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ROUTE SYSTEMS OF A CONNECTED GRAPH

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Summary. The concept of a route system was introduced by the present author in [3]. Route systems of a connected graph $G$ generalize the set of all shortest paths in $G$. In this paper some properties of route systems are studied.

Keywords: route systems, shortest paths, geodetic graphs

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0. Before giving the definition of a route system we need to introduce some auxiliary notions.

Let $G$ be a graph (in the sense of [1], for example, i.e. a finite undirected graph with no loops or multiple edges) with a vertex set $V(G)$. We denote by $\mathcal{W}_N(G)$ the set of all sequences

$$(0) \quad u_0, \ldots, u_i,$$

where $i \geq 0$ and $u_0, \ldots, u_i \in V(G)$. Similarly as in [4], instead of (0) we write $u_0 \ldots u_i$. If $v_0, \ldots, v_j \in V(G)$ and $\alpha = v_0 \ldots v_j$, where $j \geq 0$, then we put $A\alpha = v_0$, $Z\alpha = v_j$, $||\alpha|| = j$ and $\bar{\alpha} = v_j \ldots v_0$. If $u_0, \ldots, u_k, w_0, \ldots, w_m \in V(G)$, $\beta = u_0 \ldots u_k$ and $\gamma = w_0 \ldots w_m$, where $k, m \geq 0$, then we write $\beta \gamma = u_0 \ldots u_k w_0 \ldots w_m$. We denote by $*$ the empty sequence in the sense that $\alpha * = \alpha = *\alpha$ for every $\alpha \in \mathcal{W}_N(G)$, $** = *$ and $* = *$. Put $\mathcal{W}(G) = \mathcal{W}_N(G) \cup \{\ast\}$. If $\mathcal{M} \subseteq \mathcal{W}_N(G)$ and $u, v \in V(G)$, then we denote

$\mathcal{M}_{(u,v)} = \{\alpha \in \mathcal{M}; A\alpha = u \text{ and } Z\alpha = v\}$

and

$\mathcal{M}^{(u,v)} = \{\alpha \in \mathcal{M}; \text{ there exist } \beta, \gamma, \delta \in \mathcal{W}(G) \text{ such that } \alpha = \beta \gamma \delta \text{ and } \gamma \in \mathcal{M}_{(u,v)}\}$. 

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Let $v_0, \ldots, v_i \in V(G)$, where $i \geq 0$; we say that $v_0 \ldots v_i$ is a path in $G$ if the vertices $v_0, \ldots, v_i$ are mutually distinct and the vertices $v_j$ and $v_{j+1}$ are adjacent in $G$ for each integer $j$, $0 \leq j < i$. We denote by $\mathcal{P}(G)$ the set of all paths in $G$. Let $\alpha \in \mathcal{W}_N(G)$; we say that $\alpha$ is a shortest path in $G$ if $\alpha \in \mathcal{P}(G)$ and $||\alpha|| \leq ||\beta||$ for every $\beta \in \mathcal{P}(G)$ such that $A\alpha = A\beta$ and $Z\alpha = Z\beta$. We denote by $S(G)$ the set of all shortest paths in $G$.

Let $G$ be a connected graph, and let $\mathcal{R} \subseteq \mathcal{P}(G)$. We will say that $\mathcal{R}$ is a semi-route system on $G$ in the following Axioms I–IV are fulfilled for arbitrary $u, v \in V(G)$ and $\alpha, \beta, \gamma, \delta \in \mathcal{W}(G)$:

I if $u$ and $v$ are adjacent, then $uv \in \mathcal{R}$;

II if $\alpha \in \mathcal{R}$, then $\alpha \in \mathcal{R}$;

III if $u\alpha v \in \mathcal{R}$, then $u\alpha \in \mathcal{R}$;

IV if $\alpha u\beta v\gamma$, $u\delta v \in \mathcal{R}$, then $\alpha u\delta v \gamma \in \mathcal{R}$.

