Jozef Džurina

Comparison theorems for functional differential equations

Mathematica Bohemica, Vol. 119 (1994), No. 2, 203–211

Persistent URL: `http://dml.cz/dmlcz/126077`

COMPARISON THEOREMS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

JOZEF DŽURINA, Košice

(Received September 24, 1992)

Summary. In this paper the oscillatory and asymptotic properties of the solutions of the functional differential equation

$$L_n u(t) + p(t) f\left(u[g(t)]\right) = 0$$

are compared with those of the functional differential equation

$$\alpha_n u(t) + q(t) h\left(u[w(t)]\right) = 0.$$

Keywords: Property (A), canonical form

AMS classification: Primary 34C10

We consider the n-th order ($n \geqslant 2$) functional differential equation

$$\left(\frac{1}{r_{n-1}(t)} \cdots \left(\frac{1}{r_1(t)} u'(t)\right)' \cdots\right)' + p(t) f\left(u[g(t)]\right) = 0, \tag{1}$$

where $r_i, g, p \in C([t_0, \infty))$, $f \in C(\mathbb{R})$, $p(t) > 0$, $r_i(t) > 0$, $i = 1, 2, \ldots, n-1$, $xf(x) > 0$ for $x \neq 0$ and $g(t) \to \infty$ as $t \to \infty$.

We introduce the notation

$$L_0 u(t) = u(t), \quad L_i u(t) = \frac{1}{r_i(t)} \left(L_{i-1} u(t)\right)', \quad L_n u(t) = \left(L_{n-1} u(t)\right)', \qquad i = 1, 2, \ldots, n-1. \tag{2}$$

The author would like to thank the referee for some useful comments.

Then equation (1) can be rewritten as

$$L_n u(t) + p(t) f(u[g(t)]) = 0.$$

The domain $D(L_n)$ of L_n is defined to be the set of all functions $u\colon [T_u, \infty) \to \mathbb{R}$ such that $L_i u(t)$, $0 \leqslant i \leqslant n$ exist and are continuous on $[T_u, \infty)$. By a proper solution of equation (1) we mean a function $u(t) \in D(L_n)$ which satisfies (1) for all sufficiently large t and $\sup\{|u(t)|\colon t \geqslant T\} > 0$ for every $T > T_u$. We make the standing hypothesis that equation (1) does possess proper solutions. A proper solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its proper solutions are oscillatory.

We say that the operator L_n is in canonical form if

$$\int^{\infty} r_i(s)\,\mathrm{d}s = \infty \quad \text{for} \quad 1 \leqslant i \leqslant n-1. \tag{3}$$

It is well known that any differential operator of the form (2) can always be represented in a canonical form in an essentially unique way (see Trench [10]). In the sequel we will suppose L_n is in canonical form.

Lemma 1. *Let* (3) *hold. If $u(t)$ is a nonoscillatory solution of* (1), *then there exists a t_1 and an integer ℓ, $0 \leqslant \ell \leqslant n-1$ such that $\ell \not\equiv n$* (mod 2) *and*

$$\begin{aligned} u(t)L_i u(t) &> 0 \quad \text{on} \quad [t_1, \infty), \quad 1 \leqslant i \leqslant \ell, \\ (-1)^{i-\ell} u(t) L_i u(t) &> 0 \quad \text{on} \quad [t_1, \infty), \quad \ell+1 \leqslant i \leqslant n. \end{aligned} \tag{4}$$

Lemma 1 generalizes a well known lemma of Kiguradze [6] and can be proved similarly.

A function $u(t)$ satisfying (4) is said to be of degree ℓ. The set of all nonoscillatory solutions of degree ℓ of (1) is denoted by $\mathcal{N}_\ell$. If we denote by $\mathcal{N}$ the set of all nonoscillatory solutions of (1), then

$$\begin{aligned} \mathcal{N} &= \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1}, \quad \text{for } n \text{ odd}, \\ \mathcal{N} &= \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1}, \quad \text{for } n \text{ even}. \end{aligned}$$

Following Kusano and Naito [7] we are interested in the situation when $\mathcal{N} = \mathcal{N}_0$, especially when every nonoscillatory solution $u(t)$ of (1) satisfies

$$\lim_{t\to\infty} u(t) = 0. \tag{5}$$

Definition 1. Let L_n be in canonical form. Equation (1) is said to have property (A) if for n even (1) is oscillatory (i.e. $\mathcal{N} = \emptyset$) and for n odd every nonoscillatory solution $u(t)$ of (1) satisfies (5).

