume that
\[
A_{ij}^{\alpha\beta} \in C(\overline{\Omega} \times \mathbb{R}^n), \quad \kappa = \sum_{i=1}^{N} \varrho(n, m_i - 1),
\]
(0.3)

there exists \(\nu > 0\) such that
\[
\sum_{i,j=1}^{N} \sum_{|\alpha| = m_i} A_{ij}^{\alpha\beta}(x, \zeta) \zeta_i \zeta_j^\beta \geq \nu \|\zeta\|^2,
\]
(0.4)

\[(x, \zeta) \in \overline{\Omega} \times \mathbb{R}^n, \quad \zeta \in \mathbb{R}^\vartheta, \quad \vartheta = \sum_{i=1}^{N} \sigma(n, m_i),
\]
(0.5)

\[g_{\alpha}^i \in L^{p_\alpha^i}(\Omega), \quad p_\alpha^i = \frac{p}{m_i - |\alpha| + 1},
\]

where \(p > n, \quad p \geq 2(\max_i \{m_i\} + 1)\).

For \(M > 0, G > 0\) let us denote
\[\{u \in H^m(\Omega) \cap H^{m-1, \infty}(\Omega) : u \text{ is a solution to (0.1)} \quad \text{and } \|u\|_{H^{m-1, \infty}(\Omega)} \leq M\},
\]
\[[G] = \{g_{\alpha}^i \in L^{p_\alpha^i}(\Omega) : \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i} \|g_{\alpha}^i\|_{L^{p_\alpha^i}(\Omega)} \leq G\},
\]

\[A = A(M) = \sup_{|\zeta| \leq M} \left\{ \sum_{i,j,\alpha,\beta} |A_{ij}^{\alpha\beta}(x, \zeta)| \right\},
\]

\[\delta_2(u) = \{D^\alpha u^i : |\alpha| = m_i - 1, i = 1, \ldots, N\}, \quad \delta_1(u) = \delta(u) \setminus \delta_2(u).
\]

Let \(g = (s_1, \ldots, s_N), \quad s_i \in \mathbb{N} \cup \{0\}, \quad i = 1, \ldots, N. \) We shall use the notation \(P_k^N = \{(P_1, \ldots, P_N) : P_i \text{ is a polynomial such that } \deg(P_i) \leq s_i\}. \) Denote \(B(x^0, R) = \{x \in \mathbb{R}^n : |x-x^0| < R\} \) and \(\tau = \sum_{i=1}^{n} \varrho(n, m_i - 2) \) (we put \(\varrho(n, -1) = 0\).
Definition 0.6. We say that the system (0.1) has Liouville's property (L), if for every \( x^0 \in \Omega, \xi \in \mathbb{R}^r \) every function \( v \in H^m_{\text{loc}}(\mathbb{R}^n) \) with bounded derivatives of order \( m-1 \), solving in \( \mathbb{R}^n \) the system

\[
(0.7) \quad \sum_{j=1}^{N} \sum_{|\alpha|=m_i, |\beta|=m_j} (-1)^{|\alpha|} D^\alpha (A_{ij}^\alpha (x^0, \xi, \delta_2(v)) D^\beta v^j(x)) = 0, \quad i = 1, \ldots, N
\]

(i.e. \( \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i, |\beta|=m_j} \int A_{ij}^\alpha (x^0, \xi, \delta_2(v)) D^\beta v^j(x) D^\alpha \varphi^i(x) \, dx = 0, \varphi \in \mathcal{D}(\mathbb{R}^n)^N \))

is a polynomial from the set \( P_{m-1}^N \).

Definition 0.8. We say that the system (0.1) has the property of regularity (R) if for every \( x^0 \in \Omega, \xi \in \mathbb{R}^r, M > 0 \) there exist \( \eta > 0, c > 0 \) and \( \mu \in (0, 1) \) such that every weak solution \( u \) (in \( \mathbb{R}^n \)) of the system (0.7) with \( |D^\alpha u^i| \leq M, i = 1, \ldots, N, |\alpha| = m_i - 1 \) belongs to the space \( C^{m-1,\mu} \left( B(0, \eta) \right) \) and \( \|u\|_{C^{m-1,\mu} \left( B(0, \eta) \right)} \leq c \).

It will be proved in this paper that the property (L) implies the interior regularity of solutions to the system (0.1), i.e. if \( u \) is a weak solution to (0.1) then \( u \in C^{m-1,\mu} (\overline{\Omega}) \), where \( \overline{\Omega} \subset \Omega, \mu \in (0, 1 - \frac{n}{p}) \).

It will be also shown that (R) \( \Rightarrow \) (L).

These results generalize the results of [4]. In [4] the analogous assertions are proved for quasilinear elliptic systems of the second order.

The history of the regularity problem and Liouville's property is described in [2], [4].

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1. SOME LEMMAS

Let us denote

\[
U(x^0, R) = R^{-n} \int_{B(x^0, R)} \left( \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} |D^\alpha u^i(x) - (D^\alpha u^i)_{x^0, R}|^2 \right) \, dx,
\]

\( u \in H^m_{\text{loc}}(B(x^0, R)), \) where by \( (D^\alpha u^i)_{x^0, R} \) we mean the integral mean value \( D^\alpha u^i \) in \( B(x^0, R) \).
Lemma 1.1. Let $A_{ij}^{\alpha \beta}$ be constants with $|A_{ij}^{\alpha \beta}| \leq L$, $L > 0$ and let (0.4) be satisfied for $A_{ij}^{\alpha \beta}$. Let $u \in H^m_{lo}(B(0,1)) \cap H^{m-1}(B(0,1))$ be a solution to the system

$$
(1.2) \sum_{i,j=1}^{N} \sum_{|\alpha| = m_i, B(0,1)} \int A_{ij}^{\alpha \beta} D^\beta u^i D^\alpha \varphi^j dx = 0, \quad \varphi \in [D(B(0,1))]^N.
$$

Then there exists a constant $\Lambda = \Lambda(n, N, L, m, \nu)$ such that for all $0 < \varrho \leq 1$

$$
(1.3) \quad U(0, \varrho) \leq \Lambda \varrho^2 U(0,1).
$$

The proof of this lemma is analogous to that of Lemma 2 in [3]. Using the Lax-Milgram lemma we could prove

Lemma 1.4. Suppose that $u \in [M]$, $x^0 \in \Omega$. Let (0.3), (0.4), (0.5) be satisfied and let the right-hand sides of the system (0.1) belong to $[G]$. Then there exists $R_0 = R_0(A, M)$, $0 < R_0 \leq \text{dist}(x^0, \partial \Omega)$ such that for all $R \in (0, R_0]$ the linear elliptic system

$$
(1.5) \quad \sum_{j=1}^{N} \sum_{|\alpha| \leq m_i} (-1)^{|\alpha|} D^\alpha (A_{ij}^{\alpha \beta}(x, \delta(u)) D^\beta v^j_R) = \sum_{|\alpha| \leq m_i} (-1)^{|\alpha|} D^\alpha g^i, \quad i = 1, \ldots, N,
$$

has a unique weak solution in $H^m_0(B(x^0, R))$.

