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Persistent URL: http://dml.cz/dmlcz/126098

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EXISTENCE OF QUASICONTINUOUS SELECTIONS FOR THE SPACE $2^\mathbb{R}$

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(Received February 7, 1995)

Summary. The paper presents new quasicontinuous selection theorem for continuous multifunctions $F: X \to \mathbb{R}$ with closed values, $X$ being an arbitrary topological space. It is known that for $2^\mathbb{R}$ with the Vietoris topology there is no continuous selection. The result presented here enables us to show that there exists a quasicontinuous and upper(lower)-semicontinuous selection for this space. Moreover, one can construct a selection whose set of points of discontinuity is nowhere dense.

Keywords: continuous multifunction, selection, quasicontinuity

AMS classification: 54C65, 54C08

1. INTRODUCTION

Up to now, papers dealing with the problem of existence of quasicontinuous selections have considered multifunctions with compact values in metric spaces (see e.g. [2, 6, 7, 8]). Another classical condition in selection theory is the convexity of values ([9, 10]).

In this paper we present a quasicontinuous selection theorem for continuous multifunctions $F: X \to \mathbb{R}$ with closed values, $X$ being an arbitrary topological space. It is shown that the graph of $F$ can be constructed as the union of graphs of quasicontinuous and upper-semicontinuous selections of $F$. Moreover, the sets of points of discontinuity of these selections are nowhere dense. Our result enables us to complete the work of [1] concerning the hyperspace $2^\mathbb{R}$.
2. Preliminaries

By $2^\mathbb{R}$ we mean the class of all nonempty closed subsets of $\mathbb{R}$ equipped with the Vietoris topology (for definition of basic notions: Vietoris topology, hyperspace, multifunction, selection, l.s.c., u.s.c., Hausdorff continuous multifunction etc. see e.g. [4] and [11]).

Let $X$ and $Y$ be two topological spaces. A multifunction $F$ from $X$ to $Y$ is called continuous, if it is l.s.c. and u.s.c. (lower and upper semicontinuous).

Let us denote $F^-(A; B) = \{x; F(x) \cap A \neq \emptyset \text{ and } F(x) \subseteq B\}$. Of course, for $B, A \subseteq Y$ open and $F$ continuous, the set $F^-(A; B)$ is an open subset of $X$.

Let $B$ be a subset of a topological space $X$. In what follows int $B$ and cl $B$ denote the interior and the closure of the set $B$, respectively. There are several equivalent definitions of quasicontinuity, we will use the following one: A function $f: X \to Y$ is said to be quasicontinuous at $x \in X$ if and only if for any open set $V$ such that $f(x) \in V$ and any open set $U$ such that $x \in U$, there exists a nonempty open set $W \subseteq U$ such that $f(W) \subseteq V$ ([5.12, 13]).

3. Results

In the next theorem the space $X$ is an arbitrary topological space, which is quite rare in the selection theory. Nevertheless, the fact that $Y = \mathbb{R}$ permits us to give a constructive proof of the assertion.

Theorem 1. Let $X$ be an arbitrary topological space. Let $F: X \to \mathbb{R}$ be a continuous multifunction with closed values. Then $F$ has a quasicontinuous and upper-semicontinuous selection $h$ such that its set of points of discontinuity is a nowhere dense set.

Proof. Let us define $g(x) = \min\{|y|; y \in F(x)\}$ for every $x$ from $X$. We denote $A = \{x \in X; g(x) \in F(x) \text{ and } -g(x) \in F(x)\}$. The set $A$ is closed. We will prove it by proving that $X - A$ is open.

Let $b \in X - A$. Let us consider the case $g(b) \in F(b)$, the other ($-g(b) \in F(b)$) being analogous. In this case $-g(b)$ is not an element of $F(b)$ and since the set $F(b)$ is closed, there exists $\delta > 0$ such that

(i) $F(b) \subseteq U = (-\infty, -g(b) - \delta) \cup (g(b) - \delta, +\infty)$

and

(ii) $g(b) - \delta > 0$. 

Let us denote $V = (g(b) - \delta, g(b) + \delta)$. Since $F$ is continuous and (i) and (ii) hold, the set $W = F^-(V; U)$ is an open neighborhood of the point $b$. Of course $W \subset X - A$. So the set $X - A$ is open.

Let us denote $B = A - \text{cl}(\text{int} A)$. For every element $x$ of $X$ one of the following assertions is true:

1. $x \in X - A$ and $g(x) \in F(x)$;
2. $x \in X - A$ and $-g(x) \in F(x)$;
3. $x \in \text{cl}(\text{int} A)$;
4. $x \in B$ and for every open neighborhood $O(x)$ of the point $x$ there exists a point $t \in O(x)$ such that $g(t) \in F(t)$ and $-g(t) \notin F(t)$ hold;
5. $x \in B$ and there exists an open neighborhood $O(x)$ of the point $x$ such that for every element $t$ of $O(x)$, $-g(t) \in F(t)$ holds.