Remark 1. Let us denote $M_0 = 1$ and

$$M_i(t) = \int_{t_0}^{t} r_i(s_i) \int_{t_0}^{s_i} \cdots \int_{t_0}^{s_2} r_1(s_1)\,\mathrm{d}s_1 \ldots \mathrm{d}s_i, \quad i = 1, \ldots, n-1.$$

Then a stronger asymptotic result can be established, namely, according to Theorem 1 in [1] if $u(t)$ satisfies (5) then also

$$\lim_{t\to\infty} M_i(t) L_i u(t) = 0, \quad i = 0, 1, \ldots, n-1.$$

The following lemma is elementary but quite useful in the sequel.

Lemma 2. *Let $p(t) \geqslant 0$, $q(t) \geqslant 0$. Suppose that a function $a(t)$ is positive, continuous and nondecreasing on $[t_0, \infty)$. If*

$$\int_t^\infty p(s)\,\mathrm{d}s \geqslant \int_t^\infty q(s)\,\mathrm{d}s, \quad t \geqslant t_0,$$

then

$$\int_t^\infty p(s)a(s)\,\mathrm{d}s \geqslant \int_t^\infty q(s)a(s)\,\mathrm{d}s, \quad t \geqslant t_0.$$

In the literature many comparison results have been established to the effect that if a differential equation with a deviating argument has property (A) then so does another related equation with larger deviating argument. Attempts in this direction have been undertaken e.g. by Kusano and Naito [7], Mahfoud [9] and Erbe [3] and [4].

The aim of this paper is to present comparison results in the opposite direction, that is we wish to derive property (A) of an equation with a deviating argument from the corresponding property of another equation with larger deviating argument. Therefore, let functions $g(t)$ and $w(t)$ be subject to the conditions

$$g, w \in C^1, \quad g'(t) > 0, \quad w'(t) > 0, \quad w(t) \geqslant g(t). \tag{6}$$

To be able to build our comparison technique we use the following differential operator which was introduced by Kusano and Naito [7].

$$\alpha_n = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{r_{n-1}[\tau(t)]\tau'(t)} \frac{\mathrm{d}}{\mathrm{d}t} \cdots \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{r_1[\tau(t)]\tau'(t)} \frac{\mathrm{d}}{\mathrm{d}t},$$

where $\tau(t) = g(w^{-1}(t))$ and $w^{-1}(t)$ is the inverse function to $w(t)$.

We compare oscillatory and asymptotic properties of solutions of equation (1) with those of the equation

$$\alpha_n u(t) + q(t)h\,(u[w(t)]) = 0, \tag{7}$$

where $q \in C([t_0, \infty))$, $h \in C(\mathbb{R})$, $q(t) > 0$, $xh(x) > 0$ for $x \neq 0$ and $w(t)$ satisfies (6).

Note that the function $\tau(t)$ expressed in terms of arguments $g(t)$ and $w(t)$ of equations (1) and (7) is the main tool for comparing (1) and (2).

Theorem 1. *Suppose that* (3) *and* (6) *hold. Let* $h(x)$ *be nondecreasing. Further assume that*

$$f(x)\operatorname{sgn} x \geqslant h(x)\operatorname{sgn} x \qquad \text{for } x \neq 0, \tag{8}$$

$$\int_t^\infty p(s)\,\mathrm{d}s \geqslant \int_t^\infty q(s)\,\mathrm{d}s \qquad \text{for } t \geqslant t_0. \tag{9}$$

Then equation (1) *has property* (A) *if so does equation* (7).

Proof. Let $u(t)$ be a nonoscillatory solution of (1). We may assume that $u(t)$ is positive (for $u(t) < 0$ we can use a similar argument). Then there exists an integer $\ell \in \{0, 1, \dots, n-1\}$ such that $n + \ell$ is odd, and a $t_1 \geqslant t_0$ associated with $u(t)$ by Lemma 1. Assume $\ell \geqslant 1$. Integrating (1) and using $\tau(t) \leqslant t$ we obtain

$$L_{n-1}u(\tau(t)) \geqslant \int_t^\infty p(s) f(u[g(s)])\,\mathrm{d}s \tag{10}$$

for $t \geqslant t_2$ ($\geqslant t_1$) provided t_2 is sufficiently large. First, note that $u(t)$ is nondecreasing as $\ell \geqslant 1$. Combining (10) with (8) one gets

$$L_{n-1}u(\tau(t)) \geqslant \int_t^\infty p(s)h(u[g(s)])\,\mathrm{d}s, \quad t \geqslant t_2.$$

