Luis B. Boza; Ana Diánez; Alberto Márquez
On infinite outerplanar graphs


Persistent URL: [http://dml.cz/dmlcz/126112](http://dml.cz/dmlcz/126112)

Terms of use:

© Institute of Mathematics AS CR, 1994

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://dml.cz](http://dml.cz)
ON INFINITE OUTERPLANAR GRAPHS

L. Boza, A. Díánez, A. Márquez, Sevilla

(Received May 21, 1993)

Summary. In this Note, we study infinite graphs with locally finite outerplane embeddings, given a characterization by forbidden subgraphs

Keywords: infinite outerplanar graphs, end of an infinite graph

AMS classification: 05C10, 05C75

1. INTRODUCTION

Clearly, the classic result about outerplanar graphs is valid even if the graph is infinite and thus a planar graph is outerplanar (it admits an embedding where all its vertices lie on the same face) if and only if has no subvision of $K_4$ or $K_{2,3}$. But, as was pointed out by many authors (see [6]), when dealing with embeddings of finite graphs it is interesting to ensure embeddings without vertices or edges accumulation (p-embeddings) and in this case the previous result fails as Figure 1 shows.

The p-embeddings of the graphs pictured in Figure 1 are unique (see [5]) and so, since they are not outerplane, we have that those graphs are not p-outerplanar (on the other hand, if they are outerplanar, it is possible to shorten some of the infinite rays accumulating the vertices and edges of these rays to a point in the plane). We are going to prove that the two graphs given in Figure 2 are the two forbidden subgraphs for outerplane p-embeddings.

As it happens with other results on p-embeddings (see [3]), it can be remarked that the one-point compactifications of the underlying topological spaces to those graphs are homeomorphic to the two forbidden graphs for outerplane embeddings.

By an infinite graph we mean a connected graph such that its vertex set is countable and the degree at every vertex is finite (a locally finite countable graph). We
will use the notation and definitions of [4], except for using vertex instead of point
and edge instead of line.

We will use an invariant of non-compact spaces, namely, the ends of Freudenthal. An infinite ray in a graph $G$ is a morphism $\varphi: J \to G$ inducing an injection on both the vertex set and the edge set, where $J$ represents a graph such that its underlying topological space is homeomorphic to the positive half-line $R^+$. Two rays in $G$ define the same Freudenthal end if for any finite subgraph $H$ of $G$, there exist vertices of both rays in $G - H$. The set of Freudenthal ends of $G$ is denoted by $\mathcal{F}(X)$ and its cardinal by $e(X)$ (see [1] for details).

2. P-OUTERPLANAR GRAPHS

**Definition 1.** A graph $G$ is said to be p-outerplanar if it can be embedded in the plane without vertices or edges accumulation so that all its vertices lie in the same face (a face is a connected component of the complement of the embedding with respect to the whole plane).

Observe that all blocks of the graphs given in Figure 1 are outerplanar (actually, the two graphs are outerplanar) but those graphs are not p-outerplanar, in fact it is possible to give graphs which are not p-outerplanar graphs and have a finite number of p-outerplanar blocks as is shown in Figure 2.

![Figure 1: Two outerplanar graphs with no outerplane p-embeddings.](image1.png)

![Figure 2: A outerplanar and non-p-outerplanar graph with two p-outerplanar blocks.](image2.png)

382
Nevertheless, we have

**Lemma 2.** A graph $G$ with only one end and infinitely many cut-vertices is p-outerplanar if and only if is outerplanar.

**Proof.** $G$ is outerplanar if and only if all its blocks are, and we can number those blocks in such a way that block $B_i$ is joined in $G$ with blocks $B_{i-1}$ and $B_{i+1}$ (block $B_1$ is only joined to block $B_2$). In this way we can glue each block to its adjacent blocks and we can ensure that all the vertices are in the unbounded face.

**Theorem 3.** A graph $G$ is p-outerplanar if and only if it has no subdivision of $K_4, K_{2,3}, L_4$ or $L_{2,3}$ (see Figure 1).

**Proof.** Obviously the four graphs given in the theorem are not p-outerplanar and thus it only remains to show that any p-outerplanar graph has no subdivision of $K_4, K_{2,3}, L_4$ or $L_{2,3}$.

As any p-outerplanar graph is outerplanar, it suffices to prove that an outerplanar graph is p-outerplanar if it has no subdivision of $L_4$ or $L_{2,3}$.

First, suppose that there exists a subgraph $H$ of $G$ homeomorphic to $R^1$ (the 1-dimensional Euclidean space). Then the restriction to $H$ of any p-embedding of $G$ if $R^2$ splits the plane in two half-planes. Thus it is straightforward to check that the two possible obstructions to be p-outerplanar are

- There exists a path in $G - H$ connecting two non-consecutive vertices of $H$. In this case, we have a subdivision of $L_{2,3}$.
- There exists an infinite ray in $G - H$. In this case, we have a subdivision of $L_4$.

If there is not such an $H$, then $G$ has only a Freudenthal end and, as a consequence of Menger's theorem for infinite graphs (see [2]), in any $H$ representative of that end (a representative of an end is a subgraph homeomorphic to the half-line $[0, +\infty)$ such that there exist its elements in the component of the complement of any compact defining that end), there exist infinitely many cut-vertices. Let $C = \{v_1, v_2, \ldots\}$ be that set of cut-vertices sorted as they appear in $H$ and let $G_i$ be the block of $G$ which contains $v_{i-1}$ and $v_i$ ($G_1$ is the block of $G$ which contains only $v_1$). Then, by Lemma 2, $G$ is p-outerplanar if and only if $G$ is outerplanar and, this happens if and only if all $G_i$'s are outerplanar.

Obviously if $G$ has more that two ends, it is always possible to find a subgraph homeomorphic to $L_4$. Then $G$ cannot be p-outerplanar and so we have

**Corollary 4.** If $G$ is p-outerplanar, then $e(G) \leq 2$.  

383
References


Authors’ addresses: *L. Boza*, E.T.S. Arquitectura, Universidad de Sevilla, Reina Mercedes 2, 41012-Sevilla (Spain); *A. Diánez*, E.T.S. Arquitectura, Universidad de Sevilla, Reina Mercedes 2, 41012-Sevilla (Spain), anarosa@sevaxu.cica.es; *A. Márquez*, Facultad de Informática, Universidad de Sevilla, Reina Mercedes s/n, 41012-Sevilla (Spain), almar@sevaxu.cica.es.