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ON TORSION OF A 3-WEB

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(Received February 10, 1994)

Summary. A 3-web on a smooth 2n-dimensional manifold can be regarded locally as a triple of integrable n-distributions which are pairwise complementary, [5]; that is, we can work on the tangent bundle only. This approach enables us to describe a 3-web and its properties by invariant (1,1)-tensor fields $P$ and $B$ where $P$ is a projector and $B^2 = \text{id}$. The canonical Chern connection of a web-manifold can be introduced using this tensor fields, [1]. Our aim is to express the torsion tensor $T$ of the Chern connection through the Nijenhuis (1,2)-tensor field $[P, B]$, and to verify that $[P, B] = 0$ is a necessary and sufficient conditions for vanishing of the torsion $T$.

Keywords: distribution, projector, manifold, connection, web

AMS classification: 53C05

All objects under considerations will be supposed to be of the class $C^\infty$ (smooth).

1. An (ordered) three-web on a manifold $M$ can be defined as an ordered triple $W = (D_1, D_2, D_3)$ of integrable distributions of dimension $n$ such that the tangent bundle is a Whitney sum of each couple of them, $TM = D_1 \oplus D_2 = D_2 \oplus D_3 = D_1 \oplus D_3$. Obviously, the web manifold has an even dimension $2n$.

It was proved in [1], [5] that an ordered 3-web on a smooth 2n-dimensional manifold $M_{2n}$ can be introduced as a couple $(P, B)$ of differentiable (1,1)-tensor fields on $M$ satisfying on $M$ the polynomial equations

\[
\begin{align*}
P^2 - P &= 0, \\
B^2 - I &= 0,
\end{align*}
\]

the identity $B = BP + PB$, and the integrability conditions

\[
\begin{align*}
[P, P] &= 0, \\
[B, B](X, Y) &= 0 \quad \text{for} \ X, Y \in \ker(B - I)
\end{align*}
\]
by which the integrability of all the three web distributions is guaranteed. From this viewpoint, a 3-web is an integrable \{P, B\}-structure introduced in [1].

Let us denote
\[ D_1 = \ker(I - P) = \text{im } P, \quad D_2 = \ker P = \text{im } (I - P), \quad D_3 = \ker(B - I). \]

Then \((D_1, D_2, D_3)\) satisfies the above definition of a 3-web, and three foliations of integral submanifolds of our distributions form a 3-web in the classical approach.

Let us denote by \(\tilde{P} = I - P\) the complementary projector. The following equalities are obvious:

\[ (3) \quad \tilde{P} \tilde{P} = \tilde{P} P = 0, \quad \tilde{P} B P = \tilde{P} B \tilde{P} = 0, \quad \tilde{P} B = B \tilde{P}, \quad B \tilde{P} = \tilde{P} B. \]

In [5], all linear connections \(\mathcal{V}\) were found with respect to which the web distributions \(D_1, D_2, D_3\) are parallel. This property is expressed by the condition saying that both \(P\) and \(B\) are covariantly constant:

\[ (4) \quad \tilde{\mathcal{V}} P = 0, \quad \tilde{\mathcal{V}} B = 0. \]

All such connections form a \(2n^3\)-parameter family, [5]. Among these distributions preserving connections, there exists a unique connection \(\mathcal{V}\) the torsion tensor of which satisfies

\[ (5) \quad T(PX, \tilde{P}Y) = 0, \]

that is, homogeneous vectors \(X \in D_{1x}\) and \(Y \in D_{2x}\) are conjugated with respect to \(T; x \in M\). The covariant derivative of this connection [1] is expressed by tensor fields \(P, B, \tilde{P}\) defining the web as follows:

\[ (6) \quad \nabla_X Y = PB[PX, BPY] + \tilde{P} B[\tilde{P}X, B\tilde{P}Y] + P[\tilde{P}X, PY] + \tilde{P}[PX, \tilde{P}Y]. \]

Its torsion tensor, \(T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]\), is given by the formula

\[ (7) \quad T(X, Y) = PB([PX, BPY] + [BPX, PY]) + \tilde{P} B([\tilde{P}X, B\tilde{P}Y] + [\tilde{P}X, \tilde{P}Y] + [PX, \tilde{P}Y] - [X, Y]). \]

Using the above notation, let us recall the proof that the formula (6) defines a covariant derivation with the properties (4), (5), and that any connection \(\tilde{\mathcal{V}}\) satisfying (4), (5) coincides with \(\mathcal{V}\) described in (6).
Let $\nabla$ be defined by (6). The additivity in both arguments follows by the additivity of tensor fields and Lie brackets occurring in the formula. We use the identities (1), (3) and

$$[fX, gY] = fg[X, Y] - Yf \cdot X + Xg \cdot Y$$

to obtain

$$\nabla_X fY = PB(f[PX, BPY] + (PXf) \cdot BPY) + \hat{P}B(f[\hat{P}X, B\hat{P}Y] + (\hat{P}Xf) \cdot \hat{B}\hat{P}Y)$$

$$+ P(f[\hat{P}X, PY] + (\hat{P}Xf) \cdot PY) + \hat{P}(f[\hat{P}X, \hat{P}Y] + (\hat{P}Xf) \cdot \hat{P}Y)$$

$$= f\nabla_X Y + (PXf) \cdot PY + (PXf) \cdot \hat{P}Y + (PXf) \cdot \hat{P}Y = f\nabla_X Y + Xf \cdot Y.$$  

