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ON THE FIXED POINTS IN AN ω-LIMIT SET

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Summary. Let M and K be closed subsets of $[0,1]$ with K a subset of the limit points of M. Necessary and sufficient conditions are found for the existence of a continuous function $f\colon [0,1] \to [0,1]$ such that M is an ω-limit set for f and K is the set of fixed points of f in M.

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In [1] it was proven that a nonvoid subset M of $[0,1] = I$ is an ω-limit set for some continuous self map of I if and only if either M is closed and nowhere dense or M is the union of finitely many non-degenerate closed intervals. In [2] a simpler proof of the main part of this result was given, one which, however, gives much less information on the possible fixed points of the function. For example, as shown in [1], a countable closed set can be realized as an ω-limit set for a continuous function for which all the limit points are fixed. The same is true for an uncountable closed nowhere dense set provided the perfect part is a subset of the set of limit points of the countable part. On the other hand [2] shows that for "most" sets the set of fixed points can be empty. Moreover, there are various examples in the literature of functions realizing the Cantor set as an ω-limit set for which there are an arbitrary finite number of fixed points in the Cantor set.

These possibilities then give rise to the following question: *Given a closed subset M of I and K a subset of the limit points of M what are necessary and sufficient conditions on M and K for the existence of a continuous function realizing M as an ω-limit set and having K as its set of fixed points in M?*

When M is an ω-limit set which is not nowhere dense the question has an obvious answer: In case M is a closed interval the set of fixed points K can be any compact nowhere dense subset. In case M is a union of two or more closed intervals K must be empty.

First of all one necessary condition is, in view of the [1] theorem, that M be nowhere dense or a union of finitely many nondegenerate closed intervals. Another necessary condition, which is easy to establish, is that K be closed and nowhere dense in M.

We give a set of necessary and sufficient conditions in Theorem 2. As a preliminary result we give necessary and sufficient conditions when the additional requirement is imposed that each point of M is eventually fixed.

The proofs of both Theorem 1 and Theorem 2 depend heavily on the results and techniques of both [1] and [2] and will be developed in the sequel. First we present the necessary preliminaries.

Notation and Terminology. The symbol I will denote the unit interval $[0,1]$. For $f: I \to I$ and $x \in I$ we define $f^0(x) = x$; and $f^{n+1}(x) = f(f^n(x))$ for each natural number n. By the *orbit* of x under f we mean the set $\gamma(x,f) = \{f^n(x)\colon n \in \omega_0\}$, where ω_0 is the set of natural numbers.

The notation $\{x_n\}_{n=0}^{\infty}$ denotes the sequence as a function whereas $\{x_n\colon n \in \omega_0\}$ is the range of the function. The *cluster set* of $\{x_n\}_{n=0}^{\infty}$ is the set of subsequential limit points of $\{x_n\}_{n=0}^{\infty}$.

Let f be a continuous function. The *ω-limit set* $\omega\ (x,f)$ (some authors use the term "attractor set") is defined to be the cluster set of $\{f^n(x)\}_{n=0}^{\infty}$. A point x is a *fixed point* for f if $f(x) = x$. A point is *eventually fixed* if there is an n for which $f^n(x)$ is fixed. A finite set of distinct points $\{x_1,\dots,x_k\}$, $k > 1$, is a *cycle* for f if $f^i(x_i) = x_{i+1} (\operatorname{mod} k)$.

Let A' or $D(A)$ denote the set of limit points of a set A. If $A \subseteq I$ we may define inductively sets $D_\alpha(A)$ for each ordinal $\alpha \leqslant \omega_1$, the first uncountable ordinal, as follows:

$$D_0(A) = A$$
$$D_{\alpha+1}(A) = D(D_\alpha(A))$$
$$D_\lambda(A) = \bigcap\{D_\alpha(A)\colon \alpha < \lambda\} \qquad \text{when } \lambda \text{ is a limit ordinal.}$$

When A is compact in I we define the *order* of A, $\varrho(A)$, as follows: If card $A = \omega_1$ (or c), put $\varrho(A) = \omega_1$. If A is countable, there exists a smallest β such $D_\beta(A) = \emptyset$. Moreover, $\beta = \alpha + 1$ for some α by compactness. We define $\varrho(A) = \alpha$. If A is any set and $x \in A$, the *order of x in A*, denoted by $\varrho_A(x)$ (or in short by $\varrho(x)$), is the smallest α such that $x \in D_\alpha(A)$.

Note that if A is countable and compact with $\varrho(A) = \alpha$ then $D_\alpha(A)$ consist of finitely (nonzero) many points of order α. We will say that $x \in A$ is a ***point of highest order*** if there are no other points at A of higher order than $\varrho(x)$.

Our first result is Lemma 13 of [3]

Lemma 0 [3]. *If $f: I \to I$ is continuous, $x_0 \in I$ and α is countable, then* $D_\alpha(\omega(x_0, f)) = f[D_\alpha(\omega(x_0, f))]$.

Lemma 1. *Let M be a nonempty closed nowhere dense subset of I and K be a nonempty closed subset of M which is nowhere dense in M.*

If $M - K$ consists of isolated points, then there exists a homeomorph N of M in $(0,1)$ satisfying the following property $(*)$: if G is any component of $I - N'$ with right hand end point b, then $(G \cap N)' = \{b\}$ when $b \in N$.

Proof. Case 1: K is countable. We show the existence of such a set N by induction on the order of K. It is obviously true when $\varrho(K) = 1$. Now assume that the assertion is true for any ordinal α less than a specific countable ordinal β. Suppose $\varrho(K) = \beta$. Then $\varrho^{-1}(\beta) = \{x_1, \ldots, x_K\}$ and we can find disjoint open intervals $I_1, \ldots, I_k$ such that $x_i \in I_i$ for each i and $M \subseteq \bigcup_{i=1}^{k} I_i$. It then follows that there exists a sequence of ordinals $\{\beta_n\}_{n=1}^{\infty}$ with $\beta_n < \beta$ for each n and for each $1 \leqslant i \leqslant k$ a sequence of open sets $\{W_{in}\}_{n=1}^{\infty}$ such that $\overline{W}_n \cap \overline{W}_j = \emptyset$ when $n \neq i$, $\overline{W}_{in} \subseteq I_i - \{x_i\}$ and $\varrho(W_{in} \cap M) = \beta_n$ for each i and n and $(M - \{x_i\}) \cap I_i \subseteq \bigcup_{n=1}^{\infty} W_{in}$ for each i.

Now pick distinct points $y_1, \ldots, y_k$ in $(0,1)$ and for each i a sequence of open intervals $\{S_{in}\}_{n=1}^{\infty}$ in $(0,1) - \{y_1, \ldots, y_k\}$ converging to y_i from the left such that $S_{in} \cap S_{jm} = \emptyset$ whenever $(i,n) \neq (j,m)$.