Since (1.5) is uniquely solvable for $R \leq R_0$ we may decompose any solution $u$ of the quasilinear system (0.1) in the following manner:

$$
(1.6) \quad u = v_R + w_R,
$$

where $v_R \in H^m_0(B(x^0, R))$ solves the system (1.5) and

$$
(1.7) \quad \sum_{i,j=1}^{N} \sum_{|\alpha| \leq m_i, B(x^0, R)} \int A_{ij}^{\alpha \beta}(x, \delta(u)) D^\beta w^j_R D^\alpha \varphi^i dx = 0, \quad \varphi \in [D(B(x^0, R))]^N.
$$

Now we shall investigate $v_R, w_R$.

Lemma 1.8. Let the assumptions of Lemma 1.4 be satisfied. Let $v_R$ be defined as above with $0 < R \leq R_0$, $\Omega' \subset \subset \Omega$. There exists a constant $c_1 = \ldots$
\( c_1(n, N, m, A, M, \nu, R_0, G) \) such that the following holds uniformly with respect to \( x^0 \in \Omega' \) and uniformly with respect to the class \([M] \cup [G]\) :

(1.9) \[ V^R(x^0, R) \leq c_1 R^{2 - \frac{2m}{p}}, \quad R \in (0, \min\{1, R_0\}] \]

**Proof.** Let \( v_R \in H^m_0(B(x^0, R)) \), \( R \in (0, \min\{1, R_0\}] \), be a weak solution to (1.5):

(1.10) \[ \sum_{i,j=1}^{N} \sum_{|\alpha| \leq m_i} \int A_{ij}^{\alpha} (x, \delta(u)) D^\alpha v_R^i D^\alpha \varphi^j dx \]

Let us denote the left-hand side of (1.10) by \( a(v_R, \varphi) \). Putting \( \varphi = v_R \) and using the Hölder inequality, the fact that the norms are equivalent and (0.4) we have

(1.11) \[ a(v_R, v_R) \geq \frac{1}{2} \nu |v_R|_{H^m(B(x^0, R))}^2 \]

where the constant \( \frac{1}{2} \nu \) is obtained by the choice of the constant \( R_0 \) in Lemma 1.4, and \( | \cdot |_{H^m(B(x^0, R))} \) includes derivatives of order \( m \) only. The relations (1.10), (1.11), the Hölder inequality and the fact that \( p^i > 2 \), \( (m_i - |\alpha|)(p - n) \geq 0 \), \( i = 1, \ldots, N \) imply

\[ \frac{1}{2} \nu |v_R|_{H^m(B(x^0, R))}^2 \leq \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i} \int g_\alpha^i D^\alpha v_R^i dx \]

\[ \leq c_2 G R^{\frac{2m}{p} - \frac{n}{p}} |v_R|_{H^m(B(x^0, R))} \]

From this inequality we have

(1.12) \[ \|D^\alpha v_R^i\|_{L^p(B(x^0, R))} \leq c_3 \left\{ \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i} \|g_\alpha^i\|_{L^{p_i}(\Omega)} \right\} R^{m_i - |\alpha| + \frac{n}{p} - \frac{m_i}{p}} \]

\[ |\alpha| \leq m_i, \quad i = 1, \ldots, N \]

and

(1.13) \[ |v_R|_{H^m(B(x^0, R))} \leq c_4 (A, M, \nu, R_0, G, n, m, N) R^{\frac{2m}{p} - \frac{n}{p}} \]

Now (1.13) and the inequality

\[ V^R(x^0, R) \leq R^{-n} c_5 R^2 |v_R|_{H^m(B(x^0, R))}^2 \]

imply (1.9). \( \square \)
Remark 1.14. In what follows we shall often extract subsequences without changing the notation, if there is no danger of misunderstanding.

We have a fundamental lemma due to E. Giusti [3]:

Lemma 1.15. Let $M > 0$, $G > 0$ and $u \in [M]$. Suppose that assumptions (0.3), (0.4), (0.5) are satisfied for the system (0.1). Let the right-hand sides of (0.1) belong to the class $[G]$ and let $\Lambda$ be the constant from Lemma 1.1.

Then for all $\tau \in (0, 1)$ there exist $\varepsilon_0 = \varepsilon_0(\tau, M)$, $R_0 = R_0(\tau, M)$ such that for $x^0 \in \Omega$ and $0 < R \leq \min\{R_0, \text{dist}(x^0, \partial\Omega)\}$ we have

\[(1.16) \quad W^R(x^0, R) < \varepsilon_0^2 \Rightarrow W^R(x^0, \tau R) \leq 2\Lambda \tau^2 W^R(x^0, R).\]

Proof. Let us suppose that the lemma is not true for some $\tau$. Then there exist $\{\varepsilon_s\}_{s=1}^{\infty}$, $\varepsilon_s \to 0$, $\{R_s\}_{s=1}^{\infty}$, $R_s \to 0$, $\{x^s\}_{s=1}^{\infty} \subset \Omega$, $x^s \to x^0 \in \Omega$ and $\{u_s\}_{s=1}^{\infty} \subset [M]$ such that

\[(1.17) \quad W^{sR_s}(x^s, R_s) > 2\Lambda \tau^2 W^{sR_s}(x^s, R_s) = 2\Lambda \tau^2 \varepsilon_s^2.\]

For $s = 1, 2, \ldots$ let $q_s \in P^N_{m-1}$ be such that

\[(1.18) \quad \int_{B(x^s, R_s)} D^\alpha q^j_s(x) \, dx = \int_{B(x^s, R_s)} D^\alpha w^j_{sR_s}(x) \, dx, \quad j = 1, \ldots, N, \quad |\alpha| \leq m_j - 1.\]

On the ball $B(0, 1)$ define

\[
\begin{align*}
    h^i_s(y) &= R_s^{1-m_j} \epsilon^{-1}_s [w^i_{sR_s}(x^s + R_s y) - q^i_s(x^s + R_s y)], \quad j = 1, \ldots, N, \\
    h_s(y) &= (h^1_s(y), \ldots, h^N_s(y)),
\end{align*}
\]

and put $z = x_s + R_s y$. We have

\[(1.19) \quad H_s(0, 1) = \int_{B(0,1)} \sum_{j=1}^{N} \sum_{|\alpha|=m_j-1} |D^\alpha h^j_s(y) - (D^\alpha h^j_s)_{0,1}|^2 dy = 1, \quad s = 1, 2, \ldots\]
and

\[ H_s(0, \tau) = \tau^{-n} \int B(0, \tau) \sum_{j=1}^{N} \sum_{|\alpha|=m_j-1} |D^\alpha h^j_s(y) - (D^\alpha h^j_s)_{0, \tau}|^2 \, dy \]

\[ = \varepsilon_s^{-2} \tau^{-n} R_n^{-n} \int B(0, \tau) \sum_{j=1}^{N} \sum_{|\alpha|=m_j-1} |D^\alpha w^j_s R_s(x) - D^\alpha q^j_s(x) - \varepsilon_s (D^\alpha h^j_s)_{0, \tau}|^2 \, dx \]

\[ \geq \varepsilon_s^{-2} \omega^{s R_s} \left( x^s, \tau R_s \right) > 2 \Lambda \tau^2. \]

Now let \( \psi \in \mathcal{D}(B(0, 1))^N \).

