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Comparison theorems for functional differential equations


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COMPARISON THEOREMS
FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Summary. In this paper the oscillatory and asymptotic properties of the solutions of the functional differential equation

\[ L_n u(t) + p(t) f(u(g(t))) = 0 \]

are compared with those of the functional differential equation

\[ \alpha_n u(t) + q(t) h(u(w(t))) = 0. \]

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We consider the \( n \)-th order \((n \geq 2)\) functional differential equation

\[ \left( \frac{1}{r_n(t)} \cdots \left( \frac{1}{r_1(t)} u'(t) \right)' \cdots \right)' + p(t) f(u(g(t))) = 0, \]

where \( r_i, g, p \in C([t_0, \infty)), f \in C(\mathbb{R}), p(t) > 0, r_i(t) > 0, i = 1, 2, \ldots, n - 1, \)
\( xf(x) > 0 \) for \( x \neq 0 \) and \( g(t) \to \infty \) as \( t \to \infty. \)

We introduce the notation

\[ L_0 u(t) = u(t), \quad L_i u(t) = \frac{1}{r_i(t)} \big( L_{i-1} u(t) \big)', \quad L_n u(t) = \big( L_{n-1} u(t) \big)', \]

\[ i = 1, 2, \ldots, n - 1. \]

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Then equation (1) can be rewritten as

\[ L_n u(t) + p(t) f(u(g(t))) = 0. \]

The domain \( D(L_n) \) of \( L_n \) is defined to be the set of all functions \( u: [T_u, \infty) \to \mathbb{R} \) such that \( L_i u(t), 0 \leq i \leq n \) exist and are continuous on \( [T_u, \infty) \). By a proper solution of equation (1) we mean a function \( u(t) \in D(L_n) \) which satisfies (1) for all sufficiently large \( t \) and \( \sup\{ |u(t)| : t \geq T \} > 0 \) for every \( T > T_u \). We make the standing hypothesis that equation (1) does possess proper solutions. A proper solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory. Equation (1) is said to be oscillatory if all its proper solutions are oscillatory.

We say that the operator \( L_n \) is in canonical form if

\[ \int_{T_u}^{\infty} r_i(s) \, ds = \infty \quad \text{for} \quad 1 \leq i \leq n - 1. \]

It is well known that any differential operator of the form (2) can always be represented in a canonical form in an essentially unique way (see Trench [10]). In the sequel we will suppose \( L_n \) is in canonical form.

**Lemma 1.** Let (3) hold. If \( u(t) \) is a nonoscillatory solution of (1), then there exists a \( t_1 \) and an integer \( \ell, 0 \leq \ell \leq n - 1 \) such that \( \ell \not\equiv n \, (\mod 2) \) and

\[ \begin{align*}
    u(t) L_i u(t) &> 0 \quad \text{on} \quad [t_1, \infty), \quad 1 \leq i \leq \ell, \\
    (-1)^{i-\ell} u(t) L_i u(t) &> 0 \quad \text{on} \quad [t_1, \infty), \quad \ell + 1 \leq i \leq n.
\end{align*} \]

Lemma 1 generalizes a well known lemma of Kiguradze [6] and can be proved similarly.

A function \( u(t) \) satisfying (4) is said to be of degree \( \ell \). The set of all nonoscillatory solutions of degree \( \ell \) of (1) is denoted by \( \mathcal{N}_\ell \). If we denote by \( \mathcal{N} \) the set of all nonoscillatory solutions of (1), then

\[ \mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_2 \cup \cdots \cup \mathcal{N}_{n-1}, \quad \text{for} \ n \ \text{odd}, \]

\[ \mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \cdots \cup \mathcal{N}_{n-1}, \quad \text{for} \ n \ \text{even}. \]

Following Kusano and Naito [7] we are interested in the situation when \( \mathcal{N} = \mathcal{N}_0 \), especially when every nonoscillatory solution \( u(t) \) of (1) satisfies

\[ \lim_{t \to \infty} u(t) = 0. \]

