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DIRECT PRODUCT DECOMPOSITIONS OF INFINITELY DISTRIBUTIVE LATTICES

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Abstract. Let \( \alpha \) be an infinite cardinal. Let \( \mathcal{T}_\alpha \) be the class of all lattices which are conditionally \( \alpha \)-complete and infinitely distributive. We denote by \( \mathcal{T}_\alpha' \) the class of all lattices \( X \) such that \( X \) is infinitely distributive, \( \sigma \)-complete and has the least element. In this paper we deal with direct factors of lattices belonging to \( \mathcal{T}_\alpha \). As an application, we prove a result of Cantor-Bernstein type for lattices belonging to the class \( \mathcal{T}_\alpha' \).

Keywords: direct product decomposition, infinite distributivity, conditional \( \alpha \)-completeness

MSC 1991: 06B35, 06D10

1. INTRODUCTION

Let \( L \) be a partially ordered set and \( s^0 \in L \). The notion of the internal direct product decomposition of \( L \) with the central element \( s^0 \) was investigated in [10] (the definition is recalled in Section 2 below).

We denote by \( F(L, s^0) \) the set of all internal direct factors of \( L \) with the central element \( s^0 \); this set is partially ordered by the set-theoretical inclusion. In the present paper we suppose that \( L \) is a lattice. Then \( F(L, s^0) \) is a Boolean algebra (cf. Section 3).

Let \( \alpha \) be an infinite cardinal. We denote by \( \mathcal{T}_\alpha \) the class of all lattices which are conditionally \( \alpha \)-complete and infinitely distributive. We prove

**Theorem 1.** Let \( L \in \mathcal{T}_\alpha \) and \( s^0 \in L \). Then the Boolean algebra \( F(L, s^0) \) is \( \alpha \)-complete.

In the particular case when the lattice \( L \) is bounded we denote by \( \text{Cen} \, L \) the center of \( L \). For each \( s^0 \in L \), \( F(L, s^0) \) is \( \alpha \)-complete and if \( \text{Cen} \, L \) is a closed sublattice of
L, then $\text{Cen} L$ is $\alpha$-complete and thus $F(L, s^0)$ is $\alpha$-complete as well. Some sufficient conditions under which the center of a complete lattice is closed were found in [2], [11], [12], [13], [14]; these results were generalized in [4]. For related results cf. also [3].

We denote by $\mathcal{T}_C$ the class of all lattices $L$ belonging to $\mathcal{T}_0$, which have the least element and are $\sigma$-complete.

As an application of Theorem 1 we prove the following result of Cantor-Bernstein type:

**Theorem 2.** Let $L_1$ and $L_2$ be lattices belonging to $\mathcal{T}_C$. Suppose that

(i) $L_1$ is isomorphic to a direct factor of $L_2$;

(ii) $L_2$ is isomorphic to a direct factor of $L_1$.

Then $L_1$ is isomorphic to $L_2$.

This generalizes a theorem of Sikorski [15] on $\sigma$-complete Boolean algebras (proven independently also by Tarski [17]).

Some results of Cantor-Bernstein type for lattice ordered groups and for MV-algebras were proved in [5], [6], [7], [8].

### 2. Internal Direct Factors

Assume that $L$ and $L_i$ ($i \in I$) are lattices and that $\varphi$ is an isomorphism of $L$ onto the direct product of lattices $L_i$; then we say that

$$\varphi: L \rightarrow \prod_{i \in I} L_i$$

is a direct product decomposition of $L$; the lattices $L_i$ are called direct factors of $L$.

For $x \in L$ and $i \in I$ we denote by $x(L_i, \varphi)$ the component of $x$ in $L_i$, i.e.,

$$x(L_i, \varphi) = \varphi(x)_i.$$

Let $s^0 \in L$ and $i \in I$. Put

$$L_i^{s^0} = \{y \in L: y(L_i, \varphi) = s^0(L_j, \varphi) \text{ for each } j \in I \setminus \{i\}\}.$$

Then for each $x \in L$ and each $i \in I$ there exists a uniquely determined element $y_i$ in $L_i^{s^0}$ such that

$$x(L_i, \varphi) = y_i(L_i, \varphi).$$

The mapping

$$\varphi^{s^0}: L \rightarrow \prod_{i \in I} L_i^{s^0}$$
defined by

\[ \varphi^{s^0}(x) = \left( \ldots, y_i, \ldots \right)_{i \in I} \]

is also a direct product decomposition of \( L \). Moreover, the following conditions are valid:

(i) For each \( i \in I \), \( L^s_i \) is a closed convex sublattice of \( L \) and \( s^0 \in L^s_i \).

(ii) For each \( i \in I \), \( L^s_i \) is isomorphic to \( L_i \).

(iii) \( H \in I \) and \( x \in L \), then \( \varphi(L^{s^0}, \varphi^s) = x \).

(iv) If \( i \in I \), \( j \in I \setminus \{i\} \) and \( x \in L^s_j \), then \( \varphi(L^s_i, \varphi^s) = s^0 \).

We say that (2) is an internal direct product decomposition of \( L \) with the central element \( s^0 \); the sublattices \( L^s_i \) are called internal direct factors of \( L