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A NOTE ON FACTORIZATION OF THE FERMAT NUMBERS
AND THEIR FACTORS OF THE FORM $3h2^n + 1$

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Summary. We show that any factorization of any composite Fermat number $F_m = 2^{2^m} + 1$ into two nontrivial factors can be expressed in the form $F_m = (k2^n + 1)(\ell2^n + 1)$ for some odd $k$ and $\ell$, $k \geq 3$, $\ell \geq 3$, and integer $n \geq m + 2$, $3n < 2^m$. We prove that the greatest common divisor of $k$ and $\ell$ is 1, $k + \ell \equiv 0 \mod 2^n$, $\max(k, \ell) \geq F_{m-2}$, and either $3 \mid k$ or $3 \mid \ell$, i.e., $3h2^{m+2} + 1 \mid F_m$ for an integer $h \geq 1$. Factorizations of $F_m$ into more than two factors are investigated as well. In particular, we prove that if $F_m = (k2^n + 1)^2(\ell2^n + 1)$ then $j = n + 1$, $3 \nmid \ell$ and $5 \nmid \ell$.

Keywords: Fermat numbers, prime numbers, factorization, squarefreeness

AMS classification: 11A51, 11Y05

Throughout the paper all variables $i, j, k, n, n_1, \ldots$ are supposed to be positive integers except for $m$ and $z$ which can moreover attain the value zero. For $m = 0, 1, 2, \ldots$, the $m$th Fermat number is defined by $F_m = 2^{2^m} + 1$. The aim of this paper is to derive some properties of factors of composite Fermat numbers.

Recall that $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, $F_4 = 65537$ are primes and other primes $F_m$ (if they exist) are not known yet. For instance, in 1732 Euler found that $F_5 = 641 \cdot 6700417$, where the both factors are prime. The Fermat number $F_6$ was factored by Landry in 1880 (see e.g. [10]), $F_7$ by Morrison and Brillhart in 1970 [8], $F_8$ by Brent and Pollard in 1980 [2], $F_9$ by Lenstra, Lenstra, Jr., Manasse, Pollard in 1990 [7] and $F_{11}$ by Brent in 1988 [1]. The complete factorizations of $F_m$ are known only for the above mentioned numbers for the time being. Their structure, however, remains a deterministic chaos. Some prime factors of $F_{10}$ and of more than 100 other Fermat numbers can be found in excellent surveys [3, 6]. From all of the above-mentioned papers we have

\begin{equation}
1 = \Omega_0 = \ldots = \Omega_4 < 2 = \Omega_5 = \ldots = \Omega_8 < 3 = \Omega_9 < 5 = \Omega_{11} < 6 < \Omega_{12},
\end{equation}
where $\Omega_m$ is the number of prime divisors of $F_m$ (counted with multiplicity). Anyhow, the monotonicity of the whole sequence $\{\Omega_m\}$ is an open problem as well as the squarefreeness of $F_m$.

In 1877, Lucas established a general form of prime divisors of the Fermat numbers, namely that: Every prime divisor $p$ of $F_m$, $m > 1$, satisfies the congruence (see e.g. [4, p. 376])

\[(2) \quad p \equiv 1 \mod 2^{m+2}.\]

The main idea of its proof is the following. As in [7, p. 320] we put $b = 2^{2^{m-2}}(2^{2^{m-1}} - 1)$. Then $b^2 = 2^{2^m-1}(2^{2^m} - 2 \cdot 2^{2^{m-1}} + 1)$ and we get

\[(3) \quad b^2 \equiv 2 \mod p,\]

since $2^{2^m} + 1 \equiv 0 \mod p$. From here we have $b^{2^m+1} \equiv 2^{2^m} \equiv -1 \mod p$ which implies that

\[(4) \quad b^{2^{m+2}} \equiv 1 \mod p.\]

According to (3), the numbers $b$ and $p$ are coprime and thus by the little Fermat theorem (i.e., $b^{p-1} \equiv 1 \mod p$) and (4) it is possible to deduce that $2^{m+2} \mid p - 1$. Therefore, (2) holds.

We start with several simple lemmas.