Moreover, we say that $\mathcal{R}$ is a route system on $G$ if it is a semi-route system on $G$ and the following Axiom V is fulfilled for arbitrary $u, v \in V(G)$:

V there exist $\alpha \in \mathcal{R}$ such that $A\alpha = u$ and $Z\alpha = v$.

Let $G$ be a connected graph. Consider a route system $\mathcal{R}$ on $G$; if $u, v \in V(G)$, then we denote

$$d_\mathcal{R}(u, v) = \min(||\alpha||; \alpha \in \mathcal{R}, A\alpha = u \text{ and } Z\alpha = v).$$

It is easy to see that $S(G)$ is a route system on $G$. Note that $S(G)$ is the only route system on $G$ if and only if $G$ is a tree, cf. [3]. Instead of $d_{S(G)}$ we will write $d$ only. Obviously, if $u, v \in V(G)$, then $d(u, v)$ is the distance between $u$ and $v$ in $G$.

The following theorem was proved in [4]:

**Theorem 0.** Let $G$ be a connected graph, and let $\mathcal{R}$ be a route system on $G$. Then $\mathcal{R} = S(G)$ if and only if the following conditions (1)–(3) hold for arbitrary $u, v, x, y \in V(G)$ and $\alpha, \beta \in \mathcal{W}(G)$:

1. if $uaxv \in \mathcal{R}$, then $uv \notin \mathcal{R}$;
2. if $uaxy$, $uv\beta y$, $vux \in \mathcal{R}$, then $v\beta yx \in \mathcal{R}$;
3. if $xy$, $u\alpha x \in \mathcal{R}$, $w\varphi yx \in \mathcal{R}$ for no $\varphi \in \mathcal{W}(G)$ and $uv\psi y \in \mathcal{R}$ for no $\psi \in \mathcal{W}(G)$, then $vaxy \in \mathcal{R}$.
1. Let $G$ be a connected graph, and let $\mathcal{R}$ be a semi-route system on $G$. We say that $\mathcal{R}$ is geodetic if the following Axiom VI is fulfilled for arbitrary $u, v \in V(G)$:

$$\text{VI} \quad |\mathcal{R}_{(u,v)}| \leq 1.$$ 

Thus, if $\mathcal{R}$ is a route system on $G$, then it is geodetic if and only if $|\mathcal{R}_{(u,v)}| = 1$ for every pair of vertices $u$ and $v$ of $G$.

**Example.** Let $G$ be a connected graph of diameter two. Put $S = S(G)$. For every pair of vertices $u$ and $v$ of distance two in $G$ we choose exactly one path in $S_{(u,v)}$, say a path $\alpha_{uv}$, such that $\alpha_{uw} = \alpha_{uw}$. Denote

$$\mathcal{R} = \{u; u \in V(G)\} \cup \{vw; v \text{ and } w \text{ are adjacent vertices of } G\} \cup \{\alpha_{xy}; x, y \in V(G) \text{ and } d(x,y) = 2\}.$$

It is not difficult to see that $\mathcal{R}$ is a geodetic route system on $G$.

Let $G$ be a connected graph. Consider a route system $\mathcal{R}$ on $G$. If $u, v \in V(G)$, then we denote by $N_{\mathcal{R}}(u, v)$ the set of all $w \in V(G)$ such that there exists $\alpha \in \mathcal{W}(G)$ with the property that $uw\alpha \in R_{(u,v)}$. Similarly as in [3] we denote

$$\#_{\mathcal{R}}(x,y) = \{x\} \cup \{z \in V(G); N_{\mathcal{R}}(z,x) - N_{\mathcal{R}}(z,y) \neq \emptyset\}$$

for any $x, y \in V(G)$. The mapping $\#_{\mathcal{R}}$ has its origin in the author's study of mathematical models in semiotics.