**Definition 1.** Let \( L_n \) be in canonical form. Equation (1) is said to have property (A) if for \( n \) even (1) is oscillatory (i.e. \( \mathcal{N} = \emptyset \)) and for \( n \) odd every nonoscillatory solution \( u(t) \) of (1) satisfies (5).
Remark 1. Let us denote $M_0 = 1$ and

$$M_i(t) = \int_{t_0}^{t} r_i(s_i) \int_{t_0}^{s_i} \cdots \int_{t_0}^{s_2} r_1(s_1) \, ds_1 \ldots ds_i, \quad i = 1, \ldots, n - 1.$$  

Then a stronger asymptotic result can be established, namely, according to Theorem 1 in [1] if $u(t)$ satisfies (5) then also

$$\lim_{t \to \infty} M_i(t) L_i u(t) = 0, \quad i = 0, 1, \ldots, n - 1.$$  

The following lemma is elementary but quite useful in the sequel.

**Lemma 2.** Let $p(t) \geq 0$, $q(t) \geq 0$. Suppose that a function $a(t)$ is positive, continuous and nondecreasing on $[t_0, \infty)$. If

$$\int_t^{\infty} p(s) \, ds \geq \int_t^{\infty} q(s) \, ds, \quad t \geq t_0,$$

then

$$\int_t^{\infty} p(s) a(s) \, ds \geq \int_t^{\infty} q(s) a(s) \, ds, \quad t \geq t_0.$$  

In the literature many comparison results have been established to the effect that if a differential equation with a deviating argument has property (A) then so does another related equation with larger deviating argument. Attempts in this direction have been undertaken e.g. by Kusano and Naito [7], Mahfoud [9] and Erbe [3] and [4].

The aim of this paper is to present comparison results in the opposite direction, that is we wish to derive property (A) of an equation with a deviating argument from the corresponding property of another equation with larger deviating argument. Therefore, let functions $g(t)$ and $w(t)$ be subject to the conditions

(6) \quad $g, w \in C^1$, \quad $g'(t) > 0$, \quad $w'(t) > 0$, \quad $w(t) \geq g(t)$.

To be able to build our comparison technique we use the following differential operator which was introduced by Kusano and Naito [7].

$$\alpha_n = \frac{d}{dt} \frac{1}{r_{n-1}[\tau(t)] \tau'(t)} \frac{d}{dt} \cdots \frac{d}{dt} \frac{1}{r_1[\tau(t)] \tau'(t)} \frac{d}{dt},$$

where $\tau(t) = g(w^{-1}(t))$ and $w^{-1}(t)$ is the inverse function to $w(t)$.
We compare oscillatory and asymptotic properties of solutions of equation (1) with those of the equation

\[ \alpha_n u(t) + q(t) h(u[w(t)]) = 0, \]

where \( q \in C([t_0, \infty)), \ h \in C(\mathbb{R}), \ q(t) > 0, \ xh(x) > 0 \) for \( x \neq 0 \) and \( w(t) \) satisfies (6).

Note that the function \( \tau(t) \) expressed in terms of arguments \( g(t) \) and \( w(t) \) of equations (1) and (7) is the main tool for comparing (1) and (2).

**Theorem 1.** Suppose that (3) and (6) hold. Let \( h(x) \) be nondecreasing. Further assume that

\[ f(x) \text{ sgn } x \geq h(x) \text{ sgn } x \quad \text{for } x \neq 0, \]

\[ \int_t^\infty p(s) \, ds \geq \int_t^\infty q(s) \, ds \quad \text{for } t \geq t_0. \]

Then equation (1) has property (A) if so does equation (7).