By inductive hypothesis we can find a subset N_{in} of S_{in} and a homeomorphism h_{in} from $M \cap W_{in}$ onto N_{in} such that property $(*)$ holds. Now define $N = \bigcup\{N_{in} : j \leqslant i \leqslant k,\ m \geqslant 1\}$ and define $h = h_{in}$ on $M \cap W_{in}$ and $h(x_i) = y_i$ for each i. Then clearly h is a homeomorphism from M onto N and property $(*)$ is satisfied.

Case 2: K *is uncountable*. Then $K = P \cup C$ where P is a Cantor set and C is a countable set disjoint from P. Without loss of generality we may assume that $K \subseteq (0,1)$; otherwise we may shrink it by a linear map to be in $(0,1)$. Let $\mathcal{G}$ be the set of components of $[o, \sup P] - P$. Let us enumerate $\mathcal{G}$ as $\{(a_n, b_n)\}_{n=1}^{\infty}$. If $(a_n, b_n) \cap M$ is infinite, put $E_n = M \cap (a_n, b_n)$. If $(a_n, b_n) \cap M$ is finite pick an isolated point e_n of $(a_n, b_n) = \cap M$ and put $E_n = \{e_n\}$. Put $E = \bigcup_{n=1}^{\infty} E_n$. Then since K is nowhere dense in M we must have $\overline{E} = P$.

By lemma 8 of [1] (and its proof) there exists a set D missing M such that $((a_n, b_n) \cap D)' = \{b_n\}$ for each n and a homeomorphism h_0 from $P \cup E$ onto $P \cup D$ which is the identity of P.

Let $\Gamma = \{n \in \omega_0 : (a_n, b_n) \cap M$ is infinite $\}$. For $n \in \Gamma$ put $H_n = (a_n, b_n) \cap (M - E_n)$. Let $\Gamma_1 = \{n \in \Gamma : a_n < \inf H_n\}$ For each $n \in \Gamma_1$ we can apply case 1 (where $\overline{H}_n$ and $\overline{H}_n \cap K$ play the role of M and K respectively) to obtain a set $N_n \subseteq (a_n, b_n)$ and homeomorphism h_n of $\overline{H}_n$ onto N_n satisfying property $(*)$. Moreover, we may choose N_n so that $N_n \cap D = \emptyset$ and $a_n < \inf N_n$ and $\sup \overline{H}_n = \sup N_n$ for each n.

Next let S consist of all components of $I - (M \cup D)$. Then for each $n \in \Gamma_2 = \Gamma - \Gamma_1$ we can find a sequence $\{S_{nk}\}_{k=0}^{\infty}$ in S such that

$$S_{nk} \cap S_{mj} = \emptyset \quad \text{when } (n, k) \neq (m, j)$$
$$\sup S_{nk} < a_n \quad \text{for each } k$$
$$\operatorname{diam}(S_{nk} \cup \{a_n\}) < 2^{-k} \quad \text{for each } n \in \Gamma_2$$

For each $n \in \Gamma_2$ pick a sequence $\{T_{nk}\}_{n=0}^{\infty}$ of mutually disjoint open intervals on (a_n, b_n) decreasing to a_n such that

$$\sup T_{n0} < b_n$$
$$\text{if } c_n = \sup(M \cap (a_n, b_n) < b_n, \text{ then } c_n \in T_{n0}$$
$$\text{each } T_{nk} \cap M \neq \emptyset$$
$$\bigcup_{k=0}^{\infty} (T_{nk} \cap M) = M \cap (a_n, \sup T_{n0})$$

By case 1 it follows that there exists a subset N_{nk} of S_{nk} and a homeomorphism h_{nk} from $T_{nk} \cap M$ onto N_{nk} either satisfying property $(*)$ for each $n \in \Gamma_2$ and $k \in \omega_0$ or such that $T_{nk} \cap M$ is finite.

For $n \in \Gamma_2$ define h_n on $\overline{H}_n$ as follows: $h_n(x) = H_{nk}$ if $x \in T_{nk} \cap M$; otherwise $h_n(x) = x$. Put $N_n = h_n(\overline{H}_n)$.

Put $N = P \cup D \cup \bigcup_{n=1}^{\infty} N_n$ and $h = \bigcup_{n=0}^{\infty} h_n$. It is easily verified that h is a homeomorphism from M onto N and property $(*)$ is satisfied.

Now we can prove the following special case of Theorem 1. □

Lemma 2. *Let M be a nonempty, closed, nowhere dense subset of I and K be a nonempty, closed subset of M which is nowhere dense in M.*

If $M - K$ consists of isolated points, then there exists $x_0 \in I$ and a continuous g: $I \to I$ such that

(1) $\omega(x_0, g) = M$

(2) *K is the set of fixed points of g in M*

(3) *each member of M is eventually fixed.*

Proof. Apply Lemma 1 to obtain a set $N \subseteq (0,1)$ and a homeomorphism h from M onto N such that property $(*)$ holds. Lemma 6 of [1] and its proof shows that if N is infinite, closed and nowhere dense and has property $(*)$ than there exists $z_0 \in I$ and a continuous f such that $\omega(z_0, f) = N$ and N' is the set of fixed points of f in N and each point of N is eventually fixed. Lemmas 2 and 4 of [1] show that the function hfh^{-1} can be extended to a continuous $g\colon I \to I$ such that $M = \omega(h^{-1}(z_0),\ g)$ and K is the set of fixed points of g in M. Moreover, it is easy to see that since points of N are eventually fixed by f, points of M are eventually fixed by g.

The following is well-know and part of its proof is sketched in [2]. □

Lemma 3. *Suppose A and B are nonempty, closed, nowhere dense subsets of I. There is a continuous function f mapping A onto B if any one of the following conditions hold*

(1) *A is uncountable*

(2) *A and B are countable and $\varrho(B) < \varrho(A)$*

(3) *A and B are countable and $\varrho(B) = \varrho(B)$ and B has exactly one point of highest order*

The next lemma is distilled from the proof of Theorem 1 of [2].

Lemma 4 [2]. *Let M be a closed nowhere dense subset of I. Let $g\colon I \to I$ be continuous such that $g(M) = M$ and for each $x \in M$ and neighborhood W of x, $g(W)$ is a neighborhood of $g(x)$.*

If there exists a countable dense subset D of M such that for each $x,\ y \in M$ and connected compact neighborhood H of x there exists n and a compact interval J such that $J \subseteq H$ and $y \in g^n(J)$, then there exists z such that $M = \omega(z, g)$.

Now we are ready to prove the characterization of those sets K which form the possible sets of fixed points of continuous functions which realize M as their ω-limit sets such that all points in M are eventually fixed.

Although the union of finitely many non-degenerate closed intervals can be an ω-limit set we did not incorporate this possibility in the statement of Theorem 1. This is because $K = \emptyset$ is the only option since such a set would have to contain an orbit point and orbit points can't be eventually fixed.