Since the composite function $h(u[g(t)])$ is nondecreasing, according to Lemma 2 we obtain

$$L_{n-1}u(\tau(t)) \geqslant \int_t^\infty q(s)h(u[g(s)])\,\mathrm{d}s, \quad t \geqslant t_2. \tag{11}$$

We multiply (11) by $r_{n-1}(\tau(t))\tau'(t)$ and integrate the resulting inequality over $[t, \infty)$. Repeating this procedure, we arrive at

$$\begin{aligned} L_\ell u(\tau(t)) \geqslant &\int_t^\infty r_{\ell+1}[\tau(s_{\ell+1})]\tau'(s_{\ell+1}) \\ &\times \int_{s_{\ell+1}}^\infty \cdots \int_{s_{n-1}}^\infty q(s_n)h\,(u[g(s_n)])\,\mathrm{d}s_n \dots \mathrm{d}s_{\ell+1}. \end{aligned} \tag{12}$$

We multiply (12) by $r_\ell(\tau(t))\tau'(t)$ and integrate over $[t_2, t]$. Continuing in this manner we obtain

$$
\begin{aligned}
(13)\qquad u[\tau(t)] \geqslant c + \int_{t_2}^{t} r_1[\tau(s_1)]\tau'(s_1) \int_{t_2}^{s_1} \cdots \int_{t_2}^{s_{\ell-1}} r_\ell[\tau(s_\ell)]\tau'(s_\ell) \\
\times \int_{s_\ell}^{\infty} \cdots \int_{s_{n-1}}^{\infty} q(s_n) h\big(u[g(s_n)]\big)\, \mathrm{d}s_n \ldots \mathrm{d}s_1, \quad t \geqslant t_2,
\end{aligned}
$$

where $c = u[\tau(t_2)] > 0$. Denote the right hand side of (13) by $z(t)$. By repeated differentiation of $z(t)$ one can verify that $z(t)$ is a function of degree ℓ and, on the other hand,

$$
(14)\qquad \alpha_n z(t) + q(t) h\big(u[g(t)]\big) = 0.
$$

As $u[\tau(t)] \geqslant z(t)$ and $\tau\big(w(t)\big) = g(t)$ we see that

$$
u\big(g(t)\big) = u\big(\tau[w(t)]\big) \geqslant z\big(w(t)\big)
$$

for all large t, say $t \geqslant t_3$. Combining this fact with (14) we see that $z(t)$ is a solution of the differential inequality

$$
\big\{\alpha_n z(t) + q(t) h\big(z[w(t)]\big)\big\} \operatorname{sgn} z[w(t)] \leqslant 0, \quad t \geqslant t_3.
$$

Then by Kusano and Naito (see [7]) equation (7) has also an eventually positive solution $x(t)$ satisfying

$$
\lim_{t\to\infty} x(t) \geqslant c > 0,
$$

which contradicts the hypotheses.

Now, let $\ell = 0$ (note that this is possible only when n is odd). To obtain a contradiction assume that $c_0 = \lim_{t\to\infty} u(t) > 0$. Integrating (1), in view of (8) we have

$$
\begin{aligned}
(15)\qquad L_0 u\big(\tau(t)\big) \geqslant c_0 + \int_{t}^{\infty} r_1[\tau(s_1)]\tau'(s_1) \\
\times \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} p(s_n) h\left(u[g(s_n)]\right) \mathrm{d}s_n \ldots \mathrm{d}s_1
\end{aligned}
$$

for all large t. Since $u(t)$ is decreasing ($\ell = 0$) one gets

$$
\frac{3}{2} c_0 \geqslant u(t) \geqslant c_0, \quad t \geqslant t_2,
$$

where t_2 is large enough. This fact together with (15) and (9) implies

$$c_0 \geqslant \frac{c_0}{2} + \int_t^{\infty} r_1[\tau(s_1)]\tau'(s_1) \times \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} q(s_n) h(c_0)\, \mathrm{d}s_n \ldots \mathrm{d}s_1 \tag{16}$$

for all large t, say $t \geqslant t_3$. Again, let us denote the right hand side of (16) by $z(t)$. Then

$$\alpha_n z(t) + q(t)h(c_0) = 0, \quad t \geqslant t_3. \tag{17}$$

Noting that $z(w(t)) \leqslant c_0$, (17) yields

$$\{\alpha_n z(t) + q(t)h(z[w(t)])\} \operatorname{sgn} z[w(t)] \leqslant 0.$$

Again according to Kusano and Naito [7] equation (7) has also an eventually positive solution $x(t)$ with the property