Put \( \psi^i = \varepsilon_s^{-1} R_m^{i-1} \psi^i \left( \varepsilon_s x^s \right), \) \( i = 1, \ldots, n \), in (1.7), where \( w_R = w_{s R_s} \). Using the transformation \( x = x^s + R_s y \) and the fact that \( D^\beta w^j_s R_s(x^s + R_s y) = \varepsilon_s R_s^j D^\beta h^j_s(y), \) \( |\beta| = m_j \), we have

\[ \sum_{i,j=1}^{N} \int_{B(0, 1)} R_m^{i-1} \alpha B_{ij}^{\alpha \beta}(y) D^\beta h^j_s(y) D^\alpha \psi^i(y) \, dy = 0, \]

where \( B_{ij}^{\alpha \beta}(y) = A_{ij}^{\alpha \beta}(x^s + R_s y, \delta(u_s(x^s + R_s y))) \).

The definition of \( h^j_s(y) \) and (1.6) imply

\[ D^\alpha u^j_s(x^s + R_s y) = R_m^{j-1-|\alpha|} \varepsilon_s D^\alpha h^j_s(y) \]

\[ + D^\alpha q^j_s(x^s + R_s y) + D^\alpha v^j_s R_s(x^s + R_s y), \]

\[ j = 1, \ldots, N, \quad \alpha : |\alpha| \leq m_j - 1. \]

From (1.12) it follows that

\[ D^\alpha v^j_s R_s(x^s + R_s y) \to 0 \quad \text{in} \quad L^2(B(0, 1)), \quad \alpha : |\alpha| \leq m_j - 1, \quad j = 1, \ldots, N. \]

and consequently

\[ D^\alpha v^j_s R_s(x^s + R_s y) \to 0 \quad \text{a.e. in} \ B(0, 1), \quad \alpha : |\alpha| \leq m_j - 1, \quad j = 1, \ldots, N. \]

Using (1.18), (1.19) we have for \( j = 1, \ldots, N \)

\[ \|h^j_s\|_{H^{m_j-1}(B(0, 1))} \leq c_s, \quad s = 1, 2, \ldots, \]

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and this inequality implies

\[(1.25)\quad R^{m_j-1-|\alpha|} \varepsilon D^\alpha h_i^j(y) \to 0 \quad \text{a.e. in } B(0,1), \]

\[|\alpha| \leq m_j - 1, \quad j = 1, \ldots, N\]

and

\[(1.26)\quad h_i^j \to h^j \text{ in } H^{m_j-1}(B(0,1)), \quad j = 1, \ldots, N, \]

i.e. \(D^\alpha h_i^j \to D^\alpha h^j \text{ in } L^2(B(0,1)), \quad \alpha \leq m_j - 1.\)

The polynomials in (1.18) may be written in the form

\[q_i^j(x) = \sum_{|\alpha| \leq m_j-1} c_{i,s}^{j,\alpha} x^\alpha, \quad x = x^0 + R_s y.\]

By induction, using the form of the coefficients \(c_{i,s}^{j,\alpha}\) and (1.22) we could prove: there exists a constant \(K > 0\) such that

\[(1.27)\quad |c_{i,s}^{j,\alpha}| \leq K, \quad j = 1, \ldots, N, \quad \alpha: |\alpha| \leq m_j - 1, \quad s = 1, 2, \ldots.

It follows from (1.27) that for \(j = 1, \ldots, N, |\alpha| \leq m_j - 1\) there exist subsequences \(\{c_{i,s}^{j,\alpha}\}_{s=1}^{\infty}\) such that

\[(1.28)\quad c_{i,s}^{j,\alpha} \to c_{i,s}^{j,\alpha}, \quad s \to \infty.\]

Put \(q_i^j(x) = \sum_{|\alpha| \leq m_j-1} c_i^j x^\alpha, j = 1, \ldots, N.\) It is clear that

\[D^\beta q_i^j \Rightarrow D^\beta q_j, \quad j = 1, \ldots, N, \quad \beta: |\beta| \leq m_j - 1\]

(in \(\Omega\)).

By the relations \(|D^\beta q_i^j(x^0 + R_s y) - D^\beta q_j(x^0 + R_s y)| \Rightarrow 0\) and \(|D^\beta q_i^j(x^0 + R_s y) - D^\beta q_i^j(x^0)| \Rightarrow 0\) in \(B(0,1)\) we have

\[(1.29)\quad D^\beta u_i^j(x^0 + R_s y) \Rightarrow D^\beta q_i^j(x^0) \text{ in } B(0,1), \]

\[j = 1, \ldots, N, \quad \beta: |\beta| \leq m_j - 1.\]

Using (1.22), (1.23), (1.25) and (1.29) we have

\[D^\beta u_i^j(x^0 + R_s y) \to D^\beta q_i^j(x^0) \text{ a.e. in } B(0,1), \]

\[\beta: |\beta| \leq m_j - 1, \quad j = 1, \ldots, N.\]
This and the fact that \{u_\ast\}_{\ast=1}^\infty \subset \{M\} imply that

$$|\delta(q(x^0))|_{r^n} \leq M, \quad q = (q^1, \ldots, q^N).$$

(0.3) implies

$$B_{ij\ast}^{\alpha\beta}(y) \to A_{ij\ast}^{\alpha\beta}(x^0, \delta(q(x^0))) \text{ a.e. in } B(0, 1).$$

Now let 0 < t < t_1 < 1, \chi \in D(B(0, t_1)), 0 \leq \chi \leq 1 in B(0, t_1) and \chi = 1 in B(0, t).

