Jiří Cerha

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A METHOD FOR DETERMINING CONSTANTS IN THE LINEAR COMBINATION OF EXPONENTIALS

J. ČERHA, Praha

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Summary. Shifting a numerically given function \( b_1 \exp a_1 t + \ldots + b_n \exp a_n t \) we obtain a fundamental matrix of the linear differential system \( \dot{y} = Ay \) with a constant matrix \( A \). Using the fundamental matrix we calculate \( A \), calculating the eigenvalues of \( A \) we obtain \( \lambda_1, \ldots, \lambda_n \) and using the least square method we determine \( b_1, \ldots, b_n \).

Keywords: fundamental matrix, linear differential system, shifted exponentials, eigenvalues, the least square method

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Let \( n \geq 1 \) denote an integer, \( a_1, \ldots, a_n, b_1, \ldots, b_n \) real numbers, \( a_i \neq a_j \) if \( i \neq j \), \( b_i \neq 0 \) for \( i = 1, \ldots, n \),

\[
f(t) = b_1 \exp a_1 t + \ldots + b_n \exp a_n t
\]

for real \( t \). Let \( h_1, \ldots, h_n; k_1, \ldots, k_n \) denote real numbers, \( h_2 = k_1 = 0, h_i \neq k_j \) and \( k_i \neq k_j \) if \( i \neq j \); \( i, j = 1, \ldots, n \). Define the \( n \times n \)-matrix valued function

\[
Y(t) = \begin{bmatrix}
f(t - h_1 - k_1) & \cdots & f(t - h_1 - k_n) \\
\cdots & \cdots & \cdots \\
f(t - h_n - k_1) & \cdots & f(t - h_n - k_n)
\end{bmatrix}
\]

for real \( t \).

Theorem. \( Y \) is a fundamental matrix of the linear differential system \( \dot{y} = Ay \) with a constant \( n \times n \)-matrix \( A \), and \( a_1, \ldots, a_n \) are the eigenvalues of \( A \).

Proof. Let us set \( y_i = \exp(-a_i); i = 1, \ldots, n \),

\[
E_1 = E(h_1, \ldots, h_n) = \begin{bmatrix}
    y_1^{h_1} & \cdots & y_1^{h_n} \\
    \vdots & \cdots & \vdots \\
    y_n^{h_1} & \cdots & y_n^{h_n}
\end{bmatrix}, \quad E_2 = E(k_1, \ldots, k_n),
\]
Using induction we shall prove that $E_1$ is regular or, equivalently, the function

$$\varphi(y) = c_1 y^{h_1} + \ldots + c_n y^{h_n}$$

has at most $n - 1$ positive roots for arbitrary $c_1, \ldots, c_n$ excluding $c_1 = \ldots = c_n = 0$ and arbitrary $h_1, \ldots, h_n$ satisfying our assumptions. This is clear for $n = 1$. Let $n > 1$, let our assertion be true for $n - 1$ and let us suppose $\varphi$ has $n$ positive roots. Hence, the derivative $\varphi'$ has $n - 1$ positive roots which, using $h_1 = 0$, contradicts the induction hypothesis. Similarly, $E_2$ is regular. Using our notation we obtain $Y = E_1 B E_1^T$, $Y = E_2 B E_2^T$. Hence $Y$ is regular and $A \equiv YY^{-1} = E_1 G E_1^{-1}$ is constant, which proves our theorem.

Let $p \geq 2n$ be an integer, $t_0, h > 0$ real numbers, $f_i = f(t_0 - (i - 1)h)$ for $i = 1, \ldots, p$. Let $n, h, f_1, \ldots, f_p$ be known, while $a_1, \ldots, a_n; b_1, \ldots, b_n$ are to be determined. We put $h_i = h_i = (i - 1)h$ for $i = 1, \ldots, n$. (However, there exist many methods for choosing $h_i, k_i$.) Now, we may calculate $Y(t)$ for

$$t \in M \equiv \{t_0 - (i - 1)h: i = 1, \ldots, q\},$$

where $q = p - 2n + 2$, $q \geq 2$. We will determine $Y(t)$ numerically for some fixed $t \in M$ and put

$$A = Y(t)Y(t)^{-1}.$$

Concerning numerical errors, we would probably obtain better results putting

$$A = \frac{1}{m} (Y(t_1)Y(t_1)^{-1} + \ldots + Y(t_m)Y(t_m)^{-1}),$$

where $m > 1$ is an integer, $t_1, \ldots, t_m \in M, t_i \neq t_j$ if $i \neq j$. We obtain $a_1, \ldots, a_n$ calculating the eigenvalues of $A$.

Alternatively, the formula

$$A = \frac{1}{t_2 - t_1} \ln Y(t_2)Y(t_1)^{-1}$$

can be used. Let $g_1, \ldots, g_n$ denote the eigenvalues of the matrix $Y(t_2)Y(t_1)^{-1}$ for some fixed $t_1, t_2 \in M, t_1 \neq t_2$. Hence, the values $a_1, \ldots, a_n$ coincide with the values

$$\frac{1}{t_2 - t_1} \ln g_i: \quad i = 1, \ldots, n.$$

Now, $b_1, \ldots, b_n$ may be determined using the least square method.

References


Author’s address: Jiří Cerha, Veletská 533, 190 00 Praha 9, Czech Republic.