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Existence of multiple solutions for a third-order three-point regular boundary value problem


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EXISTENCE OF MULTIPLE SOLUTIONS FOR A THIRD-ORDER THREE-POINT REGULAR BOUNDARY VALUE PROBLEM

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Summary. In the paper we prove an Ambrosetti-Prodi type result for solutions \( u \) of the third-order nonlinear differential equation, satisfying \( u'(0) = u'(1) = u(\eta) = 0, \ 0 \leq \eta \leq 1 \).

Keywords: Boundary value problem, lower and upper solutions, coincidence degree, Nagumo functions, Ambrosetti-Prodi results

AMS classification: 34B15

1. INTRODUCTION

In a recent paper, Fabry, Mawhin and Nkashama [3] have considered periodic problems of the form

\[
\begin{align*}
u'' + f(x,u) &= s, \\
u(0) - u(2\pi) &= u'(0) - u'(2\pi) = 0
\end{align*}
\]

and have proved that if

\[ f(x,u) \to \infty \ \text{as} \ |u| \to \infty \]

uniformly in \( x \in [0,2\pi] \), an Ambrosetti-Prodi type result [1] holds, namely, there exists \( s_1 \) such that the above problem has no solution if \( s < s_1 \), at least one solution if \( s = s_1 \), and at least two solutions if \( s > s_1 \). A similar result holds for

\[
\begin{align*}
u' + f(x,u) &= s, \\
u(0) &= u(2\pi)
\end{align*}
\]

(see [5]) and the corresponding proofs rely on a combination of the techniques of lower and upper solutions and the degree theory.
In [2] a somewhat weakened Ambrosetti-Prodi-like [1] result is given only for the following special case of a higher order boundary value problem (BVP):

\[ u^{(n)} + g(u) = s + e(x, u), \]
\[ u(0) - u(2\pi) = \ldots = u^{(n-1)}(0) - u^{(n-1)}(2\pi) = 0. \]

In this paper we prove an Ambrosetti-Prodi-like result [1] for the third-order BVP

(1) \[ u''' + f(t, u, u', u'') = s, \]
(2) \[ u'(0) = u'(1) = u(\eta) = 0, \quad 0 \leq \eta \leq 1. \]

This problem models the static deflection of a three-layered elastic beam.

The proofs in this chapter are based on a combination of the techniques of lower and upper solutions and the degree theory.

2. NOTATIONS AND DEFINITIONS

\[ \|x\| = \max \{|x(t)|, \quad t \in [0, 1]\}. \]

Functions \( \sigma_1 \) and \( \sigma_2 \in C^3(0, 1) \) satisfying

\[ \sigma_1''' \geq s - f(t, x, \sigma_1'(t), \sigma_1''(t)), \]
\[ \sigma_2''' \leq s - f(t, x, \sigma_2'(t), \sigma_2''(t)) \]

for \( t \in [0, 1], x \in [\min\{\sigma_1(t), \sigma_2(t)\}, \max\{\sigma_1(t), \sigma_2(t)\}] \) and

\[ \sigma_1(\eta) = \sigma_2(\eta) = 0, \]
\[ \sigma_1'(0) \leq 0, \quad \sigma_1'(1) \leq 0, \]
\[ \sigma_2'(0) \geq 0, \quad \sigma_2'(1) \geq 0, \]

will be called a lower and an upper solution of the BVP (1), (2), respectively.

By replacing the above inequalities with strict inequalities we obtain the definition of a strict lower and a strict upper solution of the BVP (1), (2).

The BVP (1), (2) is equivalent to

\[ Lu + N_s u = 0, \]
where

$$L: \text{dom} \ L \to C^0(0,1), \quad Lu = u''', \quad X = \{x \in C^2(0,1), \ x \text{ satisfies (2)}\}, \ \text{dom} \ L = C^3(0,1) \cap X, \quad N_s: X \to C^0(0,1), \quad N_s u = f(t, u, u', u'') - s, \quad s \in \mathbb{R}.$$  

It can be easily proved (see [4]) that $L + N_s$ is $L$-compact on $\overline{\Omega}$ (with $\overline{\Omega}$ the closure of $\Omega$), where $\Omega$ is an open bounded subset of $X$.