Let us define a function $h: X \rightarrow \mathbb{R}$ as follows:

$$h(x) = \begin{cases} 0 & \text{if } 0 \notin F(x), \\ g(x) & \text{if (1) or (3) or (4) is true and } 0 \notin F(x), \\ -g(x) & \text{if (2) or (5) holds and } 0 \notin F(x). \end{cases}$$

It is easy to see that $h$ is a selection of $F$. We will prove that $h$ is quasicontinuous at every $x$ in $X$.

First, let $x \in X$ be such that $h(x) = 0$. Let $V$ be an open neighborhood of $h(x)$. Then there exists $\varepsilon > 0$ such that $U = (-\varepsilon, \varepsilon) \subset V$. The set $W = F^-(U; \mathbb{R})$ is an open neighborhood of $x$ and for every element $w$ of $W \setminus (-g(w), g(w)) \subset U$ holds. So $h(w) \in V$, $\forall w \in W$ is true, and the function $f$ is continuous at the point $x$.

If $0 \notin F(x)$, we distinguish five cases:

(I) Let $x \in X$ and let (1) hold. Let $O \subset \mathbb{R}$ be an open set such that $h(x) \in O$. Then there exists $\delta > 0$ such that

$$F(x) \subset G = (-\infty, -h(x) - \delta) \cup (h(x) - \delta, +\infty)$$

and

$$h(x) - \delta > 0 \quad \text{and} \quad H = (h(x) - \delta, h(x) + \delta) \subset O$$

is true.

Hence $x$ is an element of the set $C = F^-(H; G)$ and since $F$ is continuous, the set $C$ is open. It is easy to verify from the definition of $C$ that (1) holds for every $t \in C$ and this implies $h(C) \subset H \subset O$. So the function $h$ is continuous at the point $x$.

(II) Quite analogously, if (2) is satisfied for an $x$ from $X$, then $h$ is continuous at the point $x$. 

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(III) Let \(x \in X\) and let (3) hold. Let \(O \subset \mathbb{R}, \ G \subset X\) be two open sets such that \(x \in G\) and \(h(x) \in O\). Then there exists \(\delta > 0\) such that following holds:

\[
\begin{align*}
  h(x) - \delta &> 0, \\
  V &= (h(x) - \delta, h(x) + \delta) \subset O, \\
  F(x) &\subset U = (\langle h(x) - \delta, +\infty\rangle) \cup (\langle -h(x) - \delta, +\infty\rangle).
\end{align*}
\]

Let us denote \(W = G \cap F^-(V; U)\). Since \(W\) is an open neighborhood of \(x\) and \(x \in \mathrm{cl}(\mathrm{int} \ A)\), the set \(P = W \cap \mathrm{int} \ A\) is nonempty open, \(P \subset G\). For every \(p\) from \(P\) we have \(p \in F^-(V; U)\), hence \(h(p) \in V \subset O\). This proves the quasicontinuity of \(h\) at \(x\). Moreover, if \(x \in \mathrm{int} \ A\), then \(x \in P\) and we see that \(h\) is continuous at the point \(x\). If \(x\) is not from \(\mathrm{int} \ A\), it is still true that for every \(\varepsilon > 0\) and for every \(v \in F^-(\langle h(x) - \varepsilon, h(x) + \varepsilon\rangle; R)\) the inequality

\[
  h(v) \leq h(x) + \varepsilon
\]

holds; so, \(h\) is upper-semicontinuous at \(x\).

(IV) Let \(x \in X\) and let (4) hold. Let \(O \subset \mathbb{R}, \ G \subset X\) be two open sets such that \(x \in G\) and \(h(x) \in O\). Then there exists \(\delta > 0\) such that

\[
\begin{align*}
  h(x) - \delta &> 0, \\
  V &= (h(x) - \delta, h(x) + \delta) \subset O \\
  F(x) &\subset U = (\langle -\infty, -h(x) + \delta\rangle) \cup (\langle h(x) - \delta, +\infty\rangle)
\end{align*}
\]

hold. Let us denote \(W = G \cap F^-(V; U)\). \(W\) is an open neighborhood of the point \(x\). From the validity of (4) we obtain that there exists \(t \in W\) such that (1) is true for \(t\) and \(h(t) = g(t)\). Since \(t \in W, h(t) \in V\) holds. By (1) the function \(h\) is continuous at the point \(t\); so, there exists an open neighborhood \(H\) of \(t\) such that \(h(s) \in V\) for every \(s \in H\). Let us denote \(P = H \cap W\). The set \(P\) is an open subset of \(G\) and \(h(p) \in V\) for every \(p \in P\). This proves the quasicontinuity of \(h\) at the point \(x\). The proof of the upper-semicontinuity of \(h\) at the point \(x\) is left to the reader.

(V) Let \(x \in X\) and let (5) hold. Let \(O \subset \mathbb{R}\) be an open set such that \(h(x) \in O\). Then there exists \(\delta > 0\) such that

\[
\begin{align*}
  h(x) + \delta &< 0, \\
  V &= (\langle h(x) - \delta, h(x) + \delta\rangle) \subset O \\
  F(x) &\subset U = (\langle -\infty, h(x) + \delta\rangle) \cup (\langle -h(x) - \delta, +\infty\rangle)
\end{align*}
\]
hold. Let us denote \( W = \left( F^{-1}(V;U) \cap O(x) \right) - \text{cl}(\text{int}A) \) where \( O(x) \) is the set mentioned in (5). Then \( W \) is an open neighborhood of the point \( x \) and for every \( w \in W \) either (2) or (5) is true. Therefore \( h(w) = -g(w) \in V \subset O \) holds for every \( w \in W \). This implies the continuity of the function \( h \) at \( x \).