$$x(t) \geqslant \frac{c_0}{2} > 0.$$

This contradicts the assumptions and the proof is complete. $\square$

Example 1. Consider an even order delay equation

$$\left(\frac{1}{r(t)} y^{(m)}\right)^{(m)} + p(t) f(y[g(t)]) = 0, \quad m \geqslant 1, \tag{18}$$

where functions r, p, f and g are the same as in Theorem 1 and $g(t) \leqslant t$. Then by Theorem 1 equation (18) is oscillatory if the ordinary equation without delay

$$\left(\frac{1}{r(g(t))g'(t)} y^{(m)}\right)^{(m)} + p(t) f(y(t)) = 0, \quad m \geqslant 1$$

is oscillatory.

Corollary 1. *Let* (3) *hold. Let $r_i(t)$; $1 \leqslant i \leqslant n-1$ be nonincreasing. Assume that f is nondecreasing. Suppose that $g \in C^1$ and $g'(t) > 0$. Then for any constant M equation* (1) *has property* (A) *if and only if so does the equation*

$$L_n u(t) + p(t) f(u[g(t) + M]) = 0. \tag{19}$$

P r o o f. Assume that $M > 0$. The "only if" part follows from Theorem 1 of Kusano and Naito [7]. Now suppose that equation (19) has property (A). We put $w(t) = g(t) + M$. Then

$$\tau(t) = g\left(w^{-1}(t)\right) = t - M \leqslant t.$$

By Theorem 1 of Kusano and Naito [7] the equation

$$\alpha_n u(t) + p(t) f\left(u[g(t) + M]\right) = 0 \tag{20}$$

has property (A) as $r_i(\tau(t))\tau'(t) \geqslant r_i(t)$. The "if" part now follows from Theorem 1 applied to equations (20) and (1).

Now let $M < 0$. We put $\tilde{g}(t) = g(t) + M$. Since $-M > 0$, by the first part of the proof of this theorem we see that the equation

$$L_n u(t) + p(t) f\left(u[\tilde{g}(t)]\right) = 0$$

has property (A) if and only if so does the equation

$$L_n u(t) + p(t) f\left(u[\tilde{g}(t) + (-M)]\right) = 0,$$

which we wanted to verify. □

E x a m p l e 2. Consider an n-order linear delay differential equation

$$\left(\frac{1}{r(t)} \cdots \left(\frac{1}{r(t)} u'(t)\right)' \cdots\right)' + p(t) u(g(t)) = 0, \tag{21}$$

where r, p and g are subject to the same conditions as in (1). Let us denote $b(t) = r(g(t))g'(t)$ and put $w(t) = t$. By Theorem 1 equation (21) has property (A) if the differential equation without delay

$$\left(\frac{1}{b(t)} \cdots \left(\frac{1}{b(t)} u'(t)\right)' \cdots\right)' + p(t) u(t) = 0$$

has property (A), which by Theorem 5 in [2] occurs if

$$\liminf_{t\to\infty} \left(\int_{t_0}^{t} b(s)\,\mathrm{d}s\right)^{n-1} \left(\int_{t}^{\infty} p(s)\,\mathrm{d}s\right) > \frac{M_1}{n-1}, \tag{22}$$

where M_1 is the maximum of all local maxima of the polynomial

$$P_n(k) = -k(k-1)\dots(k-n+1).$$

If we set $g(s) = x$ then (22) reduces to

$$\liminf_{t\to\infty} \left(\int_{t_0^*}^{g(t)} r(x)\,\mathrm{d}x \right)^{n-1} \left(\int_t^{\infty} p(s)\,\mathrm{d}s \right) > \frac{M_1}{n-1},$$

which is a weaker sufficient condition for equation (21) to have property (A) than Kusano and Naito have required in [7].

For n even, as we see from Definition 1, property (A) of equation (1) reduces to oscillation of (1). In [8] Kusano and Naito discussed the oscillatory character of a special case of (1), namely the even order linear differential equation

$$L_n u(t) + p(t)u(t) = 0, \tag{23}$$

by comparing (23) with a set of second order differential equations (see Theorem B in [8]). We adapt their method together with Theorem 1 and make use of the results obtained to extend the results of Kusano and Naito [8] and Trench [11].