Let us put \psi^i = h_i^k \chi^{2k}, k = \max\{m_i\}, i = 1, \ldots, N, in (1.21). Using the Leibniz formula we have

$$\sum_{i,j=1}^N \sum_{|\alpha| \leq m_i, \gamma \leq \alpha_B(0,1)} \int \left(\frac{\alpha}{\gamma}\right)^{R^m_i - |\alpha|} B_{ij\ast}^{\alpha\beta}(y) D^\beta h_i^k(y) D^\gamma h_i^k(y) D^{\alpha - \gamma}(\chi^{2k}) \, dy = 0.$$

Using the equality $D^{\alpha - \gamma}(\chi^{2k}) = \chi^k \cdot \mathcal{Z}^{\alpha - \gamma}(\chi^{2k})$ where \mathcal{Z}^{\alpha - \gamma} contains derivatives of the function \chi, we have

$$\sum_{i,j=1}^N \sum_{|\alpha| = m_i, |\beta| = m_j} \int A_{ij}^{\alpha\beta}(x^\ast + R_s y, \delta(u_\ast(x^\ast R_s y))) (D^\beta h_i^k(y) \chi^k) (D^\alpha h_i^k(y) \chi^k) \, dy$$

$$= - \sum_{i,j=1}^N \sum_{|\alpha| = m_i, |\beta| = m_j} \left( \sum_{|\alpha| = m_i, \gamma \leq \alpha} \sum_{|\alpha| < m_i, \gamma \leq \alpha} \int \left(\frac{\alpha}{\gamma}\right)^{R^m_i - |\alpha|} B_{ij\ast}^{\alpha\beta}(y) (D^\beta h_i^k(y) \chi^k) \right)$$

$$\times D^\gamma h_i^k(y) \mathcal{Z}^{\alpha - \gamma}(\chi^{2k}) \, dy.$$

Denoting the left-hand side of (1.31) by (LS) and the right-hand side by (RS) and using (0.4) and the Hölder inequality we have

$$(LS) \geq \nu \sum_{i=1}^N \sum_{|\alpha| = m_i B(0,1)} \int |D^\alpha h_i^k \cdot \chi^k|^2 \, dy = \nu \cdot J_s$$

$$|(RS)| \leq c_7(t)(J_s)^{\frac{1}{2}} \|h_s\|_{H^{m-2}(B(0,1))}.$$

It follows from these inequalities that

$$J_s \leq c_8(t)(J_s)^{\frac{1}{2}} \|h_s\|_{H^{m-2}(B(0,1))},$$

and using (1.19) we have

$$J_s \leq c_9\|h_s\|^2_{H^{m-2}(B(0,1))} \leq c_{10}\|h_s\|^2_{H^{m-2}(B(0,1))} \leq c_{11}H_s(0, 1) = c_{11}(t).$$
The inequality $|h_s|^2_{H^m(B(0,t))} \leq J_s \leq c_{11}(t)$ and Poincaré's inequality imply

$$||h_s||_{H^m(B(0,t))} \leq c_{12}(t), \quad t \in (0,1), \quad s = 1, 2, \ldots.$$  \hspace{1cm} (1.33)

Using the imbedding theorem we obtain from (1.33) that

$$\begin{cases}
h_s \rightharpoonup h & \text{in} \ H^m(B(0,t)) \\
D^\alpha h^i \rightharpoonup D^\alpha h^i & \text{in} \ L^2(B(0,t)), \quad |\alpha| = m_i, \quad i = 1, \ldots, N \\
h_s \to h & \text{in} \ H^{m-1}(B(0,t)).
\end{cases}$$  \hspace{1cm} (1.34)

Now let us choose $t = t_r = 1 - \frac{1}{r+1}$, $r = 1, 2, \ldots$. Thanks to the diagonalization process there exists a subsequence $\{h_s\}_{s=1}^\infty$ such that

$$h_s \to h \quad \text{in} \ H^m(B(0,t_r)), \quad r \in \mathbb{N},$$  \hspace{1cm} (1.35)

$$D^\alpha h^i \to D^\alpha h^i \quad \text{in} \ L^2(B(0,t_r)), \quad r \in \mathbb{N}, \quad |\alpha| = m_i, \quad i = 1, \ldots, N,$$  \hspace{1cm} (1.36)

$$h_s \to h \quad \text{in} \ H^{m-1}(B(0,t_r)), \quad r \in \mathbb{N}.$$  \hspace{1cm} (1.37)

Let $\psi \in [\mathcal{D}(B(0,1))]^N$. The Dominated Convergence Theorem and (1.30) imply

$$B_{ij}^{\alpha\beta} D^\alpha \psi^j \to \tilde{A}_{ij}^{\alpha\beta}(x^0, \delta(q(x^0))) D^\alpha \psi^j \quad \text{in} \ L^2(B(0,1)), \quad i, j = 1, \ldots, N, \quad \alpha, \beta: |\alpha| = m_i, \quad |\beta| = m_j,$$  \hspace{1cm} (1.38)

$$R^{m_i-|\alpha|}_{ij} B_{ij}^{\alpha\beta} D^\alpha \psi^j \to 0 \quad \text{in} \ L^2(B(0,1)), \quad i, j = 1, \ldots, N, \quad \alpha, \beta: |\alpha| < m_i, \quad |\beta| = m_j.$$  \hspace{1cm} (1.39)

It is clear that for $\psi \in [\mathcal{D}(B(0,1))]^N$ there exists $r \in \mathbb{N}$ such that $\text{supp} \, \psi \subset B(0,t_r)$. Now using the limiting process and (1.36), (1.38), (1.39) we conclude from (1.21) that

$$\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \int_{B(0,1)} A_{ij}^{\alpha\beta}(x^0, \delta(q(x^0))) D^\beta h^j(y) D^\alpha \psi^i(y) \, dy = 0.$$  \hspace{1cm} (1.40)

The Hölder inequality, (1.19) and (1.26) imply that $H(0,1) \leq 1$. Using the fact that $H_s(0, \tau) \to H(0, \tau)$, $\tau \in (0,1)$ and (1.20) we have

$$H(0, \tau) \geq 2\Lambda \tau^2 > 0.$$  \hspace{1cm} (1.41)
and

\[(1.42)\]

\[H(0, 1) \geq \tau^n H(0, \tau) > 0.\]

Now (1.41) and Lemma 1.1 (applied to the system (1.40)) imply

\[2\Lambda \tau^2 H(0, 1) \leq 2\Lambda \tau^2 \leq H(0, \tau) \leq \Lambda \tau^2 H(0, 1),\]

i.e. \(H(0, 1) = 0\). This assertion contradicts (1.42). \(\square\)

**Remark 1.43.** Let \(M > 0\), \(G > 0\), \(u \in [M]\). It follows from the inequality (1.12) that there exist constants \(\gamma_1(G)\) and \(\overline{R}(M)\) such that for all \(x \in \Omega\) and \(0 < R \leq \min\{\text{dist}(x, \partial\Omega), \overline{R}\}\)

\[(1.44)\]

\[\left(\sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} R^{-n} \int_{B(x, R)} |D^\alpha v_x R(y)|^2 dy\right)^{\frac{1}{2}} \leq \gamma_1 R^\omega,\]

where

\[\omega = 1 - \frac{n}{p}.\]

Let \(A\) be the constant from Lemma 1.1 and let \(\tau \in (0, 1)\) be such that

\[(1.45)\]

\[\sqrt{2\Lambda \tau} \leq \tau^\omega < \frac{1}{2}.\]

Put \(\gamma_2 = \gamma_1(\tau^\omega + \tau^{-\frac{n}{p}})\). It is clear that there exists \(k_0 \in \mathbb{N}\) such that \(k_0 \tau^{\omega(k_0-1)} = \max_{k \in \mathbb{N}} k \tau^{\omega(k-1)} = c_0 \geq 1\). Now let \(\varepsilon_0 = \varepsilon_0(\tau, M), R_0 = R_0(\tau, M)\) be the constants from Lemma 1.15 and let \(R_1\) be chosen in such a way that

\[(1.46)\]

\[0 < R_1 \leq \min\{R_0, \overline{R}\},\]

\[(1.47)\]

\[c_0 \gamma_2 R_1^\omega < \frac{\varepsilon_0}{2},\]

\[(1.48)\]

\[\gamma_1 R_1^\omega < \frac{\varepsilon_0}{6}.\]

Put \(\delta = R_1(1 - 2^{-\frac{1}{2}})\).