It is not difficult to see that if $G$ is a connected graph and $\mathcal{R}$ is a geodetic route system on $G$, then $\#_{\mathcal{R}}(u, v) = \#_{\mathcal{R}}(v, u)$.

**Lemma 1.** Let $G$ be a connected graph, and let $\mathcal{R}$ be a route system on $G$. Assume that $\mathcal{R}$ is not geodetic. Then there exists a pair of adjacent vertices $u$ and $v$ of $G$ such that $\#_{\mathcal{R}}(u, v) \neq \#_{\mathcal{R}}(v, u)$.

**Proof.** Since $\mathcal{R}$ is not geodetic, there exist $v, w \in V(G)$ such that $|\mathcal{R}_{(w,v)}| \geq 2$ and $|\mathcal{R}_{(x,y)}| = 1$ for any $x, y \in V(G)$ with the property that $d_{\mathcal{R}}(x,y) < d_{\mathcal{R}}(w,v)$. Since $|\mathcal{R}_{(w,v)}| \geq 2$, there exist distinct $\alpha, \beta \in \mathcal{R}_{(w,v)}$ such that $\|\alpha\| = d_{\mathcal{R}}(w,v)$. Then $\alpha$ and $\beta$ have no common vertex different from $v$ and $w$ (otherwise, combining Axioms II and III, we easily get $\alpha = \beta$, which is a contradiction). We distinguish two cases:

1. Let $d_{\mathcal{R}}(w,v) = 1$. Then $\alpha = wv$. Since $\beta \neq \alpha$, there exist $u \in V(G)$ and $\gamma \in \mathcal{W}(G)$ such that $\beta = w\gamma uv$. Axiom IV implies that if $\delta \in \mathcal{R}_{(w,u)}$, then
δv ∈ \( R_{(w,v)} \). Hence \( w \notin \#R(u,v) \). Recall that \( \alpha = wv \). We have \( wvu \notin R_{(w,u)} \) (otherwise, Axiom IV would imply that \( w\gamma wu \in R \), which is a contradiction). Thus \( w \in \#R(v,u) \).

2. Let \( d_R(w,v) \geq 2 \). Then there exist \( u \in V(G) \) and \( \gamma \in W(G) \) such that \( \alpha = w\gamma wu \). According to Axiom III, \( w\gamma u \in R_{(w,u)} \). It follows from the definition of \( d_R \) that \( d_R(w,u) < d_R(w,v) \). This implies that \( R_{(w,u)} = \{w\gamma u\} \). Hence, \( w \notin \#R(u,v) \).

Clearly, \( u \) does not lie on \( \beta \). Moreover, we see that \( \beta u \notin R_{(w,u)} \). Thus \( w \in \#R(v,u) \), which completes the proof of lemma.

Combining Lemma 1 with the above observation we get:

**Theorem 1.** Let \( G \) be a connected graph, and let \( R \) be a route system on \( G \). Then \( R \) is geodetic if and only if \( \#R(u,v) = \#R(v,u) \) for every pair of vertices \( u \) and \( v \) of \( G \).

If \( G \) is a connected graph and \( u,v \in V(G) \), then instead of \( \#S(G)(u,v) \) we will write \( \#(u,v) \). Note that the mapping \( \# \) was introduced in [2].

A connected graph \( G \) is called geodetic if \( S(G) \) is a geodetic route system on \( G \).

**Corollary 1.** A connected graph \( G \) is geodetic if and only if \( \#(u,v) = \#(v,u) \) for every pair of vertices \( u \) and \( v \) of \( G \).

2. In this section we will prove that if \( G \) is a connected graph and \( R \) is a route system on \( G \), then there exists a subset of \( R \) which is a geodetic route system on \( G \). In fact, we will prove a more general result for semi-route systems.

If \( G \) is a connected graph, then we define \( b(G) = |E(G)| - |V(G)| + 1 \), where \( E(G) \) is the edge set of \( G \).