Theorem 1. *Let M be a nonempty, closed, nowhere dense subset of I and K be a closed subset of M which is nowhere dense in M. Then, there exists a contuinuous $g\colon I \to I$ and $x_0 \in I$ such that*

(1) $\omega(x_0, g) = M$

(2) *K consists of the fixed points of g in M*

(3) *when $K \neq \emptyset$ each point of M is eventually fixed*

if and only if one of the following hold

(a) *$K = \emptyset$ and there is more than one point of highest order in M*

(b) *$K \neq \emptyset$, $M - K$ is countable and the set $\{y \in M - K : \varrho(y) \geqslant \varrho(x)\}$ is infinite whenever $x \in M - K$*

(c) *$K \neq \emptyset$, $M - K$ is uncountable and $M - K$ has a c-limit point in K.*

Proof. *Part 1: The Necessity.* Suppose conditions (1), (2) and (3) hold. Then we have three cases.

Case 1: $K = \emptyset$. Let $H(M)$ denote the set of points of highest order. If $H(M) = \{z\}$ for some z, then M must be countable. By Lemma 0 $H(M) \subseteq g(H(M))$. Hence $\{z\} \subseteq g(\{z\})$ and $g(z) = z$, a contradiction. Hence, card $H(M) > 1$.

Case 2: $K \neq \emptyset$ and $M - K$ is countable. Suppose $x_0 \in M - K$ and $\varrho(x_0) = \beta$. By Lemma 1 there exists a sequence $\{x_n\}_{n=0}^{\infty}$ in M such that $\varrho(x_{n+1}) \geqslant \varrho(x_n) \geqslant \beta$ and $g(x_{n+1}) = x_n$ for all n. Clearly each $x_n \in M - K$. If $x_{n+m} = x_n$ for some m, then $\{x_0, \ldots, x_m\}$ forms a cycle. Since $g^i(x_0) \in K$ for some i, this forces $x_0 \in K$, a contradiction. Hence, $\{x_n : n \in \omega_0\}$ is infinite.

Case 3: $K \neq \emptyset$ and $M - K$ is uncountable. Let us show that $M - K$ has a c-limit point in K. Suppose the contrary. Then for each $x \in K$ there exists an open set $W(x)$ containing x such that $W \cap M - K$ is countable. Using the compactness of K there exists an open W such that $K \subseteq W$ and $W \cap M - K$ is countable. Then $M - W$ is closed and uncountable, so there exists a nonempty perfect set P and a countable set E such that $M - W = P \cup E$.

Let $Q = (M - W) - \bigcup_{n=1}^{\infty} f^{-n}(M \cap W)$. Since each $f^{-n}(M \cap W)$ is open, Q is closed and $f(Q) \subseteq Q$. If $Q = \emptyset$, then by compactness there would exist a k such that $M - W \subseteq \bigcup_{i=1}^{k} f^{-i}(M \cap W)$. Let m be the greatest $i < k$ such that $f^{-i}(M \cap W)$ is countable. Let $T = \left(\bigcup_{i=1}^{m-1} (f^{-i}(M \cap W) - K)\right) \cup \left(\bigcup_{i=m+1}^{k} f^{-i}(M \cap W)\right)$. Then T is countable and since $f(M) = M$ it follows that $f^{-m}(M \cap W) \subseteq f(T)$ so that $f^{-m}(M \cap W)$ is countable, a contradiction.

Hence, $Q \neq \emptyset$ and $f^n(Q) \subseteq Q$ for all n. This violates condition (3). Therefore, condition (c) is valid.

Part 2: The Sufficiency. First suppose (a) holds, that is, $K = \emptyset$ and there exist points a and b in M of highest order. According to [2] there exists $x_0 \in I$ and a continuous $g : I \to I$ for which $M = \omega(x_0, g)$, $g(a) = b$, $a = g(b)$ and for each $x \in M$

there exists n such that $g^n(x) = a$. Hence, there are no fixed points and conditions (1), (2) and (3) are satisfied.

Let us now consider the case when $K \neq \emptyset$. Without loss of generality we may assume K is infinite. Enumerate the one sided limit points of K as $\{e_n\}_{n=0}^{\infty}$. By induction we may pick sequences $\{e_{nk}\}_{k=0}^{\infty}$ in $M - K$ such that $e_{nk} \to e_n$ and $e_{nk} \neq e_{ni}$, whenever $(n, k) \neq (m, j)$. Also we may pick these sequences so that the following hold: (1) if $e_i \in (\inf K, \sup K)$, then $e_{ik} \in (\inf K, \sup K)$ for all K and (2) if $e_i = \inf K$ and $e_i \in [M \cap (0, e_i)]'$, then $e_{ik} < e_i$ for all k. Similarly for the case when $e_i = \sup K$.

Now put $M_1 = K \cup \{e_{nk}: n,\ k \in \omega_0,\ k \text{ is even}\}$ and $M_2 = M - M_1$. Then $K = M_1' \cap M_2'$ and both M_1 and M_2 are infinite.

Now apply Lemma 2 to M_1 to get $x_0 \in I$ and continuous $h: I \to I$ such that $\omega(x_0 h) = M_1$ and K is the set of fixed points of h in M_1 and each point of M_1 is eventually fixed. Moreover, by Lemma 1 of [1] we can assume that $\gamma(x_0, h) \cap M = \emptyset$.

Let us define a *section* to be any non-empty closed F of M_2 which can be expressed as $M_2 \cap J$ where J is open set disjoint from K and either F is uncountable or F is countable and has exactly one point of highest order.

The sufficiency of condition (b). Suppose $K \neq \emptyset$, $M - K$ is countable and for each $x \in M - K$ the set $\{y \in M - K: \varrho(y) \geqslant \varrho(x)\}$ is infinite.

First it is clear that we can find a family $\mathcal{H}$ of mutually disjoint open intervals having end points in $I - M$ with the properties that (i) if J is any closed interval inside some component of $I - K$, then J intersects only finitely many members of $\mathcal{H}$ and (ii) $(I - K) \cap M \subseteq \bigcup \mathcal{H}$.

For each $H \in \mathcal{H}$, $M_2 \cap H$ is closed and is either empty or has a finite number of points of highest order. It follows that each $M_2 \cap H$ is empty or a union of finitely many mutually disjoint sections. Hence M_2 is the union of a family $\mathcal{A}$ of mutually disjoint sections. From the assumptions it follows that for each $A \in \mathcal{A}$, $\{C \in \mathcal{C}: \varrho(C) \geqslant \varrho(A)\}$ is infinite.

Let B be the set of all $x \in M$ such that there exists a sequence $\{T_n\}_{n=0}^{\infty}$ in $\mathcal{A}$ such that $\operatorname{diam}(\{x\}) \cup T_n) \to 0$. By the construction $\emptyset \neq B \subseteq K$. Without loss of generality we may assume B is infinite. For each $b \in B$ let $\xi(b) = \varrho_c(b)$ where $C = \{b\} \cup (\bigcup \mathcal{A})$. For $b \in B$ and a sequence $\{C_m\}_{m=0}^{\infty}$ of closed sets we write $C_m \xrightarrow{*} b$ if (i) $\operatorname{diam}(\{b\}) \cup C_m) \to 0$ (ii) $\varrho(C_m) \leqslant \varrho(C_{m+1})$ for each m and (iii) $\varrho(C_m) \to \xi(b)$ whenever $\xi(b)$ is a limit ordinal and $\varrho(C_m)$ is eventually equal to α whenever $\xi(b) = \alpha + 1$.