3. LEMMAS AND THEOREMS

**Lemma 1.** (On a priori estimates) Let $u$ be a solution of $(1)_s$, $(2)$ and let $\|u'\| \leq R, \ R \in \mathbb{R}, \ R > 0$. Assume that for every $R \in \mathbb{R}, \ R > 0$ there exists a continuous function $h_R: \mathbb{R}^+ \to [a_R, \infty)$ ($a_R > 0$) such that

$$|f(t, x, y, z)| \leq h_R(|z|)$$

for $x, y \in [-R, R], \ t \in [0, 1], \ z \in \mathbb{R}$, where

$$\int_0^\infty \frac{t \, dt}{h_R(t)} = \infty.$$  

Then there exists $r^*$ (depending only on $s, R, h_R$) such that

$$\|u''\| \leq r^*.$$  

**Proof.** Let $u$ be a solution of $(1)_s$, $(2)$ and $\|u'\| \leq R$. We define

$$\Omega(x) = \int_x^z \frac{t \, dt}{h_R(|t|) + |s|}.$$  

From (4) it follows that $\Omega$ is a bijective mapping of $\mathbb{R}^+$ onto itself. From (2) it follows that there exists $a_0 \in (0, 1)$ such that $u''(a_0) = 0$. Let $r^* = \Omega^{-1}(\Omega(1) + 2R)$ and assume that $|u''(t_1)| > r^*$, where $t_1 \in (a_0, 1]$. Let $[a_1, b_1] \subset [a_0, 1]$ be the maximal interval containing $t_1$ in which $|u''(t)| \geq 1$ and let $s_1 \in (a_1, b_1]$ be such that

$$|u''(s_1)| = \varepsilon_1 = \max \{|u''(t)|: a_1 \leq t \leq b_1\}.$$  

From (3) and $(1)_s$ it follows that

$$|u'''| = |s - f(t, u, u', u'')| \leq h_R(|u''|) + |s|.$$  

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If \( u''(t) \geq 1 \), then
\[
\int_{a_1}^{t} u''u'' \leq \int_{a_1}^{t} u'' \, dt.
\]
The last inequality implies that \( \Omega(\varphi_1) - \Omega(1) \leq 2R \) and \( \varphi_1 \leq r^* \) which contradicts (5). We can obtain a similar contradiction if \( u''(t) \leq -1 \) on \([a_1, s_1] \). For \( t_1 \in [0, a_0] \) the proof is analogous. Lemma 1 is proved.

**Theorem 2.** Let \( \sigma_1 \) be a lower solution and \( \sigma_2 \) an upper solution of the BVP (1), (2) and let \( \sigma_1' (t) \leq \sigma_2' (t) \) for every \( t \in [0, 1] \). If the function \( f \) satisfies (3), then the BVP (1), (2) has a solution \( u \) such that
\[
\sigma_1' (t) \leq u'(t) \leq \sigma_2' (t) \quad \text{for each } t \in [0, 1].
\]

**Proof.** The theorem follows from Lemma 1 (On a priori estimates) and from the results given in [6]. \( \square \)

**Remark.** [6] deals with the BVP
\[
u''' = f(t, u, u', u''), \quad (2).
\]
The existence of a solution \( u \) satisfying
\[
\sigma_1' (t) \leq u'(t) \leq \sigma_2' (t),
\]
where \( \sigma_1, \sigma_2 \) is a lower and an upper solution, respectively, is proved under a more general growth condition than (3).

