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ON LIOUVILLE THEOREM AND HÖLDER CONTINUITY OF WEAK
SOLUTIONS TO SOME QUASILINEAR ELLIPTIC SYSTEMS
OF HIGHER ORDER

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Summary. The aim of this paper is to show that the Liouville-type property is a sufficient
and necessary condition for the regularity of weak solutions of quasilinear elliptic systems
of higher orders.

Keywords: regularity of weak solutions, quasilinear elliptic systems

AMS classification: 35J60, 35D10

INTRODUCTION

In this paper we shall deal with quasilinear elliptic systems. More precisely we
shall consider the following problem.

Let \( \Omega \) be a bounded domain with Lipschitz boundary in \( \mathbb{R}^n \), \( n \geq 2 \). Let us
denote \( \sigma(n, k) = \binom{n+k-1}{k} \), \( \varphi(n, k) = \binom{n+k}{k} \), \( n, k \in \mathbb{N} \). We shall study weak solutions
\( u \in H^{m_i}(\Omega) \cap H^{m_{i+1} - \frac{1}{2}}(\Omega) \) to the system

\[
\sum_{i=1}^{N} \sum_{|\alpha| \leq m_i, |\beta| = m_j} (-1)^{|\alpha|} D^\alpha (A_{ij}^\alpha (z, \delta(u)) D^\beta u^j) = \sum_{|\alpha| \leq m_i} (-1)^{|\alpha|} D^\alpha g^i,
\]

(0.1)

\( i = 1, \ldots, N \), in \( \Omega \).

By a weak solution of (0.1) we mean a function \( u \in H^{m_i}(\Omega) \) \( H^{m_i}(\Omega) = H^{m_1}(\Omega) \times \ldots \times H^{m_N}(\Omega) \), \( H^{m_i}(\Omega) \) — Sobolev space, \( m_i \geq 1 \) for \( i = 1, \ldots, N \), \( u = (u^1, \ldots, u^N) \)—see
such that

\[ \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i, |\beta|=m_j} \int A_{ij}^{\alpha\beta}(x, \delta(u)) D^{\alpha} u^i D^{\beta} \varphi^j dx = \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i, \Omega} \int g_{\alpha}^i D^{\alpha} \varphi^j dx, \]

(0.2)

\[ \varphi \in [D(\Omega)]^N. \]

\[ \delta(u) = \{ D^{\alpha} u^i : |\alpha| \leq m_i - 1, i = 1, \ldots, N \}. \]

We shall assume that

(0.3)

\[ A_{ij}^{\alpha\beta} \in C(\overline{\Omega} \times \mathbb{R}^n), \quad \kappa = \sum_{i=1}^{N} \sigma(n, m_i - 1), \]

there exists \( \nu > 0 \) such that

(0.4)

\[ \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i} A_{ij}^{\alpha\beta}(x, \zeta) \xi_i^\alpha \xi_j^\beta \geq \nu \| \xi \|^2, \]

\[ (x, \zeta) \in \overline{\Omega} \times \mathbb{R}^n, \quad \xi \in \mathbb{R}^\vartheta, \quad \vartheta = \sum_{i=1}^{N} \sigma(n, m_i), \]

(0.5)

\[ g_{\alpha}^i \in L^{p_i^\vartheta}(\Omega), \quad p_{\alpha}^i = \frac{p}{m_i - |\alpha| + 1}, \]

where \( p > n, p \geq 2(\max_i(m_i) + 1) \).

For \( M > 0, G > 0 \) let us denote

\[ [M] = \{ u \in H^m(\Omega) \cap H^{m-1,\infty}(\Omega) : u \text{ is a solution to (0.1)} \]

and \( \| u \|_{H^{m-1,\infty}(\Omega)} \leq M \}, \]

\[ [G] = \{ g_{\alpha}^i \in L^{p_i^\vartheta}(\Omega) : \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i} \| g_{\alpha}^i \|_{L^{p_i^\vartheta}(\Omega)} \leq G \}, \]

\[ A = A(M) = \sup_{|\zeta| \leq M} \left\{ \sum_{i,j,\alpha,\beta} |A_{ij}^{\alpha\beta}(x, \zeta)| \right\}, \]

\[ \delta_2(u) = \{ D^{\alpha} u^i : |\alpha| = m_i - 1, i = 1, \ldots, N \}, \]

\[ \delta_1(u) = \delta(u) \setminus \delta_2(u). \]

Let \( \mathbf{s} = (s_1, \ldots, s_N), s_i \in \mathbb{N} \cup \{0\}, i = 1, \ldots, N. \) We shall use the notation \( P_{L}^N = \{(P_1, \ldots, P_N) : P_i \text{ is a polynomial such that } \deg(P_i) \leq s_i \}. \) Denote

\[ B(x^0, R) = \{ x \in \mathbb{R}^n : |x - x^0| < R \} \]

and \( \tau = \sum_{i=1}^{N} \sigma(n, m_i - 2) \) (we put \( \sigma(n, -1) = 0 \).
Definition 0.6. We say that the system (0.1) has Liouville's property (L), if for every \( x^0 \in \Omega, \xi \in \mathbb{R}^r \) every function \( v \in H_{\text{loc}}^{m\text{-loc}}(\mathbb{R}^n) \) with bounded derivatives of order \( m-1 \), solving in \( \mathbb{R}^n \) the system

\[
(0.7) \quad \sum_{i=1}^{N} \sum_{|\alpha|=m_i} (-1)^{|\alpha|} D^\alpha (A_{ij}^\alpha (x^0, \xi, \delta_2(v)) D^\beta v^i (x)) = 0, \quad i = 1, \ldots, N
\]

(i.e. \( \sum_{i,j=1}^{N} \sum_{|\alpha|=m_i, |\beta|=m_j} \int A_{ij}^\alpha (x^0, \xi, \delta_2(v)) D^\beta v^j (x) D^\alpha \varphi^i (x) \, dx = 0, \varphi \in [D(\mathbb{R}^n)]^N \))

is a polynomial from the set \( P_{m-1}^N \).

Definition 0.8. We say that the system (0.1) has the property of regularity (R) if for every \( x^0 \in \Omega, \xi \in \mathbb{R}^r, M > 0 \) there exist \( \eta > 0, c > 0 \) and \( \mu \in (0, 1) \) such that every weak solution \( u \) (in \( \mathbb{R}^n \)) of the system (0.7) with \( |D^\alpha u^i| \leq M, i = 1, \ldots, N, |\alpha| = m_i - 1 \) belongs to the space \( C^{m-1,\mu}(\overline{B(0, \eta)}) \) and \( \|u\|_{C^{m-1,\mu}(\overline{B(0, \eta)})} \leq c \).

It will be proved in this paper that the property (L) implies the interior regularity of solutions to the system (0.1), i.e. if \( u \) is a weak solution to (0.1) then \( u \in C^{m-1,\mu}(\Omega^\nu), \) where \( \Omega^\nu \subset \Omega, \mu \in (0, 1 - \frac{n}{p}) \).

It will be also shown that (R) \( \Rightarrow \) (L).

These results generalize the results of [4]. In [4] the analogous assertions are proved for quasilinear elliptic systems of the second order.

The history of the regularity problem and Liouville's property is described in [2], [4].