**Lemma 1.** If $2^n + 1$ divides $F_m$ for some $n \geq 1$ and $m \geq 0$ then $F_m = 2^n + 1$.

**Proof.** Set $Q_n = 2^n + 1$, i.e., $F_m = Q_{2^m}$. From the binomial theorem we obtain

\[Q_{ij} = 2^{ij} + 1 = (Q_j - 1)^i + 1 \equiv 1 + (-1)^i \mod Q_j\]

and thus

\[(5) \quad \gcd(Q_{ij}, Q_j) = \begin{cases} 1 & \text{for } i \text{ even}, \\ Q_j & \text{for } i \text{ odd}. \end{cases}\]

Hence,

\[(6) \quad \gcd(F_z, F_m) = 1 \quad \text{for } z \neq m,\]

i.e., no two different Fermat numbers have a common divisor greater than 1 (see also [5, p. 14]).

Suppose that $Q_n \mid F_m$ for some $n < 2^m$. Then $n = i2^z$, where $i$ is odd and $z < m$. Using (5) for $j = 2^z$, we see that $Q_{2^z} \mid Q_n$. However, this contradicts (6), since $Q_{2^z} = F_z$ and $Q_n \mid F_m$. Therefore, $n = 2^m$. \qed
Lemma 2. Let $F_m$ be composite. Then there exist natural numbers $j, k, \ell, n$ such that

\[ F_m = (k2^n + 1)(\ell2^j + 1), \quad k \geq 3, \ell \geq 3, k \text{ and } \ell \text{ are odd}. \]  

Proof. Since $F_m$ is odd and composite, it can be written as a product of two odd numbers $k2^n + 1$ and $\ell2^j + 1$ for some natural numbers $n, j$ and odd integers $k, \ell$. However, according to Lemma 1 the case $k = 1$ or $\ell = 1$ is not possible. Hence, $k \geq 3$ and $\ell \geq 3$. \qed

Definition 3. Let $q > 1$ be an odd integer. A uniquely determined exponent $n$ from the decomposition $q = k2^n + 1$, where $k$ is odd, is called the order of $q$.

In the next lemma we prove that the orders of two odd factors are not greater than the order of their product.

Lemma 4. Let

\[ k2^n + 1 = (k_12^{n_1} + 1)(k_22^{n_2} + 1), \]

where $k, k_1, k_2$ are odd. Then $n \geq \min(n_1, n_2)$, where the sharp inequality holds if and only if $n_1 = n_2$. Moreover, $k > k_1k_22^{\max(n_1, n_2)}$ whenever $n_1 \neq n_2$.

Proof. Without loss of generality assume that $n_1 \geq n_2$. Then

\[ k2^n + 1 = (k_1k_22^{n_1} + k_12^{n_1-n_2} + k_2)2^{n_2} + 1. \]

Since $k$ is odd, $n \geq n_2 = \min(n_1, n_2)$. The number in the brackets from (9) is even if and only if $n_1 = n_2$. If $n_1 > n_2$ then $n = n_2$ and thus $k > k_1k_22^{n_1}$ by (9). \qed

Theorem 5. Let $F_m$ be composite and let $k2^n + 1$ be its arbitrary factor (not necessarily prime) where $k$ is odd. Then $k \geq 3$, $n$ is an integer for which

\[ m + 2 \leq n < \frac{1}{3}2^m \]

and there exists an odd $\ell \geq 3$, such that

\[ F_m = (k2^n + 1)(\ell2^n + 1), \]

i.e., the both factors have the same order. Moreover,

\[ k + \ell \equiv 0 \mod 2^n, \]
\( k \) and \( \ell \) are coprime, i.e.,

\[(13) \quad \gcd(k, \ell) = 1, \]
\[(14) \quad \max(k, \ell) \geq F_{m-2} \]

and

\[(15) \quad \text{either } 3 \mid k \text{ or } 3 \mid \ell, \]
i.e., for any composite Fermat number \( F_m \) there exists a natural number \( h \) such that \( 3h2^n + 1 \mid F_m \).

**Proof.** Let \( \ell 2^j + 1 \) be a cofactor to \( k2^n + 1 \) such that \( \ell \) is odd. According to (7), we have

\[ F_m = k\ell 2^{n+j} + k2^n + \ell 2^j + 1. \]

Without loss of generality we may assume that \( n \geq j \). Then

\[ 2^{2^n-j} = k\ell 2^n + k2^n - j + \ell, \]

where the terms \( 2^{2^n-j} \) and \( k\ell 2^n \) are even because \( 2^n > j \) and \( n \geq 1 \). This implies that \( n = j \), since \( \ell \) is odd. (The role of \( k \) and \( \ell \) is thus the same.)