**Theorem 3.** Let \( f \) be nonincreasing (or nondecreasing) for \( t \in [0, \eta] \) (for \( t \in [\eta, 1] \)) as a function of \( x \) for every fixed \( y, z \in \mathbb{R} \). Further suppose there exist \( R_1, s_1 \in \mathbb{R}, R_1 > 0 \) such that
\[
f (t, R_1 (t - \eta), 0, 0) < s_1 \quad \text{for } t \in [0, 1],
\]
and for any \( r_1 \geq R_1 \) the inequality
\[
s_1 < f (t, -r_1 (t - \eta), y, 0) \quad \text{for } t \in [0, 1], \ y \leq -r_1,
\]
is valid. If the function \( f \) satisfies (3), then there exists \( s_0 < s_1 \) (with the possibility that \( s_0 = -\infty \)) such that for \( s < s_0 \) the BVP (1), (2) has no solution and for \( s \in (s_0, s_1] \) the BVP (1), (2) has at least one solution.
Proof. Let \( s^* = \max \{ f(t,0,0,0); t \in [0,1] \} \). From (7) and (8) it follows that \( s^* - f(t,x,0,0) \geq 0 \) and \( s^* - f(t,x,-R_1,0) \leq 0 \) for \( t \in [0,1] \), \( x \in \left[ \min\{0,-R_1(t-\eta)\}, \max\{0,-R_1(t-\eta)\} \right] \). From the last two inequalities we get that \( \sigma_1 = -R_1(t-\eta) \) is a lower solution of \((1)_{s^*}, (2)\) and \( \sigma_2 = 0 \) is an upper solution of the BVP \((1)_{s^*}, (2)\), so Theorem 2 implies that the BVP \((1)_{s^*}, (2)\) has a solution.

Next we show that if the BVP \((1)_{s}, (2)\) has a solution \( u \) for \( s = s < s_1 \) then it also has a solution for \( s \in [s,s_1] \). If \( s \in [s,s_1] \) then \( u''' = s - f(t,u,u',u'') \) and \( u''' \leq s - f(t,x,u',u'') \) for \( t \in [0,\eta], x \geq u \) or for \( t \in [\eta,1], x \leq u \). It is easily seen that for \( s \leq s_1 \) all solutions of \((1)_{s}, (2)\) satisfy the relation \(-R_1 \leq u'\). If \( u'(t_0) \leq -R_1 \) for some \( t_0 \in (0,1) \), then there exists \( t_1 \in (0,1) \) such that \( \min\{u'(t), t \in (0,1)\} = u'(t_1), u''(t_1) = 0, u'''(t_1) \geq 0 \). If \( t_1 \in [\eta,1] \) then \( u'(t_1) = -r_1 \leq -R_1, u'(t) \geq -r_1 \) for \( t \in [\eta,1] \) and \( u(t_1) \geq -r_1(t_1 - \eta) \). From (8) it follows that \( s_1 < f(t_1,u(t_1),-r_1,0), u'''(t_1) < 0 \) and this contradicts our assumption. A similar contradiction can be obtained for \( t_1 \in (0,\eta) \).

(8) implies that \( s - f(t,x,-R_1,0) \leq 0 \) for \( t \in [0,1] \), \( x \in \left[ \min\{u(t),-R_1(t-\eta)\}, \max\{u(t),-R_1(t-\eta)\} \right] \). Setting \( \sigma_1 = -R_1(t-\eta), \sigma_2 = u \) and using Theorem 2 we can see that the BVP \((1)_{s}, (2)\) has a solution.

Taking \( s_0 = \inf \{ s \in \mathbb{R}: (1)_{s}, (2) \text{ has a solution} \} \) with \( s_0 = -\infty \) if the BVP \((1)_{s}, (2)\) has a solution for any \( s \leq s_1 \), it follows from the above discussion that \( s_0 \leq s^* < s_1 \) and that \((1)_{s}, (2)\) has a solution for any \( s \in (s_0,s_1] \). Theorem 3 is proved. 

Lemma 4. Let \( \Omega = \{ x \in \text{dom } L: \sigma_1'(t) < x'(t) < \sigma_2'(t), \|x''\| < k \} \), where \( \sigma_1 < \sigma_2, \sigma_1 \) is a strict lower solution and \( \sigma_2 \) is a strict upper solution of \((1)_{s}, (2)\). If \( f \) satisfies (3) then there exists \( k \in \mathbb{R} \) such that the coincidence degree of \( L + N_s \) in \( \Omega \) relative to \( L \) (see [4]) satisfies

\[ d_L(L + N_s, \Omega) = \pm 1 \pmod{2}. \]