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1. SOME LEMMAS

Let us denote

\[
U(x^0, R) = R^{-n} \int_{B(x^0, R)} \left( \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} |D^\alpha u^i(x) - (D^\alpha u^i)_{x^0, R}|^2 \right) \, dx,
\]

\( u \in H^{m-1}(B(x^0, R)), \) where by \( (D^\alpha u^i)_{x^0, R} \) we mean the integral mean value \( D^\alpha u^i \)

in \( B(x^0, R) \).
Lemma 1.1. Let $A_{ij}^{\alpha\beta}$ be constants with $|A_{ij}^{\alpha\beta}| \leq L$, $L > 0$ and let (0.4) be satisfied for $A_{ij}^{\alpha\beta}$. Let $u \in H_{\text{loc}}^m(B(0,1)) \cap H^{m-1}(B(0,1))$ be a solution to the system

\begin{equation}
\sum_{i,j=1}^{\infty} \sum_{|\alpha|=m_i} A_{ij}^{\alpha\beta} D^\alpha u^i D^\beta \varphi^i \, dx = 0, \quad \varphi \in [\mathcal{D}(B(0,1))]^N.
\end{equation}

Then there exists a constant $\Lambda = \Lambda(n, N, L, m, \nu)$ such that for all $0 < \eta \leq 1$

\begin{equation}
U(0, \eta) \leq \Lambda \eta^2 U(0,1).
\end{equation}

The proof of this lemma is analogous to that of Lemma 2 in [3]. Using the Lax-Milgram lemma we could prove

Lemma 1.4. Suppose that $u \in [M], x^0 \in \Omega$. Let (0.3), (0.4), (0.5) be satisfied and let the right-hand sides of the system (0.1) belong to $[G]$. Then there exists $R_0 = R_0(A, M), 0 < R_0 \leq \text{dist}(x^0, \partial \Omega)$ such that for all $R \in (0, R_0]$ the linear elliptic system

\begin{equation}
\sum_{\alpha} (-1)^{|\alpha|} D^\alpha (A_{ij}^{\alpha\beta}(x, \delta(u)) D^\beta v_R^i) = \sum_{\alpha} (-1)^{|\alpha|} D^\alpha g^i, \quad i = 1, \ldots, N,
\end{equation}

has a unique weak solution in $H_0^m(B(x^0, R))$.

Since (1.5) is uniquely solvable for $R \leq R_0$ we may decompose any solution $u$ of the quasilinear system (0.1) in the following manner:

\begin{equation}
u = v_R + w_R,
\end{equation}

where $v_R \in H_0^m(B(x^0, R))$ solves the system (1.5) and

\begin{equation}
\sum_{i,j=1}^{\infty} \sum_{|\alpha|=m_i} \int_{B(x^0, R)} \mathcal{A}_{ij}^{\alpha\beta}(x, \delta(u)) D^\alpha w_R^i D^\beta \varphi^i \, dx = 0, \quad \varphi \in [\mathcal{D}(B(x^0, R))]^N.
\end{equation}

Now we shall investigate $v_R, w_R$.

Lemma 1.8. Let the assumptions of Lemma 1.4 be satisfied. Let $v_R$ be defined as above with $0 < R \leq R_0$, $\Omega' \subset \subset \Omega$. There exists a constant $c_1 = \ldots$
$c_1(n, N, m, A, M, \nu, R_0, G)$ such that the following holds uniformly with respect to $x^0 \in \Omega'$ and uniformly with respect to the class $[M] \cup [G]$:

$$(1.9) \quad V^R(x^0, R) \leq c_1 R^{2 - \frac{2n}{m}} , \quad R \in (0, \min\{1, R_0\}).$$

Proof. Let $v_R \in H^m_0(B(x^0, R))$, $R \in (0, \min\{1, R_0\})$, be a weak solution to (1.5):

$$\sum_{i,j=1}^N \sum_{|\alpha| \leq m_i B(x^0, R)} A_{ij}^\alpha (z, \delta(v)) D^\alpha v_R^i D^\alpha \varphi^j dz$$

$$= \sum_{i=1}^N \sum_{|\alpha| \leq m_i B(x^0, R)} \int g^i_\alpha D^\alpha \varphi^i dx, \quad \varphi \in [D(B(x^0, R))]^N. \tag{1.10}$$

Let us denote the left-hand side of (1.10) by $a(v_R, \varphi)$. Putting $\varphi = v_R$ and using the Hölder inequality, the fact that the norms are equivalent and (0.4) we have

$$\tag{1.11} a(v_R, v_R) \geq \frac{1}{2} |v_R|_{H^m(B(x^0, R))}^2,$$

where the constant $\frac{1}{2} \nu$ is obtained by the choice of the constant $R_0$ in Lemma 1.4, and $|\cdot|_{H^m(B(x^0, R))}$ includes derivatives of order $m$ only. The relations (1.10), (1.11), the Hölder inequality and the fact that $p_\alpha \geq 2$, $(m_i - |\alpha|)(p - n) \geq 0, i = 1, \ldots, N$ imply

$$\frac{1}{2} \nu |v_R|_{H^m(B(x^0, R))}^2 \leq \sum_{i=1}^N \sum_{|\alpha| \leq m_i B(x^0, R)} \int g^i_\alpha D^\alpha v_R^i dz$$

$$\leq c_2 G R^{\frac{3}{2} - \frac{n}{p}} |v_R|_{H^m(B(x^0, R))}.$$ 

From this inequality we have

$$\tag{1.12} \|D^\alpha v_R^i\|_{L^2(B(x^0, R))} \leq c_3 \left\{ \sum_{i=1}^N \sum_{|\alpha| \leq m_i} \|g^i_\alpha\|_{L^1(\Omega)} \right\} R^{m_i - |\alpha| - \frac{3}{2} - \frac{n}{p}},$$

$$|\alpha| \leq m_i, \quad i = 1, \ldots, N$$

and

$$\tag{1.13} |v_R|_{H^m(B(x^0, R))} \leq c_4 (A, M, \nu, R_0, G, n, m, N) R^{\frac{3}{2} - \frac{n}{p}}.$$ 

Now (1.13) and the inequality

$$V^R(x^0, R) \leq R^{-n} c_5 R^2 |v_R|_{H^m(B(x^0, R))}^2$$

imply (1.9). \hfill \Box
Remark 1.14. In what follows we shall often extract subsequences without changing the notation, if there is no danger of misunderstanding.

We have a fundamental lemma due to E. Giusti [3]:

Lemma 1.15. Let $M > 0$, $G > 0$ and $u \in [M]$. Suppose that assumptions (0.3), (0.4), (0.5) are satisfied for the system (0.1). Let the right-hand sides of (0.1) belong to the class $[G]$ and let $\Lambda$ be the constant from Lemma 1.1.