From the relation

\[ 2^{2^n-n} = k\ell 2^n + k + \ell, \]

we deduce that \( 2^n - n > n \) which implies (12). Moreover, if \( q \mid k \) and \( q \mid \ell \) for some odd \( q \) then \( q \mid 2^{2^n-n} \). Hence, \( q = 1 \) and we observe that (13) holds.

Further we establish the proposed bounds (10) for \( n \). By (12), \( k + \ell \geq 2^n \). Since \( k \neq \ell \) due to (13), we have

\[(16) \quad \max(k, \ell) > 2^{n-1}, \]

and thus

\[ F_m = (k2^n + 1)(\ell 2^n + 1) > (2^{n-1}2^n + 1)(2 \cdot 2^n + 1) > 2^{3n} + 1. \]

Consequently, \( 3n < 2^m \).

By (2) each prime factor of \( F_m \) is of the form \( r2^{m+2} + 1 \) for some integer \( r \). Hence, if \( k2^n + 1 \) is a prime factor then \( m + 2 \leq n \), since \( k \) is odd. Suppose that \( k2^n + 1 \) is a product of two primes which is of the form (8). Then Lemma 4 implies \( m + 2 \leq \min(n_1, n_2) \leq n \). By induction we find that \( m + 2 \leq n \) for any factor of \( F_m \), i.e., (10) is valid.

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If \( n \leq 2^{m-2} \) then by (11), (13) and (10)
\[
\max(k, \ell) > 2^{-n}\left(\sqrt{F_m} - 1\right) > 2^{-2^{m-2}}(2^{2^{m-1}} - 1) = 2^{2^{m-2}} - 2^{2^{m-2}}
\]
and thus \( \max(k, \ell) \geq F_{m-2} \), since \( \max(k, \ell) \geq 2^{2^{m-2}} \) and \( k \) and \( \ell \) are odd. Conversely, if \( n \geq 2^{m-2} + 1 \) then by (16),
\[
\max(k, \ell) > 2^{n-1} \geq 2^{2^{m-2}},
\]
i.e., (14) holds.

Finally we prove (15). Obviously,
\[
(17) \quad 3 \mid 2^n + (-1)^{n+1}.
\]

Hence, \( 3 \mid F_m - 2 \) (taking \( n = 2^m \)) and thus \( (k2^n + 1)(\ell2^n + 1) \equiv 2 \mod 3 \). This and (17) imply
\[
(18) \quad (1 + (-1)^n k)(1 + (-1)^n \ell) \equiv 2 \mod 3.
\]

We easily find that \( xy \equiv 2 \mod 3 \) if and only if \( x \equiv 2 \mod 3 \) and \( y \equiv 1 \mod 3 \) or \( x \equiv 1 \mod 3 \) and \( y \equiv 2 \mod 3 \). From here and (18) we observe that just one of the numbers \( k \) and \( \ell \) is divisible by 3. \( \square \)

**Corollary 6.** Let the assumptions of Theorem 5 be satisfied and let \( 3 \mid \ell \). Then

\[
(19) \quad k = 3u + 1 \quad \text{for some } u \text{ even} \iff n \text{ is even},
\]
\[
(20) \quad k = 3u + 2 \quad \text{for some } u \text{ odd} \iff n \text{ is odd}.
\]

**Proof.** As \( 3 \mid \ell \), we have from (15) that \( k = 3u + y, 1 \leq y \leq 2 \) and from (18)
\[
1 + (-1)^n k \equiv 2 \mod 3.
\]

This yields (19) and (20). \( \square \)

**Remark 7.** Although the upper bound on \( n \) in (10) is too rough, we observe that no \( n \) satisfies (10) if \( m \leq 4 \) (which implies that \( F_0, \ldots, F_4 \) are primes without carrying out any trial divisions). For the prime factor 641 = 5 \cdot 2^7 + 1 of \( F_5 \) we have the equality \( n = m + 2 \). On the other hand, the sharp inequality \( n > m + 2 \) holds e.g. for the factorization of \( F_8 \) into two primes with \( n = 11 \). By (11) and (10)
\[
\min(k, \ell) < (2^n \min(k, \ell) + 1)/2^n < \sqrt{F_m}/2^n < F_{m-1}/2^{m+2}.
\]