**Proof.** Let \( u(t) \) be a nonoscillatory solution of (1). We may assume that \( u(t) \) is positive (for \( u(t) < 0 \) we can use a similar argument). Then there exists an integer \( \ell \in \{0, 1, \ldots, n - 1\} \) such that \( n + \ell \) is odd, and a \( t_1 \geq t_0 \) associated with \( u(t) \) by Lemma 1. Assume \( \ell \geq 1 \). Integrating (1) and using \( \tau(t) \leq t \) we obtain

\[ L_{n-1} u(\tau(t)) \geq \int_t^\infty p(s) f(u[g(s)]) \, ds \]

for \( t \geq t_2 \) (\( \geq t_1 \)) provided \( t_2 \) is sufficiently large. First, note that \( u(t) \) is nondecreasing as \( \ell \geq 1 \). Combining (10) with (8) one gets

\[ L_{n-1} u(\tau(t)) \geq \int_t^\infty p(s) h(u[g(s)]) \, ds, \quad t \geq t_2. \]

Since the composite function \( h(u[g(t)]) \) is nondecreasing, according to Lemma 2 we obtain

\[ L_{n-1} u(\tau(t)) \geq \int_t^\infty q(s) h(u[g(s)]) \, ds, \quad t \geq t_2. \]

We multiply (11) by \( r_{n-1}(\tau(t)) \tau'(t) \) and integrate the resulting inequality over \([t, \infty).\)

Repeating this procedure, we arrive at

\[ L_{\ell} u(\tau(t)) \geq \int_t^\infty r_{\ell+1}[\tau(s_{\ell+1})] \tau'(s_{\ell+1}) \]

\[ \times \int_{s_{\ell+1}}^\infty \cdots \int_{s_{n-1}}^\infty q(s_n) h(u[g(s_n)]) \, ds_n \cdots ds_{\ell+1}. \]
We multiply (12) by \( r_\ell(\tau(t))\tau'(t) \) and integrate over \([t_2, t]\). Continuing in this manner we obtain

\[
\begin{align*}
\tau(t) & \\
& \geq c + \int_{t_2}^{t} r_1(\tau(s_1))\tau'(s_1) \int_{t_2}^{s_1} \cdots \int_{t_2}^{s_{\ell-1}} r_\ell(\tau(s_\ell))\tau'(s_\ell) \\
& \times \int_{s_\ell}^{\infty} \cdots \int_{s_{n-1}}^{\infty} q(s_n)h(u[g(s_n)]) \, ds_n \cdots ds_1,
\end{align*}
\]

(13)

where \( c = \tau(t_2) > 0 \). Denote the right hand side of (13) by \( z(t) \). By repeated differentiation of \( z(t) \) one can verify that \( z(t) \) is a function of degree \( \ell \) and, on the other hand,

\[
\alpha_n z(t) + q(t)h(u[g(t)]) = 0.
\]

(14)

As \( \tau(t) \geq z(t) \) and \( \tau(w(t)) = g(t) \) we see that

\[
u(g(t)) = \tau(w(t)) \geq z(w(t))
\]

for all large \( t \), say \( t \geq t_3 \). Combining this fact with (14) we see that \( z(t) \) is a solution of the differential inequality

\[
\{\alpha_n z(t) + q(t)h(z[w(t)])\} \, \text{sgn} \, z[w(t)] \leq 0, \quad t \geq t_3.
\]

Then by Kusano and Naito (see [7]) equation (7) has also an eventually positive solution \( x(t) \) satisfying

\[
\lim_{t \to \infty} x(t) \geq c > 0,
\]

which contradicts the hypotheses.

Now, let \( \ell = 0 \) (note that this is possible only when \( n \) is odd). To obtain a contradiction assume that \( c_0 = \lim_{t \to \infty} u(t) > 0 \). Integrating (1), in view of (8) we have

\[
L_0 u(\tau(t)) \geq c_0 + \int_{t}^{\infty} r_1(\tau(s_1))\tau'(s_1) \\
\times \int_{s_1}^{\infty} \cdots \int_{s_{n-1}}^{\infty} p(s_n)h(u[g(s_n)]) \, ds_n \cdots ds_1
\]