Then for all $\tau \in (0,1)$ there exist $\epsilon_0 = \epsilon_0(\tau, M)$, $R_0 = R_0(\tau, M)$ such that for $x^0 \in \Omega$ and $0 < R \leq \min\{R_0, \text{dist}(x^0, \partial \Omega)\}$ we have

\[(1.16) \quad W^R(x^0, R) < \epsilon_0^2 \Rightarrow W^R(x^0, \tau R) \leq 2\Lambda \tau^2 W^R(x^0, R).\]

Proof. Let us suppose that the lemma is not true for some $\tau$. Then there exist $\{\epsilon_s\}_{s=1}^\infty$, $\epsilon_s \to 0$, $\{R_s\}_{s=1}^\infty$, $R_s \to 0$, $\{x^s\}_{s=1}^\infty \subset \Omega$, $x^s \to x^0 \in \overline{\Omega}$ and $\{u_s\}_{s=1}^\infty \subset [M]$ such that

\[W^{sR_s}(x^s, R_s) = \epsilon_s^2\]

and

\[(1.17) \quad W^{sR_s}(x^s, \tau R_s) > 2\Lambda \tau^2 W^{sR_s}(x^s, R_s) = 2\Lambda \tau^2 \epsilon_s^2.\]

For $s = 1, 2, \ldots$ let $q_j \in P^N_{m_j}$ be such that

\[(1.18) \quad \int_{B(x^s, R_s)} D^\alpha q_j^s(x) \, dx = \int_{B(x^s, R_s)} D^\alpha u_{jR_s}^s(x) \, dx, \quad j = 1, \ldots, N, \quad |\alpha| \leq m_j - 1.\]

On the ball $B(0, 1)$ define

\[h_j^s(y) = R_s^{1-m_j} \epsilon_s^{-1} [u_{jR_s}(x^s + R_s y) - q_j^s(x^s + R_s y)], \quad j = 1, \ldots, N,\]

\[h_s(y) = (h_1^s(y), \ldots, h_N^s(y)),\]

and put $z = x_s + R_s y$. We have

\[(1.19) \quad H_s(0, 1) = \int_{B(0,1)} \sum_{j=1}^N \sum_{|\alpha| = m_j - 1 \atop B(0,1)} |D^\alpha h_j^s(y) - (D^\alpha h_j^s)_{0,1}|^2 \, dy = 1, \quad s = 1, 2, \ldots\]
and

\[(1.20)\quad H_s(0, \tau) = \tau^{-n} \int \sum_{j=1}^{N} \sum_{|\alpha|=m_j-1} |D^{\alpha} h^j_s(y) - (D^{\alpha} h^j_s)_{0,\tau}|^2 \, dy \]

\[= \epsilon_s^{-2} \tau^{-n} R_s^{-n} \sum_{j=1}^{N} \sum_{|\alpha|=m_j-1} \int |D^{\alpha} w^j_{sR_s}(x) - D^{\alpha} q^j_s(x) - \epsilon_s (D^{\alpha} h^j_s)_{0,\tau}|^2 \, dx \]

\[\geq \epsilon_s^{-2} W^{sR_s}(x^s, \tau R_s) > 2\Lambda \tau^2.\]

Now let \( \psi \in [D(B(0,1))]^N \).

Put \( \varphi^i = \epsilon_s^{-1} R_s^{m_j+1} \psi^i(\frac{\cdot - x^s}{R_s}), \) \( i = 1, \ldots, n, \) in (1.7), where \( w_R = w_{sR_s} \). Using the transformation \( x = x^s + R_s y \) and the fact that \( D^{\beta} w^j_{sR_s}(x^s + R_s y) = \epsilon_s R_s^{-1} D^{\beta} h^j_s(y) \), \( |\beta| = m_j \), we have

\[(1.21)\quad \sum_{i,j=1}^{N} \sum_{|\alpha|\leq m_j} \int_{B(0,1)} R_s^{m_j-|\alpha|} B_{ijj}^{\alpha\beta} (y) \int \sum_{|\beta|\leq m_j} |D^{\alpha} h^j_s(y) D^{\alpha} \psi^i(y) \, dy = 0,\]

where \( B_{ijj}^{\alpha\beta}(y) = A_{ijj}^{\alpha\beta}(x^s + R_s y, \delta(u^s(x^s + R_s y))). \)

The definition of \( h^j_s(y) \) and (1.6) imply

\[(1.22)\quad D^{\alpha} u^j_s(x^s + R_s y) = R_s^{m_j-1-|\alpha|} \epsilon_s D^{\alpha} h^j_s(y) + D^{\alpha} q^j_s(x^s + R_s y) + D^{\alpha} v^j_{sR_s}(x^s + R_s y), \]

\[j = 1, \ldots, N, \quad \alpha: |\alpha| \leq m_j - 1.\]

From (1.12) it follows that

\[D^{\alpha} v^j_{sR_s}(x^s + R_s y) \to 0 \quad \text{in } L^2(B(0,1)), \quad \alpha: |\alpha| \leq m_j - 1, \quad j = 1, \ldots, N.\]

and consequently

\[(1.23)\quad D^{\alpha} v^j_{sR_s}(x^s + R_s y) \to 0 \quad \text{a.e. in } B(0,1), \quad \alpha: |\alpha| \leq m_j - 1, \quad j = 1, \ldots, N.\]

Using (1.18), (1.19) we have for \( j = 1, \ldots, N \)

\[(1.24)\quad ||h^j_s||_{H^{-\frac{1}{2}}(B(0,1))} \leq c_6, \quad s = 1, 2, \ldots,\]
and this inequality implies

\begin{equation}
R_s^{m_j-1-|\alpha|} \epsilon \alpha \delta^{H_j} h^j_i(y) \to 0 \quad \text{a.e. in } B(0,1), \quad |\alpha| \leq m_j - 1, \quad j = 1, \ldots, N
\end{equation}

and

\begin{equation}
h^j_i \to h^j \quad \text{in } H^{m_j-1}(B(0,1)), \quad j = 1, \ldots, N,
\end{equation}
i.e. \(D^\alpha h^j_i \to D^\alpha h^j \quad \text{in } L^2(B(0,1)), \quad \alpha \leq m_j - 1.

The polynomials in (1.18) may be written in the form

\[ q^j_i(x) = \sum_{|\alpha| \leq m_j - 1} c^{j,s}_\alpha x^\alpha, \quad z = z + R_s y. \]

By induction, using the form of the coefficients \(c^{j,s}_\alpha\) and (1.22) we could prove: there exists a constant \(K > 0\) such that

\begin{equation}
|c^{j,s}_\alpha| \leq K, \quad j = 1, \ldots, N, \quad \alpha: |\alpha| \leq m_j - 1, \quad s = 1, 2, \ldots
\end{equation}

It follows from (1.27) that for \(j = 1, \ldots, N, \quad |\alpha| \leq m_j - 1\) there exist subsequences \(\{c^{j,s}_\alpha\}_{s=1}^\infty\) such that

\begin{equation}
c^{j,s}_\alpha \to c^{j}_\alpha, \quad s \to \infty.
\end{equation}

Put \(q^j_i(x) = \sum_{|\alpha| \leq m_j - 1} c^{j}_\alpha x^\alpha, j = 1, \ldots, N\). It is clear that

\[ D^\beta q^j_i \Rightarrow D^\beta q^j, \quad j = 1, \ldots, N, \quad \beta: |\beta| \leq m_j - 1 \]

(in \(\Omega\)).