Proof. We define

\[
g(t,x,y,z) = f(t,\alpha(t,x),\beta(t,y),z) - y + \beta(t,y),
\]

\[
\alpha(t,x) = \begin{cases} 
\min\{\sigma_1(t),\sigma_2(t)\} & \text{for } x < \min\{\sigma_1(t),\sigma_2(t)\}, \\
x & \text{for } \min\{\sigma_1(t),\sigma_2(t)\} \leq x \leq \max\{\sigma_1(t),\sigma_2(t)\}, \\
\max\{\sigma_1(t),\sigma_2(t)\} & \text{for } x > \max\{\sigma_1(t),\sigma_2(t)\}, 
\end{cases}
\]

\[
\beta(t,y) = \begin{cases} 
\sigma_1'(t) & \text{for } y' < \sigma_1'(t), \\
y & \text{for } \sigma_1'(t) \leq y \leq \sigma_2'(t), \\
\sigma_2'(t) & \text{for } y' > \sigma_2'(t). 
\end{cases}
\]
The BVP

(9) \quad u'' + g(t,u,u',u'') = s, \quad (2)

can be written in the form of an operator equation

\[ Lu + G_s u = 0 \quad \text{in} \quad \text{dom} L, \]

where \( G_s : X \rightarrow C^0(0,1), G_s u = g(t,u,u',u'') - s. \)

In \( \Omega \) the BVP (1), (2) is equivalent to the BVP (9), (2), the operator equation \( Lu + N_s u = 0 \) is equivalent to the operator equation \( Lu + G_s u = 0 \) and

\[ d_L(L + G_s, \Omega) = d_L(L + N_s, \Omega). \]

We define \( \Omega_1 = \{ x \in \text{dom} L : \|x'\| < r^*, \|x''\| < k \} \), where \( r^* > \max\{\|\sigma_1\|, \|\sigma_2\|\}. \)

We shall prove that for \( \lambda \in [0,1] \) every solution of the equation

(10) \quad \[ Lu - (1 - \lambda)Iu + \lambda G_s u = 0, \]

where \( Iu = u' \), satisfies \( u \notin \Omega_1 \). If \( \|u''\| > r^* \), then there exists \( t_0 \in (0,1) \) such that

\[
\begin{align*}
  u'(t_0) &> r^* \quad (\text{or} \quad u'(t_0) \leq -r^*), \\
  u''(t_0) &= 0, \\
  u'''(t_0) &\leq 0 \quad (u'''(t_0) \geq 0).
\end{align*}
\]

If \( r^* \) is large enough, then

\[
\begin{align*}
  f(t, \alpha(t,x), \sigma_1', 0) - s + r^* + \sigma_1' > 0 \quad \text{and} \\
  f(t, \alpha(t,x), \sigma_2', 0) - s - r^* + \sigma_2' < 0 \quad \text{for} \ x \in \mathbb{R}, \ t \in [0,1].
\end{align*}
\]

For \( u'(t_0) \leq -r^* \) we obtain

\[
u'''(t_0) - (1 - \lambda)u'(t_0) + \lambda \left( f(t_0, \alpha(t_0, u(t_0), \sigma_1'(t_0), 0) - s - u'(t_0) + \sigma_1'(t_0) \right) = 0.
\]

It follows from the last equality that \( u'''(t_0) < 0 \) which contradicts \( u'''(t_0) \geq 0. \) A similar contradiction can be obtained if we suppose that \( u'(t_0) \geq r^*. \) We have proved that \( \|u''\| < r^* \). Since (3) is valid we get the inequality

\[
- (1 - \lambda)y - \lambda \left( f(t, \alpha(t,x), \beta(t,y), z) - s - y + \beta(t,y) \right) |z| \leq h_R(|z|) + 2r^* + |s|
\]

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for $y < r^*$, and

$$\int_0^\infty \frac{s \, ds}{h_R(s) + 2r^* + |s|} \geq \frac{1}{1 + \frac{2r^* + |s|}{a_R}} \int_0^\infty \frac{s \, ds}{h_R(s)} = \infty.$$ 

The last inequality implies that we can use Lemma 1 and for $k$ large enough also $\|u''\| < k$ is satisfied.