Now enumerate $\mathcal{A}$ as $\{A_n\}_{n=0}^{\infty}$. For each n define λ_n to be the point in the closed set $\{b \in B: \xi(b) \geqslant \varrho(A_n)\}$ closest to A_n. Put $\varepsilon_n = \operatorname{diam}(\{\lambda_n\} \cup A_n)$.

Then $\varepsilon_n \to 0$. To show this assume the contrary. Then there exists a subsequence $\{n_k\}_{k=0}^{\infty}$, $\varepsilon > 0$ and λ, μ in I for which $\operatorname{diam}(\{\lambda_{n_k}\} \cup A_{n_k}) \geqslant \varepsilon$, $\xi(\lambda_{n_k}) \geqslant \varrho(A_{n_k})$,

$\lambda_{n_k} \to \lambda$ and $\operatorname{diam}(A_{n_k} \cup \{\mu\}) \to 0$. From the definitions it follows that eventually $\mu < \varrho(A_{n_k})$. But since $\operatorname{diam}(A_{n_k} \cup \{\mu\}) \to 0$ we also have $\varrho(A_{n_k}) \leqslant \mu$ eventually, a contradiction.

Next let $\mathcal{C}_0 = \{A_n : \lambda_n = \lambda_0\}$ and $\delta_0 = \operatorname{diam}(\{\lambda_0\} \cup (\bigcup \mathcal{C}_0))$. Since $\varepsilon_n \to 0$ we must have $\delta_0 = \varepsilon_{k_0} = \max\{\varepsilon_n : A_n \in \mathcal{C}_0\}$. Let W_0 be the δ_0-neighborhood of λ_0.

Then we may find a sequence $\{S_{0m}\}_{m=0}^{\infty}$ of mutually disjoint sections in W_0 such that $S_{0m} \xrightarrow{*} \lambda_0$ and $\bigcup \mathcal{C}_0 \subseteq \bigcup_{m=0}^{\infty} S_{0m}$. To show this we first pick a subsequence $\{A_{n_k}\}_{k=0}^{\infty}$ in W_0 such that $A_{n_k} \xrightarrow{*} \lambda_0$. If $\mathcal{C}_0 - \{A_{n_k} : k \in \omega_0\}$ is finite we can clearly enumerate $\mathcal{C}_0 \cup \{A_{n_k} : k \in \omega_0\}$ as $\{S_{0m}\}_{m=0}^{\infty}$ in such a way that $S_{0m} \xrightarrow{*} \lambda_0$. If, on the other hand, $\mathcal{C}_0 - \{A_{n_k} : k \in \omega_0\} = \mathcal{B}$ is infinite then λ_0 is a limit point of $\bigcup \mathcal{B}$. Then clearly we can form a sequence of sections $\{S_{0m}\}_{n=0}^{\infty}$ made up of members of $\{A_{n_k} : k \in \omega_0\}$ union a finite number of members of $\mathcal{C}_0$ in such a way that $S_{0m} \xrightarrow{*} \lambda_0$.

As the next step let n_1 be the first member of $\omega_0 - \{n : \lambda_n = \lambda_0\}$. Let $\mathcal{C}_1 = \{A_n : \lambda_n = \lambda_{n_1}\}$ and $\delta_1 = \operatorname{diam}(\{\lambda_{n_1}\} \sup(\bigcup \mathcal{C}_1))$. Then $\delta_1 = \varepsilon_{k_1} = \max\{\varepsilon_n : A_n \in \mathcal{C}_1\}$. Let W_1 be the δ_1-neighborhood of λ_{n_1}. Then as before we can find a sequence $\{S_{im}\}_{n=0}^{\infty}$ of sections in W_1 such that $S_{im} \xrightarrow{*} \lambda_n$. Moreover, we can choose the sections S_{im} to be disjoint from the S_{0j} sections.

If we continue in this way by induction we obtain a sequence $\{b_n\}_{n=0}^{\infty}$ in B and for each n a sequence of sections $\{S_{nm}\}_{m=0}^{\infty}$ such that

$$S_{nm} \cap S_{ij} = \emptyset \quad \text{whenever } (n,m) \neq (i,j)$$
$$\operatorname{diam}(\{b_n\} \cup S_{nm}) \to 0 \quad \text{for each } n$$
$$\varrho(S_{nm}) \leqslant \varrho(S_{nm+1}) \quad \text{for each } m,\ n$$
$$M_2 = \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} S_{nm}$$
$$\operatorname{diam}\left(\{b_n\} \cup \left(\bigcup_{m=0}^{\infty} S_{nm}\right)\right) \to 0$$

The latter convergence follows from the facts that $\bigcup \mathcal{A} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$, $\varepsilon_n \to 0$ and $\delta_n = \max\{\varepsilon_m : A_m \in \mathcal{C}_n\}$.

Define f on $M \cup \gamma(x_0, h)$ as follows:

$$f(x) = \begin{cases} f_{kn}(x) & \text{if } x \in S_{kn} \text{ and } n > 1 \\ b_k & \text{if } x \in S_{ko} \\ h(x) & \text{if } x \in M_1 \cup \gamma(x_0, h) \end{cases}$$

Then $f(M) = M$ and it is easy to verify that f is continuous on its domain, $M \cup \gamma(x_0, h)$.

By Lemma 1 of [2] we may extend f to a continuous $g: I \to I$ such that $g(W)$ is a neighborhood of $g(x)$ whenever W is a neighborhood of x and $x \in M \cup \gamma(x_0, h)$. Then $g(M) = M$ and conditions (2) and (3) are obviously satisfied. It remains to find z_0 such that $\omega(z_0, g) = M$. For this we must invoke Lemma 4.

Let us verify the hypothesis of Lemma 4. Let x and y belong to $M - K$ which in dense in M. Let H be any connected compact neighborhood of x (i.e. a closed interval having x in its interior). First there exists n such that $g^n(x) = e \in K$ and $g^n(H)$ is a connected compact neighborhood of e. Since $\text{int}(g^n(H))$ contains an orbit point $h^j(x_0)$ and there are orbit points of the form $h^{j+m}(x_0)$ arbitrarily close to $\inf M_1$ and $\sup M_1$ and $h^{j+m}(x_0) = g^m(h^j(x_0) \in \text{int}\, g^{m+n}(H)$ it follows that $(\inf M_1, \sup M_1) \subseteq \bigcup_{m=n}^{\infty} \text{int}(g^m(H))$ and each $\text{int}(g^m(H))$ for $m \geqslant n$ is an open interval containing e.