(15)

for all large \( t \). Since \( u(t) \) is decreasing (\( \ell = 0 \)) one gets

\[
\frac{3}{2} c_0 \geq u(t) \geq c_0, \quad t \geq t_2,
\]
where \( t_2 \) is large enough. This fact together with (15) and (9) implies

\[
\sigma_0 \geq \frac{\sigma_0}{2} + \int_t^\infty r_1[\tau(s_1)]\tau'(s_1) \times \int_{s_1}^\infty \cdots \int_{s_{n-1}}^\infty q(s_n)h(c_0)ds_n \cdots ds_1
\]

for all large \( t \), say \( t \geq t_3 \). Again, let us denote the right hand side of (16) by \( z(t) \). Then

\[
(17) \quad \alpha_n z(t) + q(t) h(c_0) = 0, \quad t \geq t_3.
\]

Noting that \( z(w(t)) \leq c_0 \), (17) yields

\[
\{\alpha_n z(t) + q(t) h(z[w(t)])\} \text{sgn} z[w(t)] \leq 0.
\]

Again according to Kusano and Naito [7] equation (7) has also an eventually positive solution \( x(t) \) with the property

\[
x(t) \geq \frac{c_0}{2} > 0.
\]

This contradicts the assumptions and the proof is complete. \( \square \)

Example 1. Consider an even order delay equation

\[
(18) \quad \left( \frac{1}{r(t)} y^{(m)} \right)^{(m)} + p(t) f \left( y[g(t)] \right) = 0, \quad m \geq 1,
\]

where functions \( r, p, f \) and \( g \) are the same as in Theorem 1 and \( g(t) \leq t \). Then by Theorem 1 equation (18) is oscillatory if the ordinary equation without delay

\[
\left( \frac{1}{r(g(t))g'(t)} y^{(m)} \right)^{(m)} + p(t) f (y(t)) = 0, \quad m \geq 1
\]

is oscillatory.

Corollary 1. Let (3) hold. Let \( r_i(t); 1 \leq i \leq n - 1 \) be nonincreasing. Assume that \( f \) is nondecreasing. Suppose that \( g \in C^1 \) and \( g'(t) > 0 \). Then for any constant \( M \) equation (1) has property (A) if and only if so does the equation

\[
(19) \quad L_n u(t) + p(t) f \left( u[g(t) + M] \right) = 0.
\]
Proof. Assume that $M > 0$. The "only if" part follows from Theorem 1 of Kusano and Naito [7]. Now suppose that equation (19) has property (A). We put $w(t) = g(t) + M$. Then

$$\tau(t) = g(w^{-1}(t)) = t - M \leq t.$$ 

By Theorem 1 of Kusano and Naito [7] the equation

$$(20) \quad \alpha_n u(t) + p(t)f(u[g(t) + M]) = 0$$

has property (A) as $r_i(\tau(t))\tau'(t) \geq r_i(t)$. The "if" part now follows from Theorem 1 applied to equations (20) and (1).

Now let $M < 0$. We put $\bar{g}(t) = g(t) + M$. Since $-M > 0$, by the first part of the proof of this theorem we see that the equation

$$L_n u(t) + p(t)f(u[\bar{g}(t)]) = 0$$

has property (A) if and only if so does the equation

$$L_n u(t) + p(t)f(u[\bar{g}(t) + (-M)]) = 0,$$

which we wanted to verify. \qed

Example 2. Consider an $n$-order linear delay differential equation

$$(21) \quad \left( \frac{1}{r(t)} \ldots \left( \frac{1}{r(t)}u'(t) \right)' \ldots \right)' + p(t)u(g(t)) = 0,$$

where $r$, $p$ and $g$ are subject to the same conditions as in (1). Let us denote $b(t) = r(g(t))g'(t)$ and put $w(t) = t$. By Theorem 1 equation (21) has property (A) if the differential equation without delay

$$\left( \frac{1}{b(t)} \ldots \left( \frac{1}{b(t)}u'(t) \right)' \ldots \right)' + p(t)u(t) = 0$$