**Lemma 1.49.** Let \(\mu \in [0, \omega]\). Then there exists a constant \(c > 0\) such that for all \(x^0 \in \Omega, R_1 \leq \text{dist}(x^0, \partial\Omega)\) (\(R_1\) satisfies (1.46), (1.47), (1.48)) and \(u \in [M]\) the following assertions hold:

\[W_{R_1}(x^0, R_1) < \left(\frac{\varepsilon_0}{4}\right)^2 \Rightarrow u \in C^{m-1,\mu}(B(x^0, \delta)),\]

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and
\[ \|u\|_{C^{m-1}\cdot\cdot\cdot(B(z_0,\delta))} \leq c. \]

**Proof.** Let \( u \in [M] \) and \( x \in B(x_0, \delta) \). Put \( R_x = R_1 - |x - x_0| > R_1 - \delta, \ R_x < R_1 \leq R_0 \). It is clear that \( B(x, R_x) \subset B(x_0, R_1) \).

We shall prove that
\[ (1.50) \quad W^{xR_x}(x, R_x) < \varepsilon_0^2. \]

Using (1.6), (1.44), (1.48) and the definition of \( \delta \) we have

\[
(W^{xR_x}(x, R_x))^{\frac{1}{2}} \leq \left( \frac{R_x^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int |D^\alpha w_{xR_x}^i(y) - (D^\alpha w_{x_0R_1}^i)_{x_0, R_1}|^2 dy \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{R_x^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int |D^\alpha w_{x_0R_1}^i(y) - (D^\alpha w_{x_0R_1}^i)_{x_0, R_1}|^2 dy \right)^{\frac{1}{2}}
\]

\[
+ \left( \frac{R_x^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int |D^\alpha v_{xR_x}^i(y)|^2 dy \right)^{\frac{1}{2}}
\]

\[
+ \left( \frac{R_x^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int |D^\alpha u_{xR_x}^i(y)|^2 dy \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{R_1}{R_1 - \delta} \right)^{\frac{3}{2}} \left[ (W^{R_1}(x_0, R_1))^\frac{1}{2} + \gamma_1 R_1^\omega \right] + \gamma_1 R_x^\omega < \varepsilon_0.
\]

It follows from Lemma 1.15 that if \( W^{xR}(x, R) < \varepsilon_0^2, 0 < R \leq R_x \), then
\[ (1.51) \quad W^{xR}(x, \tau R) \leq 2\Lambda \tau^2 W^{xR}(x, R). \]

Using (1.45), (1.44) and (1.51) we have

\[
(W^{x\tau R}(x, \tau R))^\frac{1}{2} \leq \left( \frac{\tau^{-n} R^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int |D^\alpha w_{x\tau R}^i(y) - (D^\alpha w_{xR}^i)_{x, \tau R}|^2 dy \right)^{\frac{1}{2}}
\]

\[
\leq \sqrt{2\Lambda \tau} (W^{xR}(x, R))^\frac{1}{2} + \gamma_1 \tau^{-\frac{3}{2}} R_x^\omega + \gamma_1 (\tau R)^\omega
\]

\[
\leq \tau^\omega (W^{xR}(x, R))^\frac{1}{2} + \gamma_2 R^\omega.
\]

By induction we obtain
\[ (1.52) \quad (W^{x\tau^k R_x}(x, \tau^k R_x))^\frac{1}{2} \leq \tau^k \omega (W^{xR_x}(x, R_x))^\frac{1}{2}
\]

\[ + \gamma_2 k \tau^{(k-1)\omega} R_x^\omega, \quad \forall k \in \mathbb{N}. \]
Using (1.52), (1.47) and (1.50) we have

\[
(W^z, r^k R_x (z, r^k R_x))^{1/2} \leq r^{\mu} \left\{ \varepsilon_0 \tau^k (w - \mu) + \frac{\varepsilon_0}{2} \tau^{-\omega} \right\}.
\]

Because \( \lim_{k \to \infty} \tau^k (w - \mu) = 0 \), \( \lim_{k \to \infty} k \tau^k (w - \mu) = 0 \), it is clear that there exists a constant \( \gamma_3 \) such that

\[
(W^z, r^k R_x (z, r^k R_x))^{1/2} \leq \gamma_3 r^{\mu}, \quad k \in \mathbb{N}.
\]

Now (1.6), (1.44) and (1.53) imply that

\[
(U(x, r^k R_x))^{1/2} \leq (W^z, r^k R_x (x, r^k R_x))^{1/2} + \gamma_1 (r^k R_x)^{\omega} \leq r^{\mu} (\gamma_3 + \gamma_1 r^k (w - \mu) R_x^{\omega}).
\]

It follows from this estimate that there exists a constant \( \gamma_4 \) such that

\[
(U(x, r^k R_x))^{1/2} \leq \gamma_4 r^{\mu}, \quad k \in \mathbb{N}.
\]

Let \( 0 < \varrho < R_1 - \delta < R_x \). Then there exists \( k \in \mathbb{N} \) such that \( r^{k+1} R_x \leq \varrho < r^k R_x \).