Next we will show that there exists α such that $y \in g^\alpha(H)$. If $y \in (\inf M_1, \sup M_1)$ this is immediate. There remain two cases.

Case 1: $y \in M_2$. Then $y \in S_{im}$ for some i and m. Consider b_i. If $b_i = \inf K$ (or $\sup K$) then by construction of M_1, $s_{i\xi} \subseteq (\inf M_1, \sup M_1)$ for some $\varepsilon > m$. The same is true when $b_i \in (\inf M_1, \sup M_1)$. Moreover we can assume that $S_{i\xi}$ is entirely on one side of e. Hence for some γ, $S_{i\xi} \subseteq \text{int}\, g^\gamma(H)$ and $y \in S_{im} = g^{m-\xi}(S_{i\xi}) \subseteq g^{m-\xi+\gamma}(H)$.

Case 2: $y \in M_1 - K$ and $y = \inf M_1$ or $\sup M_1$. We have $g = h$ on M_1 and $(M_1 - K) \subseteq g(M_1 - K) \subseteq g^2(M_1 - K)$. So there exist z and w such that $y = g(z) = g^2(w)$. Clearly y, z and w are all distinct. Hence, z or w lies in $(\inf M_1, \sup M_1)$. For example, if $w \in (\inf M_1, \sup M_1)$ then $w \in \inf g^m(H)$ for some m so that $y \in g^{m+2}(H)$.

Now applying Lemma 4 the sufficiency of (b) is proven.

The sufficiency of (c): Assume $K \neq \emptyset$, $M - K$ is uncountable and at least one point of K is a c-limit point of $M - K$.

Let $\mathcal{A}$ consist of the sections in M_2 as constructed in the sufficiency-of-(b) part. Let $\mathcal{A}_1$ consist of all those uncountable members of $\mathcal{A}$. Let $\mathcal{A}_2 = \mathcal{A} - \mathcal{A}_1$. By assumption and the construction of $\mathcal{A}$ we have that $\mathcal{A}_1$ is infinite. Let B be the set of all $x \in K$ for each there exists a sequence $\{J_n\}_{n=1}^{\infty}$ in $\mathcal{A}_1$ such that $\text{diam}(J_n \cup \{x\}) \to 0$. Then $\emptyset \neq B \subseteq K$. Without loss of generality we may suppose B is infinite.

Now carrying out a simplified version of the argument in the sufficiency-of-part-b proof, we can find a sequence $\{b_n\}_{n=0}^{\infty}$ in B and for each n a sequence of uncountable sections $\{T_{nm}\}_{m=0}^{\infty}$ such that

$$T_{nm} \cap T_{ij} = \emptyset \quad \text{whenever } (n,m) \neq (ij)$$
$$\operatorname{diam}(\{b_n\} \cup T_{nm} \to 0 \quad \text{for each } n$$
$$\cup \mathcal{A}_1 = \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} T_{nm}$$
$$\operatorname{diam}\left(\{b_n\} \cup \left(\bigcup_{m=0}^{\infty} T_{nm}\right)\right) \to 0$$

If $\mathcal{A}_2$ is finite put $\mathcal{C}_0 = \mathcal{A}_2$ and $\mathcal{B}_0 = \emptyset$. If $\mathcal{A}_2$ is infinite, then either (1) there is no member of highest order in $\mathcal{A}_2$ of highest order in $\mathcal{A}_2$ in which case we put $\mathcal{B}_1 = \mathcal{A}_2$ and $\mathcal{C}_1 = \emptyset$ or (2) there exists a member of $\mathcal{A}_2$ of highest order α_1.

In case (2) let $\mathcal{C}_1$ consists of all members of $\mathcal{A}_2$ of order α_1. Consider $\mathcal{A}_2 - \mathcal{C}_1$. If $\mathcal{A}_2 - \mathcal{C}_1$ is finite put $\mathcal{C}_2 = \mathcal{A}_2$ and $\mathcal{B}_2 = \emptyset$. If $\mathcal{A}_2 - \mathcal{C}_1$ is infinite then either (1) there is no member if highest order of $\mathcal{A}_2 - \mathcal{C}_1$ in which case we put $\mathcal{B}_2 = \mathcal{A}_2 - \mathcal{C}_1$ and $\mathcal{C}_2 = \mathcal{C}_1$ or (2) there is a member of $\mathcal{A}_2 - \mathcal{C}_1$ of highest order α_2.

In case (2) we have $\alpha_2 < \alpha_1$ and we may continue the argument on $\mathcal{A}_2 - \mathcal{C}_2$. Since there is no decreasing sequence of ordinals, this process must stop at a finite stage and for some k, $\mathcal{A}_2 = \mathcal{B}_k \cup \mathcal{C}_k$ where $\mathcal{C}_k$ is finite and disjoint from $\mathcal{B}_k$ which, if it is not empty, has the property that for all $A \in \mathcal{B}_k$, $\{B \in \mathcal{B}_k : \varrho(B) \geqslant \varrho(A)\}$ is infinite.

Since $\mathcal{C}_k$ is finite we may order it by increasing order and incorporate it as an initial segment of the sequence $\{T_{o_k}\}_{k=0}^{\infty}$ so that $\varrho(T_{nk}) \leqslant \varrho(T_{nk+1})$ for each k and n.

If $\mathcal{B}_k \neq \emptyset$ we may carry out the construction in the sufficiency-of-(b) proof to obtain a sequence $\{e_n\}_{n=1}^{\infty}$ and for each n a sequence $\{S_{nm}\}_{m=0}^{\infty}$ of sections such that

$$S_{nm} \cap S_{ij} = \emptyset \quad \text{whenever } (n,m) \neq (i,j)$$
$$\operatorname{diam}(\{e_n\} \cup S_{nm}) \to 0 \quad \text{for each } n$$
$$\varrho(S_{nm}) \leqslant \varrho(S_{nm+1}) \quad \text{for each } m,\ n$$
$$\bigcup \mathcal{B}_k = \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} S_{nm}$$
$$\operatorname{diam}\left(\{e_n\} \cup \left(\bigcup_{m=0}^{\infty} S_{nm}\right)\right) \to 0$$

Moreover $S_{nm} \cap T_{ij} = \emptyset$ whenever $(n,m) \neq (i,j)$ and

$$M_2 = \bigcup \mathcal{A} = \bigcup_{n=0}^{\infty} \bigcup_{m=0}^{\infty} (T_{nm} \cup S_{nm})$$

Let f_{kn} be a continuous function mapping S_{nk+1} onto S_{nk} and g_{nk} be a continuous function mapping T_{nk+1} onto T_{nk}, using Lemma 3. Define f as follows

$$f(x) = \begin{cases} f_{nk}(x) & \text{if } x \in S_{nk} \text{ if } k \geqslant 1 \\ e_n & \text{if } x \in S_{n0} \\ g_{nk}(x) & \text{if } x \in T_{nk} \text{ if } k \geqslant 1 \\ b_n & \text{if } x \in T_{n0} \\ h(x) & \text{if } x \in M_1 \cup \gamma(x_0, h) \end{cases}$$

The rest of the proof is identical with that of the sufficiency-of(b) part. This completes the proof of the sufficiency of condition (c) and consequently finishes the proof of Theorem 1. □

Now we present several lemmas which culminate in Theorem 2.