To complete the proof, it suffices now to show that the set of points of discontinuity of \( h \) is nowhere dense. But it is easy to see that this set is a subset of the set

\[
B \cup (\text{cl}(\text{int}A) - \text{int}A) = (A - \text{cl}(\text{int}A)) \cup (\text{cl}(\text{int}A) - \text{int}A).
\]

Since \( A \) is closed, this set is the union of two nowhere dense sets.

Now we present two examples relevant to Theorem 1.

**Example 1** ([2]). We show that the assumption "\( F \) is u.s.c." in Theorem 1 cannot be omitted.

Let \( X = \{a,b,c\} \), let \((X,T)\) be a topological space with the topology \( T = \{\emptyset\} \cup \{\{a\},\{c,a\},\{b,a\},X\} \). Define \( F: X \rightarrow \mathbb{R} \) as follows:

\[
F(a) = \{1,2\}, \quad F(b) = \{1\}, \quad F(c) = \{2\}.
\]

\( F \) is a l.s.c. multifunction with compact values and \( F \) has no quasicontinuous selection.

**Example 2.** Let \( X = \mathbb{N} = \{1,2,\ldots\} \) be a topological space with the topology \( T = \{A; A \subset \mathbb{N}, \mathbb{N} - A \text{ is a finite set}\} \cup \{\emptyset,\mathbb{N}\} \). Let us define a multifunction \( F: X \rightarrow \mathbb{R} \) as follows:

\[
F(k) = \mathbb{N} - \{1,2,\ldots,k\}.
\]

The multifunction \( F \) is u.s.c., it has closed values, but it is not l.s.c. It is easy see that it has no quasicontinuous selection, because all quasicontinuous functions from \((X,T)\) to \( \mathbb{R} \) are constant ones.

Reading the proof of Theorem 1 we see that for every \( x \in X \), \( 0 \in F(x) \) implied \( h(x) = 0 \). This fact will be used in the proof of the following assertion:

**Theorem 2.** Let \( X \) be an arbitrary topological space. Let \( G: X \rightarrow \mathbb{R} \) be a continuous multifunction with closed values. Let \((x,y)\) be an element of the graph of \( G \). Then there exists a quasicontinuous and upper-semicontinuous selection \( g: X \rightarrow Y \) such that \( g(x) = y \), \( g \) is continuous at \( x \) and the set of points of discontinuity of \( g \) is nowhere dense.

**Proof.** Let us define a multifunction \( F: X \rightarrow \mathbb{R} \) as follows: \( F(t) = G(t) - y \) for \( t \in X \). Then \( F: X \rightarrow \mathbb{R} \) is a continuous multifunction with closed values and
according to Theorem 1 there exists a quasicontinuous and upper-semicontinuous selection $h$ of $F$. Since 0 is an element of $F(x)$, $h(x) = 0$ holds and $h$ is continuous at $x$. Let us define a function $g : X \to \mathbb{R}$ in the following way: $g(t) = h(t) + y$ for $t \in X$. The function $g$ is quasicontinuous, upper-semicontinuous and it is a selection of $G$.

Moreover, $g(x) = h(x) + y = y$ holds. \hfill \Box

It is well known that there is no continuous selection for the hyperspace of nonempty closed subsets of $\mathbb{R}$ with the Vietoris topology ([1]). However, Theorem 2 gives us the following result:

**Corollary 1.** Let $I$ be the "identity multifunction" from $2^\mathbb{R}$ to $\mathbb{R}$, such that $I(A) = A$ holds for every $A \in 2^\mathbb{R}$. Then for every point $(x,y)$ of the graph of $I$ there exists a quasicontinuous and upper-semicontinuous selection $f$ of $I$ such that $f(x) = y$ and the set of points of discontinuity of $g$ is nowhere dense.

**Remark 1.** Theorem 1 and Corollary 1 also imply (under the same conditions) the existence of a quasicontinuous selection which is lower-semicontinuous. It suffices to consider a multifunction $G = -F$. Then there exists an upper-semicontinuous (and quasicontinuous) selection $g$ of $G$. Then $h = -g$ is the lower-semicontinuous selection of $F$ we wanted.

**Remark 2.** It is easy to check that Theorem 1 and Theorem 2 are true also if the assumption "$F$ is a continuous multifunction" is replaced by the assumption "$F$ is Hausdorff continuous". In this case Corollary 1 can be reformulated: $2^\mathbb{R}$ can be replaced by the hyperspace of nonempty closed subsets of $\mathbb{R}$ with the topology derived from Hausdorff metric.

Another example relevant to our results, an example of a continuous and Hausdorff continuous multifunction $F : [-1,0] \to \mathbb{R}$ with closed values which has no continuous selection can be found in [3].

**References**


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