Using (1.54) we obtain

\[
\varrho^n U(x, \varrho) \leq \sum_{i=1}^{N} \sum_{|\alpha|=m-1} \int_{B(x, \varrho)} |D^\alpha u^i(y) - (D^\alpha u^i)_{x, r^k R_x}|^2 dy
\]

\[
\leq (r^k R_x)^n \cdot U(x, r^k R_x)
\]

and

\[
\tau^n U(x, \varrho) \leq U(x, r^k R_x) \leq \gamma_4^2 \left( \frac{\varrho}{r^k R_x} \right)^{2\mu}.
\]

The latter estimate implies that

\[
U(x, \varrho) \leq \frac{\gamma_4^2}{\tau^{n+2\mu} (R_1 - \delta)^{2\mu}} \cdot \varrho^{2\mu}
\]

and

\[
\varrho^{-(n+2\mu)} \sum_{i=1}^{N} \sum_{|\alpha|=m-1} \int_{B(x, \varrho)} |D^\alpha u^i(y) - (D^\alpha u^i)_{x, \varrho}|^2 dy \leq \frac{\gamma_4^2}{\tau^{n+2\mu} (R_1 - \delta)^{2\mu}},
\]

\( \varrho \in (0, R_1 - \delta), \quad x \in B(x^0, \delta). \)

This estimate, the definition of Campanato space and the imbedding theorem imply that the assertion of our lemma is true. \( \square \)
Remark 1.55. Using (1.6), (1.44) it is a matter of simple calculation to prove that for $x \in \Omega$, $0 < R \leq \min\{\text{dist}(x, \partial \Omega), \mathcal{R}\}$ and $u \in [M]$

\[(W^R(x, R))^\frac{1}{2} \leq (U(x, R))^\frac{1}{2} + \gamma_1 R^\omega,
(U(x, R))^\frac{1}{2} \leq (W^R(x, R))^\frac{1}{2} + \gamma_1 R^\omega.\]

From these estimates we obtain the identity

\[\liminf_{R \to 0^+} U(x, R) = \liminf_{R \to 0^+} W^R(x, R).\]

Lemma 1.49 and Remark 1.55 immediately imply

Lemma 1.56. Suppose that $u \in [M]$ and the right-hand sides of the system (0.1) belong to $[G]$. Let (0.3), (0.4), (0.5) be satisfied. Let $\Omega'$ be a domain such that $\overline{\Omega'} \subset \Omega$. Let

\[\liminf_{R \to 0^+} U(x, R) = 0\]

uniformly with respect to $x \in \overline{\Omega'}$ and $u \in [M]$.

Then $u \in C^{m-1, \mu}(\overline{\Omega'})$, $\mu \in (0, 1 - \frac{n}{p})$ and the a-priori estimate

\[\|u\|_{C^{m-1, \mu}(\overline{\Omega'})} \leq c(M, G, A, \nu, \Omega', \text{dist}(\Omega', \partial \Omega))\]

holds uniformly with respect to the class $[M] \cup [G]$.

2. MAIN RESULTS

Theorem 2.1. Let $u \in [M]$ and let the right-hand sides of the system (0.1) belong to $[G]$. Let $\Omega'$ be a domain such that $\overline{\Omega'} \subset \Omega$. Suppose that (0.3), (0.4), (0.5) and the condition (L) is satisfied. Then there exists a constant $c = c(M, G, A, \nu, \Omega')$ such that

\[\|u\|_{C^{m-1, \mu}(\overline{\Omega'})} \leq c, \quad \mu \in (0, 1 - \frac{n}{p}).\]

Proof. For all $x^0 \in \overline{\Omega'}$ and $R > 0$ we shall define the transformation $T_{x^0 R}$:

\[y = T_{x^0 R}(x) = \frac{x - x^0}{R} + x^0.\]

For $u \in [M]$ we define on $O_{x^0 R} = T_{x^0 R}(\Omega)$:

\[
\begin{cases}
    u^{i}_{x^0 R}(y) = \frac{u^{i}(x^0 + Ry)}{R^{m_i - 1}} - \sum_{|\gamma| < m_i - 1} \frac{D^\gamma u^{i}(x^0)}{R^{m_i - 1 - |\gamma|}} y^\gamma \quad \text{if } m_i > 1 \\
    u^{i}_{x^0 R}(y) = u^{i}(x^0 + Ry) \quad \text{if } m_i = 1.
\end{cases}
\]

\[\text{if } m_i > 1 \]

\[\text{if } m_i = 1.\]
From (2.2) it follows that

\[(2.3)\] \[D^{\alpha}u^i_{x^0R}(0) = 0, \quad |\alpha| \leq m_i - 2, \quad m_i > 1, \quad i = 1, \ldots, N,\]

\[(2.4)\] \[D^{\alpha}u^i(x^0 + Ry) = R^{m_i - 1 - |\alpha|}D^{\alpha}u^i_{x^0R}(y) + \sum_{\gamma < m_i - 1, \alpha \in \gamma} R^{\gamma - |\alpha|} B^{\gamma, \alpha} \frac{D^{\gamma}u(x^0)}{\gamma!} y^{\gamma - \alpha},\]

\[|\alpha| \leq m_i - 2, \quad m_i > 1, \quad i = 1, \ldots, N,\]

\[B^{\gamma, \alpha}\] being constants which are related to the derivative of \(y^\gamma\).

\[(2.5)\] \[
\left\{ \begin{array}{l}
D^{\alpha}u^i(x^0 + Ry) = D^{\alpha}u^i_{x^0R}(y) \text{ a.e. in } O_{x^0R}, \quad |\alpha| = m_i - 1, \quad i = 1, \ldots, N, \\
RD^{\alpha}u^i(x^0 + Ry) = D^{\alpha}u^i_{x^0R}(y) \text{ a.e. in } O_{x^0R}, \quad |\alpha| = m_i, \quad i = 1, \ldots, N.
\end{array} \right.
\]

Let us choose a number \(a > 0\). Then there exists \(R_0 > 0\) such that for all \(x^0 \in \Omega^r\) and \(0 < R \leq R_0\), \(B(0, 2a) \subset O_{x^0R}\). From (2.5) and (2.3) it follows that \(D^{\alpha}u^i_{x^0R}, \quad |\alpha| \leq m_i - 1, \quad i = 1, \ldots, N,\) are bounded uniformly with respect to \(x^0 \in \Omega^r\) and \(0 < R \leq R_0\). Clearly there exists a constant \(t > 0\) such that for all \(x^0 \in \Omega^r\) and \(0 < R \leq R_0\)

\[(2.6)\] \[\|u_{x^0R}\|_{H^{\frac{m_i - 1}{2}}(B(0, 2a))} \leq t.\]

Now let \(\varphi \in [D(O_{x^0R})]^N\). Putting \(R^{m_i + 1} \varphi^i(x - x^0) \in D(\Omega)\) in (0.2) as a test function and using the transformation \(x = x^0 + Ry\) we have

\[(2.7)\] \[
\sum_{i,j=1}^N \sum_{|\alpha| \leq m_i O_{x^0R}} \int R^{m_i - |\alpha| + 1} A^{\alpha}_{ij} (x^0 + Ry, \delta(u(x^0 + Ry))) D^\alpha u^j(x^0 + Ry) D^\alpha \varphi^i(y) \, dy
\]

\[= \sum_{i=1}^N \sum_{|\alpha| \leq m_i O_{x^0R}} \int R^{m_i - |\alpha| + 1} g^i_\alpha (x^0 + Ry) D^\alpha \varphi^i(y) \, dy.
\]