Moreover, \( \min(k, \ell) \geq 3 \), where the equality is achieved e.g. for prime factors of \( F_{38} \) and \( F_{207} \) (see [3, p. lxxxviii]). According to (11) and (13), no Fermat number is a square of a natural number.
Theorem 8. Let \( n_1 \leq n_2 \leq n_3 \) and let

\[
F_m = \prod_{j=1}^{3} (k_j 2^n + 1),
\]

where \( k_j \) are odd. Then \( k_j \geq 3 \) for \( j = 1, 2, 3 \),

and either no \( k_j \) is divisible by 3 or just two \( k_j \) are divisible by 3.

Moreover, if \( k_1 = k_2 \) (i.e., if \( F_m \) is not squarefree) then \( n_3 = n_1 + 1 \), \( 3 \nmid k_3 \) and \( 5 \nmid k_3 \).

Proof. Obviously \( k_j \geq 3 \) and \( n_j \geq m + 2 \) by Theorem 5. Let us rewrite (21) as a product of two factors

\[
F_m = (k_1 2^{n_1} + 1)[(k_2 k_3 2^{n_3} + k_2 + k_3 2^{n_3-n_2})2^{n_2} + 1].
\]

The number \( k_2 k_3 2^{n_3} + k_2 + k_3 2^{n_3-n_2} \) cannot be even, since then \( n_3 = n_2 \) and by Theorem 5 we would get \( n_1 \geq n_2 + 1 \) which contradicts the assumption \( n_1 \leq n_2 \). Therefore, \( k_2 k_3 2^{n_3} + k_2 + k_3 2^{n_3-n_2} \) is an odd number and thus \( k_3 2^{n_3-n_2} \) is even. This implies that \( n_3 > n_2 \). By Theorem 5 and (23) we have \( n_1 = n_2 \).

From (23) and (15) we see that all three \( k_j \) cannot be divisible by 3. Suppose now that just one \( k_j \) is divisible by 3. Let for instance \( 3 \mid k_1 \), \( 3 \nmid k_2 \) and \( 3 \nmid k_3 \). Then \( k_2 k_3 2^{n_3} + k_2 + k_3 2^{n_3-n_2} \) is not divisible by 3 which contradicts (15) and (23). In a similar way we get a contradiction for the cases \( 3 \nmid k_1 \), \( 3 \mid k_2 \), \( 3 \nmid k_3 \), and \( 3 \nmid k_1 \), \( 3 \nmid k_2 \), \( 3 \mid k_3 \).

Finally, suppose that \( k_1 = k_2 \) in (21). Then obviously \( 3 \nmid k_3 \) and from (11) and the relation

\[
F_m = [k_1 (k_1 2^{n_1-1} + 1) 2^{n_1+1} + 1](k_3 2^{n_3} + 1)
\]

we find that \( n_3 = n_1 + 1 \).

Recall that the last digit of \( k_1 2^{n_1} + 1 \) belongs to the set \( \{1, 3, 7, 9\} \), since \( 5 \nmid F_m \) for \( m \neq 1 \) by (6). Hence,

\[
(k_1 2^{n_1} + 1)^2 \mod 10 \in \{1, 9\}.
\]

From here, (21) and the trivial fact that \( F_m \equiv 7 \mod 10 \) for \( m > 1 \), we have \( k_3 2^{n_3} + 1 \mod 10 \in \{3, 7\} \) which yields \( 5 \nmid k_3 \). \( \square \)
Remark 9. The Fermat number $F_9$ is a product of three prime factors $k_j 2^{n_j} + 1$, $j = 1, 2, 3$, cf. (1). According to [7, p. 321], their orders are $n_1 = n_2 = 11 = m + 2$ and $n_3 = 16$ and thus by (11), we get

\[(24) \quad F_9 = (k_1 2^{11} + 1)(\ell_1 2^{11} + 1) = (k_2 2^{11} + 1)(\ell_2 2^{11} + 1) = (k_3 2^{16} + 1)(\ell_3 2^{16} + 1)\]