Further, (5) follows by a direct calculation, and

$$\nabla fX Y = PB[PX, BPY] - (BPYf) \cdot PBX + fPB[PX, BPY] - (BPXf) \cdot PBX$$

$$+ fP[PX, PY] - (PYf) \cdot PXP + fP[PX, PBY] - (PBYf) \cdot PXP$$

$$= f\nabla X Y.$$  

On the other hand, let $\tilde{\nabla}$ be a connection satisfying (4) and (5). To prove that $\nabla$ and $\tilde{\nabla}$ coincide, it suffices to calculate the formula (6) for couples $X, Y$ of homogeneous vector fields belonging to the distribution $D_1$ or $D_2$, and to compare it with the identities obtained for $\nabla$, [1].

(a) Let $X \in D_1$, $Y \in D_2$. Then $PY = 0$, $\hat{P}X = 0$, and $T(X, Y) = 0$. Using $0 = (\nabla f)(X, Y) = \nabla_X (PY) - P(\nabla_Y X)$ we obtain

$$\tilde{\nabla}_X Y = P(\nabla_Y X) = 0,$$

that is $\nabla_X Y \in D_2$. In a similar way, $\nabla \hat{P} = 0$ yields $\tilde{\nabla}_Y X \in D_1$. By our assumption,

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$
Since the decomposition of the Lie bracket \([X, Y] = P[X, Y] + P_X Y\) corresponding to the decomposition of the tangent bundle \(TM = D_1 \oplus D_2\) is uniquely determined we can write

\[-\nabla_Y X = P[X, Y] \in D_1, \quad \nabla_X Y = \tilde{P}[X, Y] \in D_2,\]

and we obtain

\[\nabla_X Y = \tilde{P}[X, Y] = \nabla_X Y.\]

(b) Suppose \(X, Y \in D_1\). In this case \(P_X = \tilde{P}_Y = 0, BY \in D_2, \nabla_X Y = B^2 X\).

By (a), \(\nabla_X Y = \tilde{P}[X, BY] \in D_2\). We can calculate

\[\nabla_X Y = BP[X, BY] = PB[X, BY],\]
\[\nabla_X Y = PB[PX, BPY] = PB[X, BY].\]

(c) Let \(X, Y \in D_2\). Then

\[\nabla_X Y = B\nabla_X (BY) = B^2[X, BY] = \tilde{P}B[X, BY],\]
\[\nabla_X Y = \tilde{P}B[\tilde{P}X, B\tilde{P}Y] = \tilde{P}B[X, BY].\]

2. It is well known that vanishing of the torsion tensor of the Chern connection is a necessary (but not sufficient) condition for parallelizability of a given 3-web. We will show now how this condition can be expressed in terms of the tensor fields \(P, B\) which determine the web.

**Proposition.** Let a 3-web on a manifold \(M\) be defined by a couple \((P, B)\) of \((1, 1)\)-tensor fields satisfying the conditions

\[P^2 = P, \quad B^2 = I, \quad B = BP + PB,\]
\[[P, P] = 0, \quad [B, B](X, Y) = 0 \text{ for } X, Y \in \ker(B - I),\]

and let \(T\) denote the torsion of the Chern connection on a given web manifold. Then

\[T|D_1 \times D_1 = B[P, B]|D_1 \times D_1, \quad T|D_2 \times D_2 = -B[P, B]|D_2 \times D_2,\]
\[T|D_1 \times D_2 = B[P, B]|D_1 \times D_2 = 0\]

and consequently,

\[T = 0 \iff [P, B] = 0.\]
Proof. Since $PB + BP = B$ we have

$$[P, B](X, Y) = [PX, BY] + [BX, PY] + B[X, Y] \]
- $P[X, BY] - B[X, PY] - P[BX, Y] - B[PY, X],

and

$$B[P, B](X, Y) = B\left( [PX, BY] + [BX, PY] \right)$$

- $BP\left( [X, BY] + [BX, Y] \right) - [X, PY] - [PX, Y] + [X, Y].$

(i) Let both $X, Y \in D_1$. A calculation shows that

$$B[P, B](X, Y) = PB\left( [X, BY] + [BX, Y] \right) - [X, Y],$$

and

$$T(X, Y) = PB\left( [PX, BY] + [BPX, PY] \right) - [X, Y].$$

We see that on $D_1$, both tensors coincide:

$$T[D_1 \times D_1] = B[P, B]|D_1 \times D_1.$$

(ii) Now let $X, Y \in D_2$. In this case

$$B[P, B](X, Y) = -B[PX, BY] - [X, Y],$$

$$T(X, Y) = PB[X, BY] + BP[BX, Y] - [X, Y]$$

- $BP\left( [X, BY] + [BX, Y] \right) - [X, Y],$

which proves that


(iii) Finally, let $X \in D_1$ and $Y \in D_2$. Then $[P, B](X, Y) = 0$, $T(X, Y) = T(PX, PY) = 0$, and

$$T[D_1 \times D_2] = B[P, B]|D_1 \times D_2 = 0.$$

Combining the above results we complete the proof of (8); (9) follows since $B$ is an isomorphism.

Following Russian authors, either the tensor field $T$, or the tensor field $[P, B]$ can be called a torsion of a given 3-web.
References


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