**Theorem 2.** Let \( G \) be a connected graph, and let \( R \) be a semi-route system on \( G \). Then there exists a geodetic semi-route system \( R^* \) on \( G \) with the properties that \( R^* \subseteq R \) and

\[
\text{(4)} \quad R^*_{(u,v)} \neq \emptyset \text{ if and only if } R_{(u,v)} \neq \emptyset \\
\text{for every pair of vertices } u \text{ and } v \text{ of } G.
\]

**Proof.** We proceed by induction on \( b(G) \). Obviously, \( b(G) \geq 0 \). First, let \( b(G) = 0 \). Then \( G \) is a tree, and therefore, \( R = S(G) \). We put \( R^* = R \).

Let now \( b(G) \geq 1 \). Then there exists \( a \in E(G) \) such that \( G - a \) is connected. Let \( r \) and \( s \) be the vertices incident with \( a \). Axiom I implies that \( R_{(r,s)} \neq \emptyset \). There exists \( \alpha \in R_{(r,s)} \) such that

\[
\text{(5)} \quad ||\alpha|| \geq ||\alpha'|| \quad \text{for every } \alpha' \in R_{(r,s)}.
\]
There exist adjacent $v, w \in V(G)$ and $\xi, \zeta \in W(G)$ such that $\alpha = \xi v w \zeta$. Then $vw \in R$. Combining Axiom IV with (5) we get

$$\mathcal{R}_{(u, w)} = \{vw\}.$$  

Let $e$ be the edge incident with $v$ and $w$. We see that $G - e$ is connected. Since $\mathcal{R} \subseteq \mathcal{P}(G)$, it is clear that $\mathcal{R}^{(v, w)} \cap \mathcal{R}^{(w, v)} = \emptyset$. Denote

$$\mathcal{R} = \mathcal{R} - (\mathcal{R}^{(v, w)} \cup \mathcal{R}^{(w, v)}).$$

It is easy to see that $\mathcal{R}$ is a semi-route system on $G - e$ such that $\mathcal{R}_{(t, u)} \subseteq \mathcal{R}_{(t, u)}$ for every pair of vertices $t$ and $u$ of $G$. Since $b(G - e) = b(G) - 1$, the induction hypothesis implies that there exists a geodetic semi-route system $T$ on $G - e$ with the properties that $T \subseteq \mathcal{R}$ and

$$T_{(t, u)} \neq \emptyset \text{ if and only if } \mathcal{R}_{(t, u)} \neq \emptyset \text{ for every pair of vertices } t \text{ and } u \text{ of } G.$$  

Consider arbitrary vertices $z$ and $z'$ of $G$ such that $T_{(z, z')} \neq \emptyset$. Recall that $T$ is geodetic. We denote by $\tau_{zz'}$ the only element of $T_{(z, z')}$; note that if $z = z'$, then $\tau_{zz'} = z$.

Consider arbitrary vertices $x$ and $y$ of $G$ such that $\mathcal{R}_{(x, y)} \neq \emptyset$ and $T_{(x, y)} = \emptyset$. As follows from (7) and (8),

$$\mathcal{R}_{(x, y)} \subseteq \mathcal{R}_{(x, y)} \cup \mathcal{R}_{(w, v)}.$$  

Recall that $\mathcal{R}_{(v, w)} \cap \mathcal{R}_{(w, v)} = \emptyset$. If $\mathcal{R}_{(x, y)} \subseteq \mathcal{R}_{(v, w)}$, then we put $\tilde{x} = v$ and $\tilde{y} = w$; if $\mathcal{R}_{(x, y)} \subseteq \mathcal{R}_{(w, v)}$, then we put $\tilde{x} = w$ and $\tilde{y} = v$. Since $\mathcal{R}_{(x, y)} \neq \emptyset$, it follows from Axioms II and III that $\mathcal{R}_{(x, \tilde{z})} \neq \emptyset \neq \mathcal{R}_{(y, \tilde{y})}$. We wish to show that

$$\mathcal{R}_{(z, \tilde{z})} \cap (\mathcal{R}_{(u, w)} \cup \mathcal{R}_{(w, v)}) = \emptyset = \mathcal{R}_{(\tilde{y}, y)} \cap (\mathcal{R}_{(v, w)} \cup \mathcal{R}_{(w, v)}).$$