Lemma 5 [1]. *Let M be a closed nowhere dense set. Suppose $\{z_n\}_{n=0}^{\infty}$ is a sequence of distinct points not in M but whose set of subsequential limit points is M. Then there exists a continuous $f\colon I \to I$ and $z_0 \in I$ such that $\omega(z_0, f) = M$ provided the following condition is fulfilled:*

For all numbers α and β, and $\lambda \in M$ and subsequences $\{n_k\}_{k=0}^{\infty}$ and $\{m_k\}_{k=0}^{\infty}$, $\alpha = \beta$ whenever $\lim\limits_{k\to\infty}(z_{n_k}, z_{n_k+1}) = (\lambda, \alpha)$ and $\lim\limits_{k\to\infty}(z_{m_k}, z_{m_k+1}) = (\lambda, \beta)$.

Lemma 6. *Let M and N be countable closed nonempty subsets of I and let α be a countable ordinal. If M and N each have exactly one point of order α, then M and N are homeomorphic.*

Proof. We will prove it by induction on ω_1. It is obviously true when $\alpha = 0$. Now assume it is true for all $\beta < \alpha$.

Let first take the case when α is a limit ordinal. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a strictly increasing sequence of ordinals having limit α. Let $z \in M$ such that $\varrho(z) = \alpha$. Pick $m_n \in M$ with $\varrho(m_n) = \alpha_n$. Then we must have $m_n \to z$. We may then choose a sequence $\{J_n\}_{n=1}^{\infty}$ of mutually disjoint open intervals with end points in $I - M$ such that the only point of order α_n which J_n contains is m_n. Put $J = \bigcup\limits_{n=1}^{\infty} J_n$. Choose a sequence $\{F_n\}_{n=1}^{\infty}$ of mutually disjoint closed intervals having end points in $I - M$ such that $Z_n = F_n \cap (M - J) \neq \emptyset$ for each n and $M - J - \{z\} = \bigcup\limits_{k=1}^{\infty} Z_n$. Since $\varrho(Z_n) < \alpha$ for each n we may pick by induction a subsequence $\{\alpha_{k_n}\}_{n=1}^{\infty}$ such that $\varrho(Z_n) < \alpha_{k_n}$ and $\alpha_n < \alpha_{k_N n} < \alpha_{k_{n+1}}$ for each n.

Put $W_m = J_m \cup Z_n$ if $\alpha_{k_n} = m$. Otherwise put $W_m = J_m$. Then $W_m \cap W_n = \emptyset$ when $m \neq n$ and $z \notin \overline{W}_m$ for each m. Moreover, $M - \{z\} \subseteq \bigcup\limits_{m=1}^{\infty} W_m$ and $\varrho(W_m) = \alpha_n$ and W_m contains a single point of order α_m.

Finally let $z^* \in N$ with $\varrho(z^*) = \alpha$ and construct a similar sequence of open intervals $\{W_m^*\}_{m=1}^{\infty}$ relative to N. By the inductive hypothesis there exists a homeomorphism h_m from $W_m \cap M$ onto $W_m^* \cap M$. Then $h = \{(z, z^*)\} \cup \left(\bigcup_{m=1}^{\infty} h_m\right)$ will be a homeomorphism from M onto N. □

Lemma 7. *Suppose M is a nowhere dense closed set for which $M = \omega(x_0, g)$ and there exist mutually disjoint open sets W, W_1, …, W_n where $n \geqslant 2$ such that $M \subseteq \overline{W} \cup \left(\bigcup_{i=1}^{n} W_i\right)$ and $W_{i+1} \cap M$ is homeomorphic to $W_i \cap M$ (mod n).*

Then there exists a continuous $h\colon I \to I$ such that $M = \omega(x_0, h)$, $h(W_i \cap M) = W_{i+1} \cap M$ (mod n) for each i and for $x \in M$, $h(x) = x$ if and only if $g(x) = x$ and $x \notin \bigcup_{i=1}^{m} W_i$.

Proof. Let $\{x_n : n \in \omega_0\} = \gamma(x_0, g)$. Lemma 1 of [1] we may assume that $x_0 \in W$ and $\gamma(x_0, g) \subseteq \left(W \cup \left(\bigcup_{i=1}^{n} W_i\right)\right) - M$. Let h_i be a homeomorphism from $(\gamma(x_0, g) \cup M) \cap W_i$ onto $(\gamma(x_0, g) \cup M) \cap W_{i+1}$ (mod n). Since isolated points map into isolated points it follows that $h_i(W_i \cap M) = W_{i+1} \cap M$ (mod n) for each i.

If $x_k \in W$ we put $z_k = x_k$. If $x_k \in W_i$ we put $z_k = h_i(x_k)$ (mod n). We will show that $\{z_k\}_{k=1}^{\infty}$ satisfies the hypothesis of Lemma 5. Clearly $z_k \notin M$ for all k and M is the cluster set of $\{z_k\}_{k=1}^{\infty}$. It remains to show that if $(z_{n_k}, z_{n_k+1}) \to (\lambda, \beta)$ then β is uniquely determined by λ, g and the h_i functions. There are numerous cases depending on which pair of sets among $\overline{W}$, W_1, …, W_n λ and β belong to. Each case is similar.

For example, suppose $\lambda \in W_1$ and $\beta \in W_2$. Then eventually $z_{n_k} \in W_1$ and $z_{n_k+1} \in W_2$. By construction $x_{n_k} \in W_n$ and $x_{n_k+1} \in W_1$. Hence, $h_n(x_{n_k}) = z_{n_k} \to \lambda$ and $h_1(x_{n_k+1}) = z_{n_k+1} \to \beta$. Therefore, $x_{n_k} \to h_n^{-1}(\lambda)$ and $x_{n_k+1} \to h_n^{-1}(\alpha)$. However, $x_{n_k+1} = g(x_{n_k}) \to g(h_n^{-1}(\lambda))$ so that $gh_n^{-1}(\lambda) = h_1^{-1}(\beta)$ and $\beta = h_1 g h_n^{-1}(\lambda)$. As another example if $\lambda \in \overline{W}$ and $\beta \in W_1$, then z_{n_k} is eventually in W and we proceed in a similar manner as above.

By Lemma 5 there exists a continuous $h\colon I \to I$ such that $h^n(x_0) = z_n$ and $\omega(x_0, h) = M$. It is clear that $h(W_i \cap M) = W_{i+1} \cap M$ (mod n for each i. It is also obvious that if $x \in M$, then $h(x) = x$ iff $g(x) = x$ and $x \notin \bigcup_{i=1}^{n} W_i$.