has property (A), which by Theorem 5 in [2] occurs if

$$(22) \quad \liminf_{t \to \infty} \left( \int_{t_0}^t b(s) \, ds \right)^{n-1} \left( \int_t^\infty p(s) \, ds \right) > \frac{M_1}{n - 1},$$

where $M_1$ is the maximum of all local maxima of the polynomial

$$P_n(k) = -k(k - 1) \ldots (k - n + 1).$$
If we set \( g(s) = x \) then (22) reduces to

\[
\lim_{t \to \infty} \inf \left( \int_{t_0}^{g(t)} r(x) \, dx \right)^{n-1} \left( \int_t^\infty p(s) \, ds \right) > \frac{M_1}{n-1},
\]

which is a weaker sufficient condition for equation (21) to have property (A) than Kusano and Naito have required in [7].

For \( n \) even, as we see from Definition 1, property (A) of equation (1) reduces to oscillation of (1). In [8] Kusano and Naito discussed the oscillatory character of a special case of (1), namely the even order linear differential equation

\[
L_n u(t) + p(t)u(t) = 0,
\]

by comparing (23) with a set of second order differential equations (see Theorem B in [8]). We adapt their method together with Theorem 1 and make use of the results obtained to extend the results of Kusano and Naito [8] and Trench [11].

Let \( 1 \leq i \leq n - 1 \) and \( t, s \in [t_0, \infty) \). We define

\[
I_0 = 1
\]

\[
I_i(t, s; r, \ldots, r_i) = \int_s^t r_i(x) I_{i-1}(x, s; r_{i-1}, \ldots, r_1) \, dx.
\]

Let us denote \( b_i(t) = r_i(g(t))g'(t) \) for \( i = 1, 2, \ldots, n - 1 \). For simplicity of notation we put

\[
J_i(t, s) = I_i(t, s; b_1, \ldots, b_i),
\]

\[
K_i(t, s) = I_i(t, s; b_{n-1}, \ldots, b_{n-i}).
\]

**Theorem 2.** Suppose that \( n \geq 4 \) is even. Assume that all the conditions of Theorem 1 are satisfied with \( h(x) = x \) and \( w(t) = t \). Define for \( i = 1, 3, \ldots, n - 3 \)

\[
a_i(t) = b_{i+1}(t) \int_t^\infty J_{i-1}(s, t) K_{n-i-2}(s, t) q(s) \, ds,
\]

\[
a_{n-1}(t) = b_{n-2}(t) \int_t^\infty J_{n-3}(s, t) q(s) \, ds.
\]

Then equation (1) is oscillatory if the second order equations

\[
\left( \frac{1}{b_i(t)} y'(t) \right)' + a_i(t)y(t) = 0, \quad i = 1, 3, \ldots, n - 1
\]

are oscillatory.

**Proof.** Applying Theorem B in [8] we conclude that equation (7) is oscillatory, and hence (1) is oscillatory by Theorem 1. \( \square \)
Corollary 2. Let all the conditions of Theorem 1 hold with \( h(x) = x \) and \( w(t) = t \).

If

\[
\liminf_{t \to \infty} \left( \int_{t_0}^{g(t)} r_i(s) \, ds \right) \left( \int_{t}^{\infty} a_i(s) \, ds \right) > \frac{1}{4}, \quad i = 1, 3, \ldots, n - 1, \tag{27}
\]

where \( a_i(t), \ i = 1, 3, \ldots, n - 1 \) are defined as in (24) and (25), then equation (1) is oscillatory.

Proof. We make a change of variables in the first integral in (27) by using the substitution \( s = g(x) \) and obtain

\[
\liminf_{t \to \infty} \left( \int_{t_0}^{t} b_i(x) \, dx \right) \left( \int_{t}^{\infty} a_i(s) \, ds \right) > \frac{1}{4}, \quad i = 1, 3, \ldots, n - 1. \tag{28}
\]

It is known (see [5]) that (28) is sufficient for all solutions of (26) to be oscillatory. Hence, Corollary 2 follows from Theorem 2. \( \square \)

References

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