By the relations \(|D^\beta q^j_i(x^s + R_s y) - D^\beta q^j_i(x^s + R_s y)| \Rightarrow 0\) and \(|D^\beta q^j_i(x^s + R_s y) - D^\beta q^j_i(x^0)| \Rightarrow 0\) in \(B(0,1)\) we have

\begin{equation}
D^\beta q^j_i(x^s + R_s y) \Rightarrow D^\beta q^j_i(x^0) \quad \text{in } B(0,1), \quad j = 1, \ldots, N, \quad \beta: |\beta| \leq m_j - 1.
\end{equation}

Using (1.22), (1.23), (1.25) and (1.29) we have

\[ D^\beta u^j_i(x^s + R_s y) \to D^\beta q^j_i(x^0) \quad \text{a.e. in } B(0,1), \quad \beta: |\beta| \leq m_j - 1, \quad j = 1, \ldots, N. \]
This and the fact that \( \{u_\ast\}_{\ast=1}^\infty \subset [M] \) imply that
\[
|\delta(q(x^0))|_{\mathbb{R}^d} \leq M, \quad q = (q^1, \ldots, q^N).
\]
(0.3) implies
(1.30)
\[
B^\alpha_\beta(y) \to A^\alpha_\beta(x^0, \delta(q(x^0))) \text{ a.e. in } B(0, 1).
\]
Now let \( 0 < t < t_1 < 1, \chi \in D(B(0, t_1)), 0 \leq \chi \leq 1 \) in \( B(0, t_1) \) and \( \chi = 1 \) in \( B(0, t) \).
Let us put \( \psi_i = h_i^k \chi^{2k}, \) \( k = \max\{m_i\}, i = 1, \ldots, N, \) in (1.21). Using the Leibniz formula we have
\[
\sum_{i,j=1}^N \sum_{|\alpha| \leq m_i, \gamma \leq \alpha} \int \left( \chi \right)^{\frac{m_i}{\gamma}} R^\gamma_{\alpha} B^\alpha_\beta(y) D^\beta h^i(y) D^\gamma h^i(y) D^{\alpha-\gamma}(\chi^{2k}) \, dy = 0.
\]
Using the equality \( D^{\alpha-\gamma}(\chi^{2k}) = \chi^k \cdot Z^{\alpha-\gamma}(\chi^{2k}) \) where \( Z^{\alpha-\gamma} \) contains derivatives of the function \( \chi \), we have
(1.31)
\[
\sum_{i,j=1}^N \sum_{|\alpha| = m_i, \beta = m_j} \int A^\alpha_\beta(x^t + R_s y, \delta(u_s(x^t R_s y))) (D^\beta h^i(y) \chi^k) (D^\alpha h^i(y) \chi^k) \, dy
\]
\[
= - \sum_{i,j=1}^N \sum_{|\alpha| = m_i, \gamma \leq \alpha} \sum_{|\alpha| > m_i, \gamma \leq \alpha} \int \left( \chi \right)^{\frac{m_i}{\gamma}} R^\gamma_{\alpha} B^\alpha_\beta(y) (D^\alpha h^i(y) \chi^k)
\]
\[
\times D^\gamma h^i(y) Z^{\alpha-\gamma}(\chi^{2k}) \, dy.
\]
Denoting the left-hand side of (1.31) by (LS) and the right-hand side by (RS) and using (0.4) and the Hölder inequality we have
\[
(LS) \geq \nu \sum_{i=1}^N \sum_{|\alpha| = m_i B(0, 1)} \int |D^\alpha h^i \cdot \chi^k|^2 \, dy = \nu \cdot J_s
\]
\[
|RS| \leq c_7(t)(J_s)^{\frac{1}{2}} \|h_s\|_{H^{m-1}(B(0, 1))}.
\]
It follows from these inequalities that
(1.32)
\[
J_s \leq c_8(t)(J_s)^{\frac{1}{2}} \|h_s\|_{H^{m-1}(B(0, 1))},
\]
and using (1.19) we have
\[
J_s \leq c_9 \|h_s\|_{H^{m-1}(B(0, 1))}^2 \leq c_{10} \|h_s\|_{H^{m-1}(B(0, 1))}^2 \leq c_{11} H_s(0, 1) = c_{11}(t).
\]
The inequality \( |h_s|^2_{H^m(B(0,t))} \leq J_s \leq c_{11}(t) \) and Poincaré’s inequality imply

\[(1.33) \quad \|h_s\|_{H^m(B(0,t))} \leq C(t), \quad t \in (0,1), \quad s = 1, 2, \ldots.\]

Using the imbedding theorem we obtain from (1.33) that

\[(1.34) \quad \begin{cases}
  h_s \rightharpoonup h & \text{in} \quad H^m(B(0,t)) \\
  D^\alpha h_s^i \rightharpoonup D^\alpha h^i & \text{in} \quad L^2(B(0,t)), \quad |\alpha| = m_i, \quad i = 1, \ldots, N \\
  h_s \rightharpoonup h & \text{in} \quad H^{m-1}(B(0,t)).
\end{cases}\]

Now let us choose \( t = t_r = 1 - \frac{1}{r+1}, \quad r = 1, 2, \ldots \). Thanks to the diagonalization process there exists a subsequence \( \{h_s\}_{s=1}^\infty \) such that

\[(1.35) \quad h_s \rightharpoonup h \quad \text{in} \quad H^m(B(0,t_r)), \quad r \in \mathbb{N},\]

\[(1.36) \quad D^\alpha h_s^i \rightharpoonup D^\alpha h^i \quad \text{in} \quad L^2(B(0,t_r)), \quad r \in \mathbb{N}, \quad |\alpha| = m_i, \quad i = 1, \ldots, N,\]

\[(1.37) \quad h_s \rightharpoonup h \quad \text{in} \quad H^{m-1}(B(0,t_r)), \quad r \in \mathbb{N}.
\]

Let \( \psi \in \mathcal{D}(B(0,1))^N \). The Dominated Convergence Theorem and (1.30) imply

\[B_{ij}^{\alpha \beta} D^\alpha \psi^i \rightarrow A_{ij}^{\alpha \beta}(x^0, \delta(q(x^0))) D^\alpha \psi^i \quad \text{in} \quad L^2(B(0,1)), \quad i, j = 1, \ldots, N, \quad \alpha, \beta: |\alpha| = m_i, \quad |\beta| = m_j,\]

\[R_s^{m_i-|\alpha|} B_{ij}^{\alpha \beta} D^\alpha \psi^i \rightarrow 0 \quad \text{in} \quad L^2(B(0,1)), \quad i, j = 1, \ldots, N, \quad \alpha, \beta: |\alpha| < m_i, \quad |\beta| = m_j.\]

It is clear that for \( \psi \in \mathcal{D}(B(0,1))^N \) there exists \( r \in \mathbb{N} \) such that \( \text{supp} \psi \subset B(0, t_r) \). Now using the limiting process and (1.36), (1.38), (1.39) we conclude from (1.21) that

\[\sum_{i,j=1}^N \sum_{|\alpha|=m_i,B(0,1)} \int_{B(0,1)} A_{ij}^{\alpha \beta}(x^0, \delta(q(x^0))) D^\beta h^i(y) D^\alpha \psi^i(y) \, dy = 0.\]

The Hölder inequality, (1.19) and (1.26) imply that \( H(0,1) \leq 1 \). Using the fact that \( H_s(0, \tau) \rightarrow H(0, \tau), \quad \tau \in (0,1) \) and (1.20) we have

\[H(0, \tau) \geq 2\Lambda \tau^2 > 0.\]
and

\[(1.42) \quad H(0, 1) \geq \tau^n H(0, \tau) > 0.\]