The next result is a special case of Theorem 1 of [2]. □

Lemma 8 [2]. *Let $c \in I$ and $\{P_k\}_{k=1}^{\infty}$ be a sequence of mutually disjoint nowhere dense perfect sets in $I - \{c\}$ such that diam $(P_k \cup \{c\}) \to 0$. Then, there exists $x_0 \in I$ and a continuous $h\colon I \to I$ such that $\omega(x_0, h) = \{c\} \cup \left(\bigcup_{k=1}^{\infty} P_k\right)$ and $h(P_1) = \{c\}$ and $h(P_{k+1}) = P_k$ for all k.*

The next two lemmas are out of logic sequence because they assume Theorem 2 part (b). They will be only used to prove part (c) of Theorem 2.

Lemma 9. *Suppose M, N and K are nowhere dense closed sets such that K is a nowhere dense subset of M, $M - K$ is countable and nonvoid, N is uncountable and has isolated N points, and $M \cap N = \varphi$. Moreover suppose it is not the case that ϱ has an absolute maximum on $M - K$ occuring at only one point.*

Then there exists $z_0 \in I$ and a continuous $h: I \to I$ such that $\omega(z_0, h) = M \cup N$ and the set of fixed points of h in $M \cup N$ is K.

Proof. Let W and V be open sets separating M and N. We may decompose N as $P \cup D$ where P is perfect and D is a non-void countable set. Put $C = M - K$.

By Theorem 2 part (b) pick a continuous $f: I \to I$ and $x_0 \in I$ such that $\omega(x_0, f) = K \cup C$ and the set of fixed points of f in $K \cup C$ is K. Moreover, by Lemma 1 of [1] we can assume $\gamma(x_0, f) = \{x_n : n \in \omega_0\} \subset W - (K \cup C)$. By Theorem 1 we may find $y_0 \in I$ and a continuous $g: I \to I$ such that $\omega(y_0, g) = P \cup D$ and g has no fixed points in $P \cup D$. Moreover, we may assume $\gamma(y_0, g) = \{y_n : n \in \omega_0\} \subset V - (P \cup D)$.

Choose a to be an isolated point of C such that $g(c) = b \in K \cup C$. Let U be an open interval such that $(U \cap (K \cup C))' = \{a\}$. Let $\{x_{m_n}\}_{n=0}^{\infty} = U \cap \gamma(x_0, f)$ and put $a_n = x_{m_n}$. Choose s to be an isolated point in D and put $c = g(d)$. Choose S to be a neighborhood of d such that $g(S) \cap S = \varphi$.

Let us symbolically represent the orbit $\gamma(x_0, g)$ by

$$x_0 \to x_1 \to x_2 \to \ldots \to x_k \to x_{k+1} \to$$

we will define a new "orbit" by inserting entries from $\gamma(y_0, g)$ in between x_k and x_{k+1} whenever $x_k = a_m$ for some m. By induction suppose we have made insertions in between all pairs $a_k \to b_k$ where $k < n$. Then insert y_α, $y_{\alpha+1})$, ..., $y_{\alpha+\beta}$ as

$$a_n \to y_\alpha \to y_{\alpha+1} \to \ldots \to y_{\alpha+\beta} \to b_n$$

where $y_{\alpha-1}$ is the last previously picked point (if $n = 1$ then $y_\alpha = y_0$) and β is the first integer such that $y_{\alpha+\beta} \in S$.

Let $\{z_n\}_{n=0}^{\infty}$ be the concatenation of the x_i points together with the above insertion of the y_i points. We will show that $\{z_n\}_{n=0}^{\infty}$ satisfies the hypothesis of Lemma 5.

First of all $z_n \notin (M \cup N)$ for all n. Secondly it is obvious that the cluster set of $\{z_n\}_{n=0}^{\infty}$ is $M \cup N$. Finally suppose $(z_{n_k}, z_{n_{k+1}}) \to (\lambda, \alpha)$. We need to shoe α is uniquely determined. We have several cases.

Case 1: $\lambda = a$. Then $\{z_{n_k}\}_{k=0}^{\infty}$ is eventually in $\{a_n : n \in \omega_0\}$ so that $z_{n_k+1} = y_{\alpha_k}$ where $y_{\alpha_{k-1}} \in S$. Since $\alpha_k \to \infty$, $a_{\alpha_{k-1}} \to d$ and $z_{n_k+1} = y_{\alpha_k} \to g(d) = c$.

Case 2: $\lambda \in W - \{a\}$. Then eventually $(z_{n_k}, z_{n_k+1}) = (x_{m_k}, x_{m_k+1})$ for some $\{m_k\}_{k=0}^{\infty}$ and $\alpha = f(\lambda)$.

Case 3: $\lambda = d$. Then $\{z_{n_k}\}_{k=0}^{\infty}$ is eventually in S so that $z_{n_k} = y_{\alpha_k}$ for some α_k with $z_{n_k+1} = \beta_{\alpha_k}$ for some β_k. Since $b_{\beta_k} \to b$ we have $\alpha = b$.

Case 4: $\lambda \in V - \{d\}$. Then eventually $z_{n_k} \notin S$ so that $z_{n_k} = y_{\alpha_k}$ where $z_{n_k+1} = y_{\alpha_k+1}$. Then $y_{\alpha_k+1} = g(y_{\alpha_k} \to g(\lambda)$ so that $\alpha = g(\lambda)$.

Applying Lemma 5 we obtain a continuous h such that $\omega(z_0, h) = M \cup N$. It is clear that the set of fixed points of h in $M \cup N$ is K. □

Lemma 10. *Suppose $c \in I$ and $\{P_k\}_{k=1}^{\infty}$ is a sequence of mutually disjoint nowhere dense perfect sets in $I - \{c\}$ such that diam $(P_k \cap \{c\}) \to 0$.*

Suppose M is a nowhere dense closed set and K is a closed set which is nowhere dense in M with $M - K$ countable. Suppose $N \cap M = \varphi$ where $N = \{c\} \cup \left(\bigcup_{k=1}^{\infty} P_k \right)$. Moreover, suppose it is not the case that ϱ has an absolute maximum on $M - K$ occuring at only one point.

Then, there exists $x_0 \in I$ and a continuous $h: I \to I$ such that $M \cup N = \omega(x_0, h)$ and the set of fixed points of h in $M \cup N$ is $K \cup \{c\}$.

Proof. The proof is similar to that of Lemma 9. Let W and V be open sets separating M and N. By Theorem 2 pick a continuous $f: I \to I$ and $x_0 \in I$ such that $\omega(x_0, f) = M$ and the set of f fixed points of in M is K and $\gamma(x_0, f) \subset W - M$.

By Lemma 8 choose $y_0 \in I$ and $g: I \to I$ such that $\omega(y_0, g) = N$; $g(P_1) = c$ and $g(P_{k+1}) = P_k$ for all k and $\gamma(y_0, y) \subset V - N$. Pick mutually disjoint open sets $\{S_n\}_{n=1}^{\infty}$ such that $P_n \subset S_n \subset V - \{c\}$ for each n with $\gamma(y_0, g) \subset \bigcup_{n=1}^{\infty} S_n$.