For $A = 0$ the equation (10) has only the trivial solution and $d_L(L - I, \Omega_1) = \pm 1$ (mod 2). By the property of invariance under a homotopy we obtain $d_L(L + G_s, \Omega_1) = \pm 1$ (mod 2). Next we prove that every solution $u$ of the equation $Lu + G_s u = 0$ satisfies $u \in \Omega \subset \Omega_1$. If $u'(t_1) > \sigma_2'(t_1)$ for some $t_1 \in (0, 1)$ then there exists an interval $(a, b) \subset (0, 1)$, $t_1 \in (a, b)$, $u'(t) > \sigma_2'(t)$ for $t \in (a, b)$ and $u'(a) = \sigma_2'(a)$, $u'(b) = \sigma_2'(b)$. This implies that there exists $t_2 \in (a, b)$ such that

$$u'(t_2) > \sigma_2'(t_2),$$

$$u''(t_2) = \sigma_2''(t_2),$$

$$u'''(t_2) \leq \sigma_2'''(t_2).$$

Since $u$ is a solution of (9) and $\sigma_2$ is a strict upper solution of (1), (2), it follows that

$$u'''(t_2) + f(t, \alpha(t_2, u(t_2), \sigma_2'(t_2), \sigma_2''(t_2))) - s - u'(t_2) + \sigma_2'(t_2) = 0,$$

$$u'''(t_2) > \sigma_2'''(t_2).$$

This contradicts the inequality $u'''(t_2) \leq \sigma_2'''(t_2)$. If $u'(t) \leq \sigma_2'(t)$ for $t \in (0, 1)$ and there exists $t_3 \in (0, 1)$ such that $u'(t_3) = \sigma_2'(t_3)$ then $u''(t_3) = \sigma_2''(t_3)$ and $u'''(t_3) \leq \sigma_2'''(t_3)$. This implies that

$$u'''(t_3) + f(t_3, \alpha(t_3, u(t_3), \sigma_2'(t_3), \sigma_2''(t_3))) - s = 0$$

and since $\sigma_2$ is a strict upper solution of (9) we obtain $u'''(t_3) > \sigma_2'''(t_3)$. This contradicts $u'''(t_3) \leq \sigma_2'''(t_3)$.

It is possible to prove in a similar way that $u'(t) > \sigma_1'(t)$ for every possible solution $u$ of the equation $Lu + G_s u = 0$ and for every $t \in [0, 1]$.

By using the excision property of the degree we obtain

$$d_L(L + G_s, \Omega) = \pm 1 \quad \text{(mod 2)}$$

and, finally,

$$d_L(L + N_s, \Omega) = \pm 1 \quad \text{(mod 2)}.$$ 

Lemma 4 is proved. \qed
Theorem 5. Let us suppose that the assumptions of Theorem 3 are fulfilled. Moreover, suppose that there exists $M(s_1) \in \mathbb{R}$ such that for $s \leq s_1$ any solution of the BVP $(1)_s$, $(2)$ satisfies the inequality

\begin{equation}
    u'(t) \leq M(s_1) \quad \text{for } t \in [0, 1]
\end{equation}

and that there exists $\alpha \in \mathbb{R}$ such that

\begin{equation}
    f(t, x, y, z) > \alpha
\end{equation}

for $t \in [0, 1]$, $x \in \left[ \min\{-R_1(t-\eta), M(s_1)(t-\eta)\}, \max\{-R_1(t-\eta), M(s_1)(t-\eta)\} \right]$, $y \in [-R_1, M(s_1)]$, $z \in \mathbb{R}$. Then the number $s_0$ provided by Theorem 3 is finite and for $s < s_0$ the BVP $(1)_s$, $(2)$ has no solution, for $s = s_0$ the BVP $(1)_s$, $(2)$ has at least one solution, for $s \in (s_0, s_1)$ the BVP $(1)_s$, $(2)$ has at least two solutions.