for some $\ell_j \geq 3$ odd. Hence, any factor $\ell 2^n + 1$ of $F_m$ for which $n = m + 2$ need not be a prime factor yet. We also see that for given $n \geq m + 2$ the Diophantine equation (11) with unknowns $k$ and $\ell$ can have no or one or more solutions. It is also interesting that no $k_j$ from (24) is divisible by 3. This can be directly verified from the explicit expressions of the prime factors of $F_9$ (see [7]) and thus $3 \mid \ell_j$ for $j = 1, 2, 3$ by (15). According to (22), no Fermat number is a cube of a natural number.

Theorem 10. Let $n_1 \leq n_2 \leq \ldots \leq n_N$, $N > 1$ and let

\[(25) \quad F_m = \prod_{j=1}^{N} (k_j 2^{n_j} + 1),\]

where $k_j$ are odd. Then $m + 2 \leq n_j$, $k_j \geq 3$ for $j = 1, \ldots, N$, and the number of factors $k_j 2^{n_j} + 1$, whose order is $n_1$, is even. No two factors from (25) form a twin prime pair.

Proof. We again have by Theorem 5 that $m + 2 \leq n_j$ and $k_j \geq 3$ for all $j = 1, \ldots, N$. For $N < 4$ the proof of the first part of Theorem 10 follows from Theorems 5 and 8. So let $N \geq 4$. Suppose, on the contrary, that $2z + 1$ (for an integer $z \geq 0$) is the number of factors of the lowest order $n_1$, i.e., $n_{2z+1} < n_{2z+2}$ if $2z + 1 < N$. Then by Lemma 4 we have for $z \geq 1$ that

\[\text{ord}((k_{2i} 2^{n_1} + 1)(k_{2i+1} 2^{n_1} + 1)) > n_1 \quad \text{for any } i = 1, \ldots, z,\]

where analogously to [7, p. 321] the operator ord denotes the order from Definition 3, i.e., $\text{ord}(k 2^n + 1) = n$ for $k$ odd. Using Lemma 4 again, we find by induction that

\[\text{ord} \left( \prod_{j=2}^{2z+1} (k_j 2^{n_1} + 1) \right) > n_1\]

and thus also

\[(26) \quad \text{ord} \left( \prod_{j=2}^{N} (k_j 2^{n_j} + 1) \right) > n_1\]
for \( z \geq 1 \). However, we easily find that (26) holds even if \( z \geq 0 \). This contradicts (25) and (11), as \( \text{ord}(k_12^{n_1} + 1) = n_1 \).

Let \( n_j \leq n_i \). Then

\[
|(k_12^{n_i} + 1) - (k_j2^{n_j} + 1)| = |(k_12^{n_i-n_j} - k_j)2^{n_j}| \geq 2^{n_j} \geq 2^{m+2}
\]

whenever \( n_i \neq n_j \) or \( k_i \neq k_j \). From here we see that the product (25) cannot contain a twin prime pair. 

Remark 11. The 21-digit factor of \( F_{11} \) (see [1]) is of order 14. The other four factors have order 13.

Two prime factors of \( F_{10} \) are already known and their orders are 12 and 14 (see [3]). The associated cofactor is known to be composite, i.e., \( \Omega_{10} = N \geq 4 \), cf. (1) and (25). Note that the first prime factor of \( F_{10} \) is of the form \( k_12^{n_1} + 1 = 11131 \cdot 2^{12} + 1 \).

By Theorem 10 there exists its another prime factor of order \( m + 2 = 12 \), \( k_22^{12} + 1 \), \( k_2 \geq 3 \) odd, where \( k_2 \) is for the time being unknown. However, by (20) and (11), \( k_2 \) cannot be of the form \( k_2 = 3v + 2 \), since \( n_2 = 12 \) is even.

From Theorem 10 we observe that there exist at least four factors of \( F_{12} \) of order \( m + 2 = 14 \), as three of them are already known [3].

Finally note that \( k_j \) in (25) need not be coprime (cf. (13)). For instance we have \( 3 \mid k_j \) for two factors of \( F_{11} \) and \( 7 \mid k_j \) for other its two factors, and \( 7 \mid k_j \) for three of the known factors of \( F_{12} \), etc.

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