We assume, to the contrary, that (9) does not hold. Without loss of generality, let $\mathcal{R}_{(x, \tilde{z})} \cap \mathcal{R}_{(u, w)} \neq \emptyset$. As follows from (6), there exist $\beta, \gamma \in W(G)$ such that $\beta v w \gamma \in \mathcal{R}_{(x, \tilde{z})}$. Since $\tilde{z} \in \{v, w\}$ and $\mathcal{R} \subseteq \mathcal{P}(G)$, we get $\gamma = \ast$. Thus $\tilde{z} = w$. This implies that $\mathcal{R}_{(x, y)} \subseteq \mathcal{R}_{(w, v)}$. Recall that $\mathcal{R}_{(x, y)} \neq \emptyset$. According to (6), there exist $\varphi, \psi \in W(G)$ such that $\varphi v w \psi \in \mathcal{R}_{(x, y)}$. Since $\beta v w \psi \in \mathcal{R}$, Axiom IV implies that $\beta v w \psi \in \mathcal{R}$, which is a contradiction. Thus (9) holds. We get $T_{(z, \tilde{z})} \neq \emptyset \neq T_{(\tilde{y}, y)}$. This implies that $\tau_{x\tilde{z}}\tau_{y\tilde{y}} \in \mathcal{R}$.

For arbitrary vertices $t$ and $u$ of $G$ such that $\mathcal{R}_{(t, u)} \neq \emptyset$ we define

$$\sigma_{t u} = \tau_{t u} \text{ if } T_{(t, u)} \neq \emptyset \text{ and } \sigma_{t u} = \tau_{t u} \tau_{t u} \text{ if } T_{(t, u)} = \emptyset.$$
We put
\[ \mathcal{R}^* = \{ \sigma_{tu}; t, u \in V(G) \text{ such that } \mathcal{R}(t, u) \neq \emptyset \} . \]
Certainly, \( \mathcal{R}^* \subseteq \mathcal{P}(G) \). It is easy to see that \( \mathcal{R}^* \) is a geodetic semi-route system on 
\( G \). Moreover, it is clear that (4) holds. Thus the theorem is proved. \( \square \)

**Corollary 2.** Let \( G \) be a connected graph. A route system \( \mathcal{R} \) on \( G \) is geodetic if
and only if no proper subset of \( \mathcal{R} \) is a route system on \( G \).

**Corollary 3.** For every connected graph \( G \) there exists a geodetic route system on \( G \).

3. Let \( G \) be a connected graph. We say that a route system \( \mathcal{R} \) on \( G \) is maximal
(or minimal) if \( \mathcal{R} \) is a proper subset of no route system on \( G \) (or no proper subset
of \( \mathcal{R} \) is a route system on \( G \), respectively). Corollary 2 asserts that a route system
on \( G \) is minimal if and only if it is geodetic. Recall that \( S(G) \) is a route system on
\( G \). We will ask when \( S(G) \) is (or is not) a maximal route system on \( G \).