Proof. First we prove that $s_0$ is finite. Let $u$ be a solution of $(1)_s$, $(2)$. From $(1)_s$ it follows that $u'''' \leq s - \alpha$. From $(2)$ it follows that

\begin{align*}
    u''(t) &\geq \frac{1}{4}(\alpha - s) \quad \text{for } t \in [0, \frac{1}{4}] \quad \text{or} \\
    u''(t) &\leq \frac{1}{4}(s - \alpha) \quad \text{for } t \in [\frac{3}{4}, 1].
\end{align*}

If we take $s$ such that $\frac{\alpha - s}{16} > M(s_1)$ we obtain a contradiction to $(10)$.

Let $s \in (s_0, s_1)$ and let $u$ be a solution of the BVP $(1)_s$, $(2)$ for $s = s$. We can assume that $R_1 \leq |M(s_1)|$.

Let $\Omega_1 = \{x \in X : \|x(t)\| < |M(s_1)|, \|x'(t)\| < |M(s_1)|, \|x''(t)\| < \varrho\}$, where $\varrho$ is taken sufficiently large. Since the BVP $(1)_s$, $(2)$ has no solution for $s-1 < s_0$, it is a consequence of the basic properties of the degree that

\begin{equation}
    d_L(L + N_{s-1}, \Omega_1) = 0.
\end{equation}

On the other hand, for $s \leq s_1$ all solutions of $(1)_s$, $(2)$ satisfy the inequality $\|u'\| < |M(s_1)|$. If $\varrho$ is large enough and $s \in [s-1, s_1]$ then we have $\|u''\| < \varrho$ for all solutions of $(1)_s$, $(2)$ (the bound given by Lemma 1 can be taken independent of $s$ for $s \in [s-1, s_1]$). From the properties of the degree and from $(13)$ it follows that $d_L(L + N_s, \Omega_1) = 0$ for $s \in [s-1, s_1] \supset (s_0, s_1]$.

Let $\Omega_\varepsilon = \{x \in X : \|x(t)\| < |M(s_1)|, -|M(s_1)| < x'(t) < u'(t) + \varepsilon \text{ for } t \in [0, 1], \|x''(t)\| < \varrho\}$, where $u(t)$ is a solution of $(1)_s$, $(2)$ for $s = s \in (s_0, s_1)$ and $u(t) = u(t) + \varepsilon(t - \eta)$. For $s \in (s, s_1]$ it is possible (because $f$ is continuous) to
take $\varepsilon$ such that $||u'|| < |M(s_1)|$ and $u(t)$ is a strict upper solution of (1)$_s$, (2). $-|M(s_1)|(t - \eta)$ is a strict lower solution of (1)$_s$, (2). According to Lemma 5 for $s \in (s, s_1]$ we have

$$d_L(L + N_s, \Omega_\varepsilon) = \pm 1 \pmod{2}.$$  

From the additivity property of the degree it follows that

$$d_L(L + N_s, \Omega_1 - \Omega_\varepsilon) = \pm 1 \pmod{2}$$  

for $s \in (s, s_1]$. Relations (14), (15) imply the existence of a solution of the BVP (1)$_s$, (2) in $\Omega_\varepsilon$ and in $\Omega_1 - \Omega_\varepsilon$. Since $s$ is arbitrary in $(s_0, s_1)$, the BVP (1)$_s$, (2) has at least two solutions for $s \in (s_0, s_1]$.

Now we prove that (1)$_s$, (2) has a solution for $s = s_0$. Let us take a sequence $\{s_n\}_{n=1}^\infty$, where $s_n \in (s_0, s_1]$, $n \in N$, $\lim_{n \to \infty} s_n = s_0$. We know that for any $s_n$ (1)$_{s_n}$, (2) has a solution $u_n$ satisfying $||u_n|| < |M(s_1)|$, $||u'_n|| < |M(s_1)|$, and according to Lemma 1 we get $||u''_n|| < \varrho$ for $\varrho$ large enough. Since $u_n$ is a solution of (1)$_{s_n}$, (2) the sequence $\{u''_n\}_{n=1}^\infty$ is bounded in $C^0(0, 1)$. By the Arzela-Ascoli lemma we can suppose that $\{u_n\}_{n=1}^\infty$ converges in $C^2(0, 1)$ to a solution of (1)$_s$, (2). Theorem 5 is proved.

References


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