Let $1 \leqslant i \leqslant n-1$ and $t, s \in [t_0, \infty)$. We define

$$I_0 = 1$$

$$I_i(t, s; r_i, \dots, r_1) = \int_s^t r_i(x) I_{i-1}(x, s; r_{i-1}, \dots, r_1)\,\mathrm{d}x\,.$$

Let us denote $b_i(t) = r_i(g(t))g'(t)$ for $i = 1, 2, \dots, n-1$. For simplicity of notation we put

$$J_i(t, s) = I_i(t, s; b_1, \dots, b_i),$$

$$K_i(t, s) = I_i(t, s; b_{n-1}, \dots, b_{n-i}).$$

Theorem 2. *Suppose that $n \geqslant 4$ is even. Assume that all the conditions of Theorem 1 are satisfied with $h(x) = x$ and $w(t) = t$. Define for $i = 1, 3, \dots, n-3$*

$$a_i(t) = b_{i+1}(t) \int_t^{\infty} J_{i-1}(s, t) K_{n-i-2}(s, t) q(s)\,\mathrm{d}s, \tag{24}$$

$$a_{n-1}(t) = b_{n-2}(t) \int_t^{\infty} J_{n-3}(s, t) q(s)\,\mathrm{d}s\,. \tag{25}$$

Then equation (1) is oscillatory if the second order equations

$$\left(\frac{1}{b_i(t)} y'(t) \right)' + a_i(t) y(t) = 0, \quad i = 1, 3, \dots, n-1 \tag{26}$$

are oscillatory.

Proof. Applying Theorem B in [8] we conclude that equation (7) is oscillatory, and hence (1) is oscillatory by Theorem 1. □

Corollary 2. *Let all the conditions of Theorem 1 hold with $h(x) = x$ and $w(t) = t$. If*

$$\liminf_{t\to\infty} \left(\int_{t_0}^{g(t)} r_i(s)\,\mathrm{d}s\right)\left(\int_t^{\infty} a_i(s)\,\mathrm{d}s\right) > \frac{1}{4}, \quad i = 1, 3, \ldots, n-1, \tag{27}$$

where $a_i(t)$, $i = 1, 3, \ldots, n-1$ are defined as in (24) and (25), then equation (1) is oscillatory.

P r o o f. We make a change of variables in the first integral in (27) by using the substitution $s = g(x)$ and obtain

$$\liminf_{t\to\infty} \left(\int_{t_0^*}^{t} b_i(x)\,\mathrm{d}x\right)\left(\int_t^{\infty} a_i(s)\,\mathrm{d}s\right) > \frac{1}{4}, \quad i = 1, 3, \ldots, n-1. \tag{28}$$

It is known (see [5]) that (28) is sufficient for all solutions of (26) to be oscillatory. Hence, Corollary 2 follows from Theorem 2. □

References

[1] *J. Džurina*: Oscillation and asymptotic properties of n-th order differential equations. Czech. Math. J. *42* (1992), 11–14.

[2] *J. Džurina*: Comparison theorems for nonlinear ODEs. Math. Slovaca *42* (1992), 299–315.

[3] *L. Erbe*: Oscillation criteria for second order nonlinear delay equations. Canad. Math. Bull. *16* (1973), 49–56.

[4] *L. Erbe*: Oscillation and asymptotic behavior of solutions of third order differential delay equations. SIAM J. Math. Anal. *7* (1976), 491–500.

[5] *E. Hille*: Non-oscillation theorems. Trans. Amer. Math. Soc. *64* (1948), 234–258.

[6] *I. T. Kiguradze*: On the oscillation of solutions of the equation $\mathrm{d}^m u/\mathrm{d}t^m + a(t)|u|^n \times \operatorname{sign} u = 0$. Mat. Sb. *65* (1964), 172–187. (In Russian.)

[7] *T. Kusano and M. Naito*: Comparison theorems for functional differential equations with deviating arguments. J. Math. Soc. Japan *3* (1981), 509–532.

[8] *T. Kusano and M. Naito*: Oscillation criteria for general linear ordinary differential equations. Pacific J. Math. *92* (1981), 345–355.

[9] *W. E. Mahfoud*: Comparison theorems for delay differential equations. Pacific J. Math. *83* (1979), 187–197.

[10] *W. F. Trench*: Canonical forms and principal systems for general disconjugate equations. Trans. Amer. Math. Soc. *189* (1974), 319–327.

[11] *W. F. Trench*: Oscillation properties of perturbed disconjugate equations. Proc. Amer. Math. Soc. *52* (1975), 147–155.

Author's address: *Jozef Džurina*, Department of Mathematical Analysis, Šafárik University, Jesenná 5, 041 54 Košice, Slovakia, e-mail: **dzurina@turing.upjs.sk**.