Now (1.41) and Lemma 1.1 (applied to the system (1.40)) imply

\[2\Lambda \tau^2 H(0, 1) \leq 2\Lambda \tau^2 \leq H(0, \tau) \leq \Lambda \tau^2 H(0, 1),\]

i.e. \(H(0, 1) = 0\). This assertion contradicts (1.42). \(\Box\)

**Remark 1.43.** Let \(M > 0, G > 0, u \in [M]\). It follows from the inequality (1.12) that there exist constants \(\gamma_1(G)\) and \(\overline{R}(M)\) such that for all \(x \in \Omega\) and \(0 < R \leq \min\{\text{dist}(x, \partial\Omega), \overline{R}\}\)

\[(1.44) \quad \left(\sum_{i=1}^N \sum_{|a|=m_i - 1} R^{-n} \int_{B(x, R)} |D^a v^i_{xR}(y)|^2 dy\right)^{\frac{1}{2}} \leq \gamma_1 R^\omega,\]

where

\[\omega = 1 - \frac{n}{p}.\]

Let \(A\) be the constant from Lemma 1.1 and let \(\tau \in (0, 1)\) be such that

\[(1.45) \quad \sqrt{2\Lambda \tau} \leq \tau^\omega < \frac{1}{2}.\]

Put \(\gamma_2 = \gamma_1(\tau^\omega + \tau^{-\frac{\omega}{2}})\). It is clear that there exists \(k_0 \in \mathbb{N}\) such that \(k_0 \tau^\omega(k_0 - 1) = \max_{k \in \mathbb{N}} k \tau^\omega(k - 1) = c_0 > 1\). Now let \(e_0 = e_0(\tau, M), R_0 = R_0(\tau, M)\) be the constants from Lemma 1.15 and let \(R_1\) be chosen in such a way that

\[(1.46) \quad 0 < R_1 \leq \min\{R_0, \overline{R}\},\]
\[(1.47) \quad c_0 \gamma_2 R_1^\omega < \frac{e_0}{2},\]
\[(1.48) \quad \gamma_1 R_1^\omega < \frac{e_0}{6}.\]

Put \(\delta = R_1(1 - 2^{-\frac{\omega}{2}})\).

**Lemma 1.49.** Let \(\mu \in [0, \omega]\). Then there exists a constant \(c > 0\) such that for all \(x^0 \in \Omega, R_1 \leq \text{dist}(x^0, \partial\Omega)\) \(R_1\) satisfies (1.46), (1.47), (1.48)) and \(u \in [M]\) the following assertions hold:

\[W_{R_1}(x^0, R_1) < \left(\frac{e_0}{4}\right)^2 \Rightarrow u \in C^{m-1, \mu}(B(x^0, \delta)),\]

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and
\[ \|u\|_{C^{m-1,p}(B(x^0, \delta))} \leq c. \]

Proof. Let \( u \in [M] \) and \( x \in B(x^0, \delta) \). Put \( R_x = R_1 - |x - x^0| > R_1 - \delta \), \( R_x < R_1 \leq R_0 \). It is clear that \( B(x, R_x) \subset B(x^0, R_1) \).

We shall prove that
\[ W^{x R_x}(x, R_x) < \varepsilon^2_0. \]

Using (1.6), (1.44), (1.48) and the definition of \( \delta \) we have
\[ (W^{x R_x}(x, R_x))^{\frac{1}{2}} \leq \left( \frac{R_x^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int_{B(x, R_x)} |D^{\alpha} w^i_{x R_x}(y) - (D^{\alpha} w^i_{x R_1})_{x^0, R_1}|^2 dy \right)^{\frac{1}{2}} \]
\[ \leq \left( \frac{R_x^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int_{B(x, R_x)} |D^{\alpha} w^i_{x R_1}(y) - (D^{\alpha} w^i_{x R_1})_{x^0, R_1}|^2 dy \right)^{\frac{1}{2}} \]
\[ + \left( \frac{R_x^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int_{B(x, R_x)} |D^{\alpha} w^i_{x R_x}(y)|^2 dy \right)^{\frac{1}{2}} \]
\[ + \left( \frac{R_x^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int_{B(x, R_x)} |D^{\alpha} w^i_{x R_x}(y)|^2 dy \right)^{\frac{1}{2}} \]
\[ \leq \left( \frac{R_1}{R_1 - \delta} \right)^{\frac{1}{2}} \left[ (W^{R_1(x^0, R_1)})^{\frac{1}{2}} + \gamma_1 R_1^\omega + \gamma_1 R_x^\omega \right] < \varepsilon_0. \]

It follows from Lemma 1.15 that if \( W^{x, R}(x, R) < \varepsilon_0^3 \), \( 0 < R \leq R_x \), then
\[ (1.51) \quad W^{x, R}(x, \tau R) \leq 2\Lambda \tau^2 W^{x, R}(x, R). \]

Using (1.45), (1.44) and (1.51) we have
\[ (W^{x, R}(x, \tau R))^{\frac{1}{2}} \leq \left( \frac{\tau^{-n} R^{-n}}{R_1} \sum_{i=1}^{N} \sum_{|\alpha| = m_i - 1} \int_{B(x, \tau R)} |D^{\alpha} w^i_{x R}(y) - (D^{\alpha} w^i_{x R})_{x, \tau R}|^2 dy \right)^{\frac{1}{2}} \]
\[ \leq \sqrt{2\Lambda \tau} (W^{x, R}(x, R))^{\frac{1}{2}} + \gamma_1 \tau^{-\frac{\omega}{2}} R_1^\omega + \gamma_1 (\tau R)^\omega \]
\[ \leq \tau^\omega (W^{x, R}(x, R))^{\frac{1}{2}} + \gamma_2 R_x^\omega. \]

By induction we obtain
\[ (1.52) \quad (W^{x, R^k R_x}(x, \tau^k R_x))^{\frac{1}{2}} \leq \tau^k \omega (W^{x, R_x}(x, R_x))^{\frac{1}{2}} \]
\[ + \gamma_2 k \tau^{-k+1} \omega R_x^\omega, \quad \forall k \in \mathbb{N}. \]
Using (1.52), (1.47) and (1.50) we have

\[
(W^x, r^k R_x(x, r^k R_x))^{1/2} \leq \varphi^k \{ \varphi_0 r^k (\varphi - \mu) + \frac{\varphi_0}{2} r^{-\omega} k r^k (\varphi - \mu) \}.
\]

Because \( \lim_{k \to \infty} r^k (\varphi - \mu) = 0 \), \( \lim_{k \to \infty} k r^k (\varphi - \mu) = 0 \), it is clear that there exists a constant \( \gamma_3 \) such that

\[
(1.53) \quad (W^x, r^k R_x(z, r^k R_x))^2 \leq \gamma_3 r^k \mu, \quad k \in \mathbb{N}.
\]

Now (1.6), (1.44) and (1.53) imply that

\[
(U(x, r^k R_x))^2 \leq (W^x, r^k R_x(x, r^k R_x))^2 + \gamma_1 (r^k R_x)^2
\leq r^k \mu (\gamma_3 + \gamma_1 r^k (\varphi - \mu) R_x^2).
\]