Let a be an isolated point of M such that $f(a) = b \in M$. Let U be an open interval of a such that $(U \cap M)' = \{a\}$. Let $\{x_{m_n}\}_{n=0}^{\infty} = U \cap \gamma(x_0, f)$ and put $a_n = x_{m_n}$.

Following the proof of Lemma 9 we insert y_α, $y_{\alpha+1}$, ..., $y_{\alpha+\beta}$ as

$$a_n \to y_\alpha \to y_{\alpha+1} \to \dots \to y_{\alpha+\beta} \to b_n$$

where $y_{\alpha-1}$ is the last previously picked point (if $n = 1$, $y_\alpha = y_0$) and β is the first integer such that $y_{\alpha+\beta} \in S_1$.

The rest of the proof parallels that of Lemma 9, and will be omitted. □

Theorem 2. *Let M be a non-empty closed nowhere dense subset of I and K be a closed subset of M which is nowhere dense in M.*

Then, there exists a continuous $g: I \to I$ and $x_0 \in I$ such that $\omega(x_0, g) = M$ and K consists of the fixed points of g in M if and only if one of the following hold

(1) *$K = \varphi$ and there is more than one point of highest order in M.*

(2) $K \neq \varphi$, $M - K$ *is countable and it is not the case that* ϱ *has an absolute maximum on* $M - K$ *occuring only at one point.*

(3) $K \neq \varphi$ *and* $M - K$ *is uncountable.*

Proof. *The Necessity.* Suppose $M = \omega(x_0, g))$ and K is the set of fixed points of g in M. Then be have 3 cases: (a) $K = \varphi$; (b) $K \neq \varphi$ and $M - K$ is countable; and (c) $K \neq \varphi$ and $M - K$ is uncountable.

In case (a) Theorem 1 applies and there is more than one point of highest order in M.

For case (b) suppose $M = \omega(x_0, g)$ where K is the set of fixed points of g in M. Suppose that ϱ has an absolute maximum on $M - K$ occuring at a single point z. Let $\varrho(z) = \alpha$. Then since $M - K$ is countable α is countable. By Lemma 1 there is a y such that $\varrho(y) \geqslant \alpha$ and $g(y) = z$. If $\varrho(y) > \alpha$, then $y \in K$ and $g(y) = y = z$ and $z \in K$, a contradiction. Hence, $\varrho(y) = \alpha$ and $y = z$ so that $g(z) = z$ and $z \in K$, again a contradiction.

The sufficiency. Case (a): Theorem 1 gives the conclusion.

Case (b): If ϱ has no absolute maximum on $M - K$ then for each $x \in M - K\{y \in M - K : \varrho(y) \geqslant \varrho(x)\}$ is infinite. Hence by Theorem 1 there is a continuous g realizing M with K as its fixed points in M.

If ϱ has an absolute maximum on $M - K$ of $\alpha \geqslant 1$, then $A = (M - K) \cap \varrho^{-1}(\alpha)$ consists of more than one point. If A is infinite, then again for all $x \in M - K\{y \in M - K : \varrho(y) \geqslant \varrho(x)\}$ is infinite and Theorem 1 does the job.

Hence we can assume A is finite and has at least two points. Let $K_1 = K \cup A$. Then clearly for each $x \in M - K_1$, $\{y \in M - K_1 : \varrho(y) \geqslant \varrho(x)\}$ is infinite. Applying Theorem 1, part b, there exists a continuous g which realizes M with $K \cup A$ as its set of fixed points in M. Suppose $A = \{x_1, \ldots, x_n\}$. Pick open intervals W_i $i = 1, \ldots, n$ such that $x_i \in W_i \subseteq I - K$, $\varrho(M \cap W_i) = \alpha$, the end points of W_i are not in M and $W_i \cap W_j = \varphi$ for $i \neq j$. Put $W = I - \bigcup_{i=j}^{\infty} \overline{W}_i$. Now apply Lemmas 6 and 7 to finish the sufficiency of part (b).

Case (c): Assume $K \neq \varphi$ and $M - K$ is uncountable. If $M - K$ has a c-limit point in K we may apply Theorem 1 to get the desired result. So let us assume $M - K$ has no c-limit point in K. Then we may find two disjoint open sets W and V such that $W \cap M = M_1 = K \cup C$ where C is countable and nonvoid and $V \cap M = P \cup D$ where P is nonvoid perfect set and D is countable.

Now we can find an open $U \subseteq W$ such that $\overline{U} \cap M = \varphi$ and it is not the case that ϱ has an absolute maximum on $M_1^* - K$ at only one point, where $M^* = M - U$. We show this as follows: suppose ϱ has an absolute maximum α_1 occuring at a single z_1 in $M_1 - K$. Choose an open U_1 such that $z_1 \in U_1 \subseteq W$ and $\overline{U} \cap K = \varphi$. Next consider ϱ on $M_2 - K$ where $M_2 = M_1 - U_1$. If ϱ has no single absolute maximum on $M_2 - K$ we are finished. Otherwise suppose ϱ has an absolute maximum α_2 occuring at a single point z_2 in $M_2 - K$. Then $\alpha_2 < \alpha$. Choose open U_2 such that $z_2 \in U_2 \subseteq W$ and $\overline{U}_2 \cap K = \varphi$. Consider ϱ on $M_3 - K$ where $M_3 = M_2 - U_2$. And continue the argument. Since there is no decreasing sequence of ordinals the process must stop at a stage where ϱ has no single absolute maximum on some $M_n - K$ and the construction of the desired U is clear.

Now we can adjoin $U \cap C$ to D so we can without loss of generality assume that ϱ has no single absolute maximum on $M_1 - K$.

We then have two cases to consider:

Case 1: $D \neq \varphi$. Apply Lemma 9 where $M = M_1$ and $N = P \cup D$ to get desired result.

Case 2: $D = \varphi$. Then let c_1 and c_2 be the inf a sup of P. Then we may decompose $P - \{c_1, c_2\}$ into two sequences of mutually disjoint Cantor sets $\{P_k^1\}_{k=1}^{\infty}$ and $\{P_k^2\}_{k=1}^{\infty}$ satisfying the conditions of the hypothesis of Lemma 10. It is clear that we can extend Lemma 10 to apply to both of these sequences in the obvious way and we obtain $x_0 \in I$ and $h\colon I \to I$ such that $\omega(x_0, h) = M$ and the set of fixed points of h in M is $K \cup \{c_1, c_2\}$. Now since $\{c_1\} \cup \left(\bigcup_{k=1}^{\infty} P_k^1\right)$ of homeomorphic to $\{c_2\} \cup \left(\bigcup_{k=1}^{\infty} P_k^2\right)$ we can apply Lemma 7 to get $h^*: i \to I$ and $W_0 \in I$ such that $\omega(W_0, h^*) = M$ and the set of fixed points of h^* in M is K, finishing the proof of Theorem 2. □

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