Let \(\chi \in D(B(0, 2a))\) be such that \(\chi = 1\) in \(B(0, a), 0 \leq \chi \leq 1\) in \(B(0, 2a)\). Using the notation from the introduction, (2.5) and putting \(\varphi^i = u^i_{x^0R} \cdot \chi_2^k, \quad i = 1, \ldots, N,\)
\[ k = \max\{m_i\} \] in (2.7) we have

\[
\sum_{i,j=1}^{N} \sum_{|\alpha|=m_i B(0,2a)} A_{ij}^{\alpha \beta} (x^0 + Ry, \delta_1 (u(x^0 + Ry)), \delta_2 (u_{x^0 R}(y))) \cdot (D^\beta u_{x^0 R}(y) \chi^k(y))
\]

\[
\times (D^\alpha u_{x^0 R}(y) \chi^k(y)) \, dy
\]

\[
= -\sum_{i,j=1}^{N} \sum_{|\alpha|=m_j} \left( \sum_{|\alpha|=m_i, \gamma<\alpha} + \sum_{|\alpha|\leq m_i, \gamma\leq\alpha} \right)
\]

\[
\int_{B(0,2a)} \left( A_{ij}^{\alpha \beta} (x^0 + Ry, \delta_1 (u(x^0 + Ry)), \delta_2 (u_{x^0 R}(y))) \right)
\]

\[
\times (D^\beta u_{x^0 R}(y) \chi^k(y)) \cdot D^\gamma u_{x^0 R}(y) Z^{\alpha-\gamma}(\chi^{2k}) \, dy
\]

\[
+ \sum_{i=1}^{N} \sum_{|\alpha|\leq m_i B(0,2a)} \int_{B(0,2a)} R^{m_i-|\alpha|+1} g_{x^0}^i (x^0 + Ry) D^\alpha (u_{x^0 R}(y) \chi^k(y)) \, dy,
\]

where \( Z^{\alpha-\gamma}(\chi^{2k}) \) is introduced in (1.31).

Let us denote the left-hand side of this equality by (LS), the first term on the right-hand side by (RS_1) and the second by (RS_2).

Using (0.4) we have

\[
\nu \cdot J =: \nu \cdot \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B(0,2a)} |D^\alpha u_{x^0 R}(y) \chi^k(y)|^2 \, dy \leq (LS).
\]

Using the fact that \( A_{ij}^{\alpha \beta} \) are uniformly bounded and the Hölder inequality we obtain

\[
|(RS_1)| \leq c_1 \| u_{x^0 R} \|_{H^{m-1}(B(0,2a))} J^\frac{1}{2}.
\]

(0.5), the Leibniz formula, the Hölder inequality and the boundedness of \( \chi^k(y) \) imply

\[
|(RS_2)| \leq c_2 J^\frac{1}{2} + c_3 \| u_{x^0 R} \|_{H^{m-\frac{1}{2}}(B(0,2a))}.
\]

Using the inequality \( (LS) \leq |(RS_1)| + |(RS_2)| \) and (2.6) we obtain the estimate

\[
J \leq c_4 J^\frac{1}{2} + c_5.
\]

This estimate and (2.6) imply that there exists a constant \( c_6 = c_6(\nu, a, M, G, A) \) such that for all \( x^0 \in \mathbb{R}^n, R \in (0, R_0], u \in [M] \)

\[
(2.8) \quad \| u_{x^0 R} \|_{H^m(B(0,a))} \leq c_6.
\]
Now we shall prove that (1.57) holds uniformly with respect to $x^0 \in \Omega^T$ and $u \in [M]$.

Let us suppose the contrary. Then there exist $\{x^k\}_{k=1}^\infty \subset \Omega^T$, $x^k \to z \in \Omega^T$, $\{R_k\}_{k=1}^\infty \subset \mathbb{R}^+$, $R_k \to 0$, $\{u_k\}_{k=1}^\infty \subset [M]$ and $\varepsilon > 0$ such that

$$U(x^k, R_k) \geq \varepsilon.$$  

(2.9)

The estimate (2.8) implies that there exists a subsequence $\{u_{k^*} R_k\}_{k=1}^\infty$ such that

$$u_{k^*} R_k \to P \text{ in } H^m(B(0,a)),$$

$$u_{k^*} R_k \to P \text{ in } H^{m-1}(B(0,a)),$$

$$D^\alpha u_{k^*} R_k \to D^\alpha P^i \text{ a.e. in } B(0,a), \quad |\alpha| \leq m_i - 1, \quad i = 1, \ldots, N.$$  

Putting $\alpha = r$, $r \in \mathbb{N}$ and using the diagonalization process we obtain for all $r \in \mathbb{N}$

$$u_{k^*} R_k \to P \text{ in } H^m(B(0,r)),$$

(2.10)

$$u_{k^*} R_k \to P \text{ in } H^{m-1}(B(0,r)),$$

(2.11)

$$D^\alpha u_{k^*} R_k \to D^\alpha P^i \text{ a.e. in } B(0,r), \quad |\alpha| \leq m_i - 1, \quad i = 1, \ldots, N.$$  

(2.12)

From (2.5) and (2.12) it follows that there exists a constant $\tau > 0$ such that

$$|D^\alpha P^i| \leq \tau, \quad |\alpha| \leq m_i - 1, \quad i = 1, \ldots, N.$$  

(2.13)

Now let $\psi \in [D(R^n)]^N$. It is clear that there exist $r, R_1$ such that $\text{supp } \psi \subset B(0,r) \subset O_{x^0 R}$ for all $x^0 \in \Omega^T$ and $0 < R \leq R_1$. Putting $\varphi = \psi$ in (2.7) we have

$$\sum_{i,j=1}^N \sum_{|\alpha| \leq m_i} \int_{\mathbb{R}^n} R_k^{m_i-|\alpha|} A^\alpha_{ij} (x^k + R_k y, \delta_1 (u_k (x^k + R_k y)), \delta_2 (u_{k^*} R_k (y)))$$

$$\times D^\alpha u_{k^*} R_k (y) D^\alpha \psi^i (y) \, dy$$

$$= \sum_{i=1}^N \sum_{|\alpha| \leq m_i} \int_{\mathbb{R}^n} R_k^{m_i-|\alpha|+1} g^i_\alpha (x^k + R_k y) D^\alpha \psi^i (y) \, dy.$$  

(2.14)

The fact that the functions $A^\alpha_{ij}$ and $D^\alpha \psi^i$ are uniformly bounded and the formula (2.10) imply

$$\sum_{i,j=1}^N \sum_{|\alpha| \leq m_i} \int_{\mathbb{R}^n} R_k^{m_i-|\alpha|} A^\alpha_{ij} (x^k + R_k y, \delta_1 (u_k (x^k + R_k y)), \delta_2 (u_{k^*} R_k (y)))$$

$$\times D^\alpha u_{k^*} R_k \cdot D^\alpha \psi^i \, dy \to 0 \text{ for } k \to \infty.$$  

(2.15)
If $|\alpha| = m_i$, $i = 1, \ldots, N$ then $R^{|\alpha|-1}_k = 1$, $x^k + R_k y \to \bar{x}$ for $k \to \infty$. Because $H^{m_i-1,\infty}(\Omega) \cap H^{m_i-1,p}(\Omega) \cap C^{m_i-2}(\Omega)$, it is clear that $u^i_k \to P^i$ in $C^{m_i-2}(\Omega)$ (i.e. $D^\alpha u^i_k \Rightarrow D^\alpha P^i$ on $\overline{\Omega}$, $|\alpha| \leq m_i - 2$, $i = 1, \ldots, N$) and $\delta_1(u_k(x^k + R_k y)) \to \delta_1(P(\bar{x}))$ in $B(0, r)$, $k \to \infty$.