It follows from this estimate that there exists a constant \( \gamma_4 \) such that

\[
(1.54) \quad (U(x, r^k R_x))^2 \leq \gamma_4 r^k \mu, \quad k \in \mathbb{N}.
\]

Let \( 0 < \varphi < R_1 - \delta < R_x \). Then there exists \( k \in \mathbb{N} \) such that \( r^{k+1} R_x \leq \varphi < r^k R_x \). Using (1.54) we obtain

\[
\varphi^n U(x, \varphi) \leq \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B(x, \varphi)} |D^\alpha u^i(y) - (D^\alpha u^i)_{x, r^k R_x}|^2 dy
\leq (r^k R_x)^n U(x, r^k R_x)
\]
and

\[
\varphi^n U(x, \varphi) \leq U(x, r^k R_x) \leq \gamma_4 \left( \frac{\varphi}{r^k R_x} \right)^{2\mu}.
\]

The latter estimate implies that

\[
U(x, \varphi) \leq \frac{\gamma_4^2}{r^n + 2\mu (R_1 - \delta)^{2\mu}} \cdot \varphi^{2\mu}
\]
and

\[
\varphi^{-(n+2\mu)} \sum_{i=1}^{N} \sum_{|\alpha|=m_i} \int_{B(x, \varphi)} |D^\alpha u^i(y) - (D^\alpha u^i)_{x, \varphi}|^2 dy \leq \frac{\gamma_4^2}{r^n + 2\mu (R_1 - \delta)^{2\mu}},
\]
\[ \varphi \in (0, R_1 - \delta), \quad x \in B(x^0, \delta). \]

This estimate, the definition of Campanato space and the imbedding theorem imply that the assertion of our lemma is true. \( \square \)
Remark 1.55. Using (1.6), (1.44) it is a matter of simple calculation to prove that for $x \in \Omega$, $0 < R \leq \min\{\text{dist}(x, \partial \Omega), R\}$ and $u \in [M]$

$$
(W^R(x, R))^\frac{1}{2} \leq (U(x, R))^\frac{1}{2} + \gamma_1 R^\omega,
(U(x, R))^\frac{1}{2} \leq (W^R(x, R))^\frac{1}{2} + \gamma_1 R^\omega.
$$

From these estimates we obtain the identity

$$
\lim_{R \to 0^+} \inf U(x, R) = \lim_{R \to 0^+} \inf W^R(x, R).
$$

Lemma 1.49 and Remark 1.55 immediately imply

Lemma 1.56. Suppose that $u \in [M]$ and the right-hand sides of the system (0.1) belong to $[G]$. Let (0.3), (0.4), (0.5) be satisfied. Let $\Omega'$ be a domain such that $\overline{\Omega'} \subset \Omega$. Let

$$(1.57) \quad \lim_{R \to 0^+} \inf U(x, R) = 0$$

uniformly with respect to $x \in \overline{\Omega'}$ and $u \in [M]$.

Then $u \in C^{m - \frac{1}{2}, \mu}(\overline{\Omega'})$, $\mu \in (0, 1 - \frac{n}{p})$ and the a-priori estimate

$$(1.58) \quad \|u\|_{C^{m - \frac{1}{2}, \mu}(\overline{\Omega'})} \leq c(M, G, A, \nu, \Omega', \text{dist}(\Omega', \partial \Omega))$$

holds uniformly with respect to the class $[M] \cup [G]$.

2. MAIN RESULTS

Theorem 2.1. Let $u \in [M]$ and let the right-hand sides of the system (0.1) belong to $[G]$. Let $\Omega'$ be a domain such that $\overline{\Omega'} \subset \Omega$. Suppose that (0.3), (0.4), (0.5) and the condition (L) is satisfied. Then there exists a constant $c = c(M, G, A, \nu, \Omega')$ such that

$$
\|u\|_{C^{m - \frac{1}{2}, \mu}(\overline{\Omega'})} \leq c, \quad \mu \in \left(0, 1 - \frac{n}{p}\right).
$$

Proof. For all $x^0 \in \overline{\Omega'}$ and $R > 0$ we shall define the transformation $T_{x^0 R}$:

$$
y = T_{x^0 R}(x) = \frac{x - x^0}{R}.
$$

For $u \in [M]$ we define on $O_{x^0 R} = T_{x^0 R}(\Omega)$:

$$(2.2) \quad \begin{cases}
u^i_{x^0 R}(y) = \frac{u^i(x^0 + Ry)}{R^{m_i - 1}} - \sum_{|\gamma|<m_i-1} \frac{D^\gamma u^i(x^0)}{R^{m_i - 1 - |\gamma|}} y^\gamma & \text{if } m_i > 1 \\
u^i_{x^0 R}(y) = u^i(x^0 + Ry) & \text{if } m_i = 1.
\end{cases}$$
From (2.2) it follows that

(2.3) \[ D^\alpha u^i_{x^0 R}(0) = 0, \quad |\alpha| \leq m_i - 2, \quad m_i > 1, \quad i = 1, \ldots, N, \]

(2.4) \[ D^\alpha u^i(x^0 + Ry) = R^{m_i-1-|\alpha|} D^\alpha u^i_{x^0 R}(y) + \sum_{|\gamma| \leq m_i-1, \alpha \leq \gamma} R^{\gamma-|\alpha|} B_{\gamma, \alpha} \frac{D^\gamma u^0(x^0)}{\gamma!} y^{\gamma-\alpha}, \]

\[ |\alpha| \leq m_i - 2, \quad m_i > 1, \quad i = 1, \ldots, N, \]

\[ B_{\gamma, \alpha} \text{ being constants which are related to the derivative of } \varphi. \]

(2.5) \[
\left\{ \begin{array}{ll}
D^\alpha u^i(x^0 + Ry) = D^\alpha u^i_{x^0 R}(y) \text{ a.e. in } O_{x^0 R}, & |\alpha| = m_i - 1, \quad i = 1, \ldots, N, \\
RD^\alpha u^i(x^0 + Ry) = D^\alpha u^i_{x^0 R}(y) \text{ a.e. in } O_{x^0 R}, & |\alpha| = m_i, \quad i = 1, \ldots, N.
\end{array} \right.
\]

Let us choose a number \( \alpha > 0. \) Then there exists \( R_0 > 0 \) such that for \( \forall x^0 \in \overline{\Omega} \)

and \( 0 < -R \leq R_0, B(0, 2a) \subset O_{x^0 R}. \) From (2.5) and (2.3) it follows that \( D^\alpha u^i_{x^0 R}, \)

\( |\alpha| \leq m_i - 1, \quad i = 1, \ldots, N, \) are bounded uniformly with respect to \( x^0 \in \overline{\Omega} \)

and \( 0 < R \leq R_0. \) Clearly there exists a constant \( t > 0 \) such that for all \( x^0 \in \overline{\Omega} \)

and \( 0 < R \leq R_0 \)

(2.6) \[ \|u_{x^0 R}\|_{H^{|\alpha|+1}(B(0,2a))} \leq t. \]