**Theorem 3.** Let \( G \) be a connected bipartite graph. Then \( S(G) \) is a maximal
route system on \( G \).

**Proof.** We assume, on the contrary, that there exists a route system \( \mathcal{R} \) on
\( G \) such that \( S(G) \subseteq \mathcal{R} \). As follows from Axioms I and II, there exist distinct
\( u, v, w \in V(G) \) and \( \alpha \in \mathcal{W}(G) \) with the properties that
\[ uavw \in \mathcal{R} - S(G) \quad \text{and} \quad uav \in S(G). \]
Hence, \( d(u, w) \neq ||uavw|| = d(u, v)+1 \). Since \( G \) has no odd cycle, it is routine to show
that \( d(u, w) = d(u, v) - 1 \). This means that there exists \( \beta \in \mathcal{W}(G) \) such that \( u\beta uvw \in S(G) \). Since \( S(G) \subseteq \mathcal{R} \), we have \( u\beta uvw \in \mathcal{R} \). Recall that \( uavw \in \mathcal{R} \). Axiom IV
implies that \( u\beta uvw \in \mathcal{R} \), and thus \( u\beta uvw \in \mathcal{P}(G) \), which is a contradiction. Thus
the theorem is proved. \( \square \)

Clearly, a geodetic graph has no odd cycle if and only if it is a tree.

**Theorem 4.** Let \( G \) be a geodetic graph different from a tree. Then \( S(G) \) is not
a maximal route system on \( G \).

**Proof.** Clearly, there exists an odd cycle in \( G \). It is routine to prove that
there exist \( x, y \in V(G) \) and \( g, \sigma \in \mathcal{W}(G) \) such that \( xgy \in S(G) \), \( x\sigma y \in \mathcal{P}(G) \),
\[ ||x\sigma y|| = ||xgy|| + 1, \] and \( \sigma \) has no common vertex with \( g \).

Consider arbitrary \( \varphi, \psi \in \mathcal{W}(G) \) such that \( \varphi xgy \psi \in S(G) \). Suppose \( \varphi x\sigma y \psi \notin \mathcal{P}(G) \). Then \( \sigma \) has a common vertex with \( \varphi \psi \). Without loss of generality, we assume
that $\sigma$ has a common vertex with $\phi$. Then there exist $\xi_1, \xi_2, \zeta_1, \zeta_2 \in W(G)$ and $t \in V(G)$ such that $\phi = \xi_1 t \xi_2$ and $\sigma = \zeta_1 t \zeta_2$. Since $\xi_1 t \xi_2 x y y \psi \in S(G)$, we have $t \xi_2 x y y \psi \in S(G)$. Hence $||x y y \psi|| > ||t \xi_2 x y y \psi|| > ||t \xi_2 x y y \psi|| + 1 + ||x y y \psi|| = ||x y y \psi||$, which is a contradiction. Thus $\phi x y y \psi \in P(G)$. Suppose $\phi x \psi \notin S(G)$. Put $\lambda = \phi x y y$ and $\mu = \phi x y y$. There exists $\omega \in S(G)$ such that $A \omega = A \mu$ and $Z \omega = E \mu$. We have $||\omega|| < ||\mu||$, and thus $||\omega y y|| < ||\mu y y|| = ||\lambda y y|| + 1$. This implies that $\omega y y \in S(G)$. Since $G$ is geodetic, $\omega y y = \lambda y y$. Thus $\omega = \lambda$. Recall that $\sigma \neq *$. We have $Z \lambda = Z \sigma$, which is a contradiction. Thus $\phi x y y \psi \in S(G)$. Analogously, $\sigma \psi \in S(G)$. We have proved the following statement:

(10) if $\phi x y y \psi \in S(G)$, then $\phi x y y \psi \in P(G)$ and $\phi x \sigma, \sigma \psi \in S(G)$ for any $\phi, \psi \in W(G)$.

Denote

$$T = \{\phi x y y \psi; \phi, \psi \in W(G) \text{ such that } \phi x y y \psi \in S(G)\},$$

$$\bar{T} = \{\bar{\alpha}; \alpha \in T\} \text{ and } \mathcal{R} = S(G) \cup T \cup \bar{T}.$$ 

Obviously, $S(G) \subset \mathcal{R}$. As follows from (10), $\mathcal{R} \subseteq P(G)$. We want to prove that $\mathcal{R}$ is a route system on $G$. Certainly, $\mathcal{R}$ fulfills Axioms I, II and V.