From (2.5), (2.12) it follows that

$$\delta_2(u_k x^k R_k (y)) \to \delta_2(P(y)) \quad \text{a.e. in } B(0, r), \ k \to \infty.$$ 

Using (0.3), Lebesgue’s dominated convergence theorem and (2.10) we obtain

$$(2.16) \quad \int_{B(0, r)} A^\alpha \beta_{ij}(x^k + R_k y, \delta_1(u_k(x^k + R_k y)), \delta_2(u_k x^k R_k (y)))$$

$$\times D^\alpha u^i_k(\bar{x}, \delta_1(P(\bar{x})), \delta_2(P(y))) D^\beta P^j(y) D^\alpha \psi^i(y) dy$$

$$\to \int_{B(0, r)} A^\alpha \beta_{ij}(\bar{x}, \delta_1(P(\bar{x})), \delta_2(P(y))) D^\beta P^j(y) D^\alpha \psi^i(y) dy$$

for $k \to \infty$, $i, j = 1, \ldots, N$, $|\alpha| = m_i$, $|\beta| = m_j$.

The fact that $p^i_\alpha \geq 2$ ($i = 1, \ldots, N$, $|\alpha| \leq m_i$), the transformation $x = x^k + R_k y$ and the Hölder inequality imply

$$(2.17) \quad \left| \int_{B(0, r)} R^{m_i-|\alpha|+1}_k g^i_\alpha(x^k + R_k y) D^\alpha \psi^i(y) dy \right|$$

$$\leq c_7 R^{m_i-|\alpha|+1}_k R_k^{-n} \int_{B(x^k, r R_k)} \left| g^i_\alpha(x) \right| dx$$

$$\leq c_8 R^{(m_i-|\alpha|+1)(1-\frac{1}{p})}_k G \to 0 \text{ if } k \to \infty.$$

From (2.15), (2.16) and (2.17) it follows that

$$(2.18) \quad \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i}^{m_i} \sum_{|\beta|=m_j}^{m_j} \int A^\alpha \beta_{ij}(\bar{x}, \delta_1(P(\bar{x})), \delta_2(P(y))) D^\beta P^j(y) D^\alpha \psi^i(y) dy = 0,$$ 

$$\psi \in \mathcal{D}(\mathbb{R}^n)^N.$$
The condition (L) and (2.13) imply that $P \in P_{m-1}^N$. Using (2.5), (2.9), (2.11) and the transformation $x = x_k + R_ky$ we have

$$0 < \varepsilon \leq \lim_{k \to \infty} \inf \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1}^{\alpha} \int |D^\alpha u_k^i(x) - (D^\alpha u_k^i)_{x^kR_k}|^2 dx$$

$$\leq \lim_{k \to \infty} \inf \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1}^{\alpha} \int |D^\alpha u_{kx^kR_k}^i(y) - D^\alpha P_i|^2 dy = 0.$$ 

This implies that (1.57) holds uniformly with respect to $x^0 \in \Omega$ and $u \in [M]$. Lemma 1.56 implies the assertion of the theorem.

By the standard method from [2], [4] we shall prove

**Theorem 2.19.** Suppose that the system (0.1) has the property of regularity (R). Then it has Liouville's property (L).

**Proof.** Let $x^0 \in \Omega$, $\xi \in \mathbb{R}^n$ and let $u$ be a solution (in $\mathbb{R}^n$) to the system

$$\sum_{i,j=1}^{N} \sum_{|\alpha|=m_i}^{\alpha} \int A_{ij}^\alpha (x^0, \xi, \delta_2(u(x))) D^\beta u^i(x) D^\alpha \varphi^j(x) dx = 0,$$

$$\varphi \in [D(\mathbb{R}^n)]^N,$$

such that for $M > 0$

$$|D^\alpha u^i| \leq M, \quad |\alpha| \leq m_i - 1, \quad i = 1, \ldots, N.$$ 

For $R > 0$ we define

$$u^i_R(y) = \frac{u^i(Ry)}{R^{m_i-1}}, \quad i = 1, \ldots, N.$$ 

It is clear that

$$\left\{ \begin{array}{l}
D^\alpha u^i_R(y) = D^\alpha u^i(Ry), \quad |\alpha| = m_i - 1, \quad i = 1, \ldots, N, \\
D^\alpha u^i_R(y) = RD^\alpha u^i(Ry), \quad |\alpha| = m_i, \quad i = 1, \ldots, N.
\end{array} \right.$$

Let $\varphi \in [D(\mathbb{R}^n)]^N$.

Putting $\varphi \left( \frac{y}{R} \right)$ as a test function in (2.20) and using the transformation $x = Ry$ we have

$$\sum_{i,j=1}^{N} \sum_{|\alpha|=m_i}^{\alpha} \int A_{ij}^\alpha (x^0, \xi, \delta_2(u_R(y))) D^\beta u^i_R(y) D^\alpha \varphi^j(y) dy = 0.$$ 

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(2.21), (2.22) and the property (R) imply

\[(2.24) \quad |D^\alpha u_R^i(y) - D^\alpha u_R^i(0)| \leq c|y|^\mu, \quad |\alpha| = m_i - 1, \quad i = 1, \ldots, N, \quad R > 0, \quad y \in \overline{B(0, \eta)}, \quad \eta > 0, \quad \mu \in (0, 1).\]

Let us choose $x \in \mathbb{R}^n$. Then there exists $R_0 > 0$ such that $y_R = \frac{x}{R} \in \overline{B(0, \eta)}$ for all $R \geq R_0$.

Using (2.24) and (2.22) we obtain

\[(2.25) \quad |D^\alpha u^i(x) - D^\alpha u^i(0)| \leq c\frac{|x|^\mu}{R^\mu}, \quad |\alpha| = m_i - 1,
\quad i = 1, \ldots, N, \quad R \geq R_0.\]

For $R$ tending to infinity we have

$D^\alpha u^i(x) = D^\alpha u^i(0)$ for all $x \in \mathbb{R}^n$, $|\alpha| = m_i - 1$, $i = 1, \ldots, N$.

This fact implies that $u \in P^N_{m-1}$.

Remark 2.26. Using the method from [2], [4] we could prove that the system (0.1) has Liouville's property (L) for $n = 2$, i.e. for plane domains.

References


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