Now let \( \varphi \in [D(O_{x^0 R})]^N. \) Putting \( R^{m_i+1} \varphi^i(\frac{x^0 - x}{R}) \in D(\Omega) \) in (0.2) as a test function

and using the transformation \( x = x^0 + Ry \) we have

(2.7) \[
\sum_{i=1}^N \sum_{|\alpha| \leq m_i} \int R^{m_i-|\alpha|+1} A_{ij}^{\alpha} (x^0 + Ry, \delta(u(x^0 + Ry))) D^\theta u^j(x^0 + Ry) D^\alpha \varphi^i(y) \, dy
\]

\[ = \sum_{i=1}^N \sum_{|\alpha| \leq m_i} \int R^{m_i-|\alpha|+1} g_{\alpha}^i(x^0 + Ry) D^\alpha \varphi^i(y) \, dy. \]

Let \( \chi \in D(B(0,2a)) \) be such that \( \chi = 1 \) in \( B(0,a) \), \( 0 \leq \chi \leq 1 \) in \( B(0,2a) \). Using

the notation from the introduction, (2.5) and putting \( \varphi^i = u^i_{x^0 R} \cdot \chi^{2k}, i = 1, \ldots, N, \)
\( k = \max\{m_i\} \) in (2.7) we have

\[
\sum_{i,j=1}^{N} \sum_{|\alpha|=m_i B(0,2a)} A_{ij}^{\alpha \beta} (x^0 + Ry, \delta_1 (u(x^0 + Ry)), \delta_2 (u^{x^0 R}(y))) \cdot (D^\beta u^{x^0 R}(y) x^k(y)) \times (D^\alpha u^{x^0 R}(y) x^k(y)) \ dy
\]

\[
= - \sum_{i,j=1}^{N} \sum_{|\alpha|=m_j} \left( \sum_{|\alpha|=m_i \gamma < \alpha} \sum_{|\alpha| \leq m_i \gamma \leq \alpha} \right) \int_{B(0,2a)} (\alpha^\gamma) R_{m_i - |\alpha|\alpha} A_{ij}^{\alpha \beta} (x^0 + Ry, \delta_1 (u(x^0 + Ry)), \delta_2 (u^{x^0 R}(y))) \times (D^\beta u^{x^0 R}(y) x^k(y)) \cdot D^\gamma u^{x^0 R}(y) Z^{\alpha - \gamma} (x^{2k}) \ dy
\]

\[
+ \sum_{i=1}^{N} \sum_{|\alpha| \leq m_i B(0,2a)} \int_{B(0,2a)} R_{m_i - |\alpha|+1} g_{i \alpha} (x^0 + Ry) D^\alpha (u^{x^0 R}(y) x^{2k}(y)) \ dy,
\]

where \( Z^{\alpha - \gamma} (x^{2k}) \) is introduced in (1.31).

Let us denote the left-hand side of this equality by \((LS)\), the first term on the right-hand side by \((RS_1)\) and the second by \((RS_2)\).

Using (0.4) we have

\[
\nu \cdot J =: \nu \cdot \sum_{i=1}^{N} \sum_{|\alpha|=m_i B(0,2a)} \int_{B(0,2a)} |D^\alpha u^{x^0 R}(y) x^k(y)|^2 \ dy \leq (LS).
\]

Using the fact that \( A_{ij}^{\alpha \beta} \) are uniformly bounded and the Hölder inequality we obtain

\[
|(RS_1)| \leq c_1 \|u^{x^0 R}\|_{H^{m-1} B(0,2a)} J^{\frac{1}{2}}.
\]

(0.5), the Leibniz formula, the Hölder inequality and the boundedness of \( \chi^k(y) \) imply

\[
|(RS_2)| \leq c_2 J^{\frac{1}{2}} + c_3 \|u^{x^0 R}\|_{H^{m-1} B(0,2a)}.
\]

Using the inequality \((LS) \leq |(RS_1)| + |(RS_2)| \) and (2.6) we obtain the estimate

\[
J \leq c_4 J^{\frac{1}{2}} + c_5.
\]

This estimate and (2.6) imply that there exists a constant \( c_6 = c_6(\nu, a, M, G, A) \) such that for all \( x^0 \in \Omega, R \in (0, R_0], u \in [M] \)

\[
(2.8) \quad \|u^{x^0 R}\|_{H^{m} B(0,a)} \leq c_6.
\]
Now we shall prove that (1.57) holds uniformly with respect to $x^0 \in \overline{\Omega}$ and $u \in [M]$. Let us suppose the contrary. Then there exist \( \{ x^k \}_{k=1}^\infty \subset \overline{\Omega}, \ x^k \to x \in \overline{\Omega}, \) \( \{ R_k \}_{k=1}^\infty \subset \mathbb{R}^+, \ R_k \to 0, \) \( \{ u_k \}_{k=1}^\infty \subset [M] \) and $\varepsilon > 0$ such that
\[
U(x^k, R_k) \geq \varepsilon.
\]
The estimate (2.8) implies that there exists a subsequence \( \{ u_{k^*} \}_{k=1}^\infty \) such that
\[
u_{k^*} \to P \text{ in } H^m(B(0, a)),
u_{k^*} \to P \text{ in } H^{m-1}(B(0, a)),
\]
\[
D^\alpha u^i_{k^*} \to D^\alpha P^i \text{ a.e. in } B(0, a), \ |\alpha| \leq m_i - 1, \ i = 1, \ldots, N.
\]
Putting $a = r, \ r \in \mathbb{N}$ and using the diagonalization process we obtain for all $r \in \mathbb{N}$
\[
u_{k^*} \to P \text{ in } H^m(B(0, r)),
\]
\[
u_{k^*} \to P \text{ in } H^{m-1}(B(0, r)),
\]
\[
D^\alpha u^i_{k^*} \to D^\alpha P^i \text{ a.e. in } B(0, r), \ |\alpha| \leq m_i - 1, \ i = 1, \ldots, N.
\]
From (2.5) and (2.12) it follows that there exists a constant $\tau > 0$ such that
\[
|D^\alpha P^i| \leq \tau, \ |\alpha| \leq m_i - 1, \ i = 1, \ldots, N.
\]
Now let $\psi \in [D(\mathbb{R}^n)]^N$. It is clear that there exist $r, \ R_1$ such that $\text{supp } \psi \subset B(0, r) \subset O_{x^0 R}$ for all $x^0 \in \overline{\Omega}$ and $0 < R \leq R_1$. Putting $\varphi = \psi$ in (2.7) we have
\[
\sum_{i,j=1}^N \sum_{|\alpha| \leq m_i} \int_{|\beta| = m_j} R_{m_i-|\alpha|}^{m_i} A_{ij}^{x,y} (x^k + R_k y, \delta_1(u_k(x^k + R_k y)), \delta_2(u_{k^*}(y)))
\]
\[
\times D^\beta u^i_{k^*} (y) D^\alpha \psi^j (y) dy
\]
\[
= \sum_{i=1}^N \sum_{|\alpha| \leq m_i} \int_{|\beta| = m_j} R_{m_i-|\alpha|+1}^{m_i} g^i_\alpha (x^k + R_k y) D^\alpha \psi^i (y) dy.
\]
The fact that the functions $A_{ij}^{x,y}$ and $D^\alpha \psi^i$ are uniformly bounded and the formula (2.10) imply
\[
\sum_{i,j=1}^N \sum_{|\alpha| \leq m_i} \int_{|\beta| = m_j} R_{m_i-|\alpha|}^{m_i} A_{ij}^{x,y} (x^k + R_k y, \delta_1(u_k(x^k + R_k y)), \delta_2(u_{k^*}(y)))
\]
\[
\times D^\beta u^i_{k^*} \cdot D^\alpha \psi^i \ dy \to 0 \text{ for } k \to \infty.
\]
If $|\alpha| = m_i$, $i = 1, \ldots, N$ then $R_k^{m_i-|\alpha|} = 1$, $x^k + R_ky \to \bar{x}$ for $k \to \infty$. Because $H^{m_i-1,\infty}(\Omega) \cap H^{m_i-1,p}(\Omega) \cap C^{m_i-2}(\Omega)$, it is clear that $u_k^i \to P^i$ in $C^{m_i-2}(\Omega)$ (i.e. $D^\alpha u_k^i \to D^\alpha P^i$ on $\Omega$, $|\alpha| \leq m_i - 2$, $i = 1, \ldots, N$) and $\delta_1(u_k(x^k + R_ky)) \to \delta_1(P(\bar{x}))$ in $B(0, r)$, $k \to \infty$.