Consider arbitrary $u, v \in V(G)$ and $\alpha \in W(G)$. Suppose $u \alpha v \in \mathcal{R}$. If $u \alpha v \in S(G)$, then $u \alpha \in S(G)$. Assume that $u \alpha v \notin S(G)$. Without loss of generality, let $u \alpha v \in T$. Then there exist $\phi, \psi \in W(G)$ such that $\phi x y y \psi \in S(G)$ and $u \alpha v = \phi x y y \psi$. If $\psi \neq *$, then $u \alpha \in T$. If $\psi = *$, then $u \alpha = \phi x \sigma$, and according to (10), $\phi x \sigma \in S(G)$. Hence $\mathcal{R}$ fulfills Axiom III.

Consider arbitrary $u, v, w \in V(G)$, $\alpha, \beta, \gamma, \delta \in W(G)$. Suppose $\alpha u \beta v, u v \in \mathcal{R}$. We distinguish two cases:

1. Let $\alpha u \beta v \in S(G)$. If $u v \in S(G)$, then $\alpha u \beta v \in S(G)$. Suppose $u v \notin S(G)$. Without loss of generality, we assume that $u v \in T$. Then there exist $\phi, \psi \in W(G)$ such that $\phi x y y \psi \in S(G)$ and $u v = \phi x y y \psi$. We have $\alpha \phi x y y \phi \psi \in S(G)$, and thus $\alpha \phi \psi = \alpha \phi x y y \psi \in T$.

2. Let $\alpha u \beta v \notin S(G)$. Without loss of generality, we assume that $\alpha u \beta v \in T$. Then there exist $\phi, \psi \in W(G)$ such that $\phi x y y \psi \in S(G)$ and $\alpha u \beta v = \phi x y y \psi$. According to (10), $\phi x \sigma, \sigma \psi \in S(G)$. Recall that $G$ is geodetic. If both $u$ and $v$ belong to $\phi \sigma$, then $u v \in S(G)$, and thus $\alpha u \beta v = \alpha u \beta v$. If both $u$ and $v$ belong to $\sigma \psi$, then we obtain the same result. Let now $u$ belong to $\phi x$ and $v$ belong to $y \psi$. There exist $\lambda, \mu \in W(G)$ such that $\beta = \lambda \sigma \mu$. Obviously, $\alpha \lambda \mu \nu \psi \in S(G)$. If $u v \in S(G)$, then $\alpha u \beta v = \phi x y y \psi \in S(G)$.

Suppose $u v \notin S(G)$. Then there exist $\xi, \zeta \in W(G)$ such that either (a) $\xi x y y \zeta \in S(G)$ and $u v = \xi x y y \zeta$ or (b) $y x y \zeta \in S(G)$ and $u v = y x y \zeta$. First, let $u v =$...
Since $G$ is geodetic, we have $\xi x_{x}z = u_{l}x_{l}u_{v}$. Hence $\alpha u_{v}n = \alpha u_{v}n$. Let now \( u_{w} = \xi y_{x}x_{x}z \). Then $u \neq x$. We have $\xi y_{x}x_{x}z \in S(G)$. Since $u \neq x$, $\lambda \neq \ast$. There exists $\tau \in W(G)$ such that $\lambda = \tau x$. Since $G$ is geodetic and $u\tau x \in S(G)$, we have $u\tau x = \xi y_{x}x_{x}z$. Recall that $\alpha u_{l}x_{l}u_{v}n \in S(G)$. Hence $\alpha \xi y_{x}x_{x}z \in S(G)$. Since $\mu \nu = y_{x}$, we conclude that $y_{x}x_{x}y_{x} \in S(G)$, which is a contradiction.

Thus $R$ fulfills Axiom IV. The proof is complete. \hfill \Box

**Conjecture.** Let $G$ be a connected graph. Then $S(G)$ is a maximal route system on $G$ if and only if $G$ is bipartite.

**References**


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