From (2.5), (2.12) it follows that

$$\delta_2(u_{kx^kR_k}(y)) \to \delta_2(P(y)) \text{ a.e. in } B(0, r), \ k \to \infty.$$ 

Using (0.3), Lebesgue’s dominated convergence theorem and (2.10) we obtain

$$\int_{B(0, r)} A_{ij}^{\alpha\beta}(x^k + R_ky, \delta_1(u_k(x^k + R_ky)), \delta_2(u_{kx^kR_k}(y))) \times D^\beta u_k^i(\overline{x_k}) D^\alpha \psi^j(y) \, dy$$

$$\to \int_{B(0, r)} A_{ij}^{\alpha\beta}(\overline{x}, \delta_1(P(\bar{x})), \delta_2(P(y))) D^\beta P^j(y) D^\alpha \psi^i(y) \, dy$$

for $k \to \infty$, $i, j = 1, \ldots, N$, $|\alpha| = m_i$, $|\beta| = m_j$.

The fact that $p_i \geq 2$ ($i = 1, \ldots, N$, $|\alpha| \leq m_i$), the transformation $x = x^k + R_ky$ and the Hölder inequality imply

$$\left| \int_{B(0, r)} R_k^{m_i-|\alpha|+1} g_{\alpha}^i(x^k + R_ky) D^\alpha \psi^j(y) \, dy \right|$$

$$\leq c_7 R_k^{m_i-|\alpha|+1} R_k^{-n} \int_{B(x^k, rR_k)} |g_{\alpha}^i(x)| \, dx$$

$$\leq c_8 R_k^{(m_i-|\alpha|+1)(1-\frac{n}{p_i})} G \to 0 \text{ if } k \to \infty.$$ 

From (2.15), (2.16) and (2.17) it follows that

$$\sum_{i,j=1}^N \sum_{|\alpha|=m_i} \sum_{|\beta|=m_j} \int_{B(0, r)} A_{ij}^{\alpha\beta}(\overline{x}, \delta_1(P(\bar{x})), \delta_2(P(y))) D^\beta P^j(y) D^\alpha \psi^i(y) \, dy = 0,$$

$$\psi \in [D(\mathbb{R}^n)]^N.$$
The condition (L) and (2.13) imply that \( P \in P_{m-1}^N \). Using (2.5), (2.9), (2.11) and the transformation \( x = x^k + R_k y \) we have

\[
0 < \varepsilon \leq \lim_{k \to \infty} \inf \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} |D^\alpha u_k^i(x) - (D^\alpha u_k^i)_{x^k R_k}|^2 dx \\
\leq \lim_{k \to \infty} \inf \sum_{i=1}^{N} \sum_{|\alpha|=m_i-1} \int |D^\alpha u_k^i(y) - D^\alpha P^i|^2 dy = 0.
\]

This implies that (1.57) holds uniformly with respect to \( x^0 \in \Omega \) and \( u \in [M] \). Lemma 1.56 implies the assertion of the theorem.

By the standard method from [2], [4] we shall prove

**Theorem 2.19.** Suppose that the system (0.1) has the property of regularity (R). Then it has Liouville's property (L).

**Proof.** Let \( x^0 \in \Omega, \xi \in \mathbb{R}^n \) and let \( u \) be a solution (in \( \mathbb{R}^n \)) to the system

\[
\sum_{i,j=1}^{N} \sum_{|\alpha|=m_i,|\beta|=m_j} \int A_{ij}^{\alpha\beta}(x^0,\xi,\delta_2(u(x))) D^\alpha u^i(x) D^\beta \varphi^i(x) dx = 0,
\]

\( \varphi \in [D(\mathbb{R}^n)]^N \),

such that for \( M > 0 \)

\[
|D^\alpha u^i| \leq M, \quad |\alpha| \leq m_i - 1, \quad i = 1, \ldots, N.
\]

For \( R > 0 \) we define

\[
u_R^i(y) = \frac{u^i(Ry)}{R^{m_i-1}}, \quad i = 1, \ldots, N.
\]

It is clear that

\[
D^\alpha u_R^i(y) = D^\alpha u^i(Ry), \quad |\alpha| = m_i - 1, \quad i = 1, \ldots, N,
\]

\[
D^\alpha u_R^i(y) = R D^\alpha u^i(Ry), \quad |\alpha| = m_i, \quad i = 1, \ldots, N.
\]

Let \( \varphi \in [D(\mathbb{R}^n)]^N \).

Putting \( \varphi(R^{-1}) \) as a test function in (2.20) and using the transformation \( x = Ry \) we have

\[
\sum_{i,j=1}^{N} \sum_{|\alpha|=m_i,|\beta|=m_j} \int A_{ij}^{\alpha\beta}(x^0,\xi,\delta_2(u_R(y))) D^\alpha u_R^i(y) D^\beta \varphi^i(y) dy = 0.
\]

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(2.1), (2.22) and the property (R) imply

\[(2.24) \quad |D^\alpha u_R^i(y) - D^\alpha u_R^i(0)| \leq c|y|^\mu, \]
\[|\alpha| = m_t - 1, \quad i = 1, \ldots, N, \quad R > 0, \quad y \in \overline{B(0, \eta)}, \quad \eta > 0, \quad \mu \in (0, 1).\]

Let us choose \(x \in \mathbb{R}^n\). Then there exists \(R_0 > 0\) such that \(y_R = \frac{x}{R} \in \overline{B(0, \eta)}\) for all \(R \geq R_0\).

Using (2.24) and (2.22) we obtain

\[(2.25) \quad |D^\alpha u^i(x) - D^\alpha u^i(0)| \leq c \frac{|x|^\mu}{R^\mu}, \quad |\alpha| = m_t - 1, \]
\[i = 1, \ldots, N, \quad R \geq R_0.\]

For \(R\) tending to infinity we have

\[D^\alpha u^i(x) = D^\alpha u^i(0) \quad \text{for all} \quad x \in \mathbb{R}^n, \quad |\alpha| = m_t - 1, \quad i = 1, \ldots, N.\]

This fact implies that \(u \in P^N_{m-1}\). \(\square\)

Remark 2.26. Using the method from [2], [4] we could prove that the system (0.1) has Liouville's property (L) for \(n = 2\), i.e. for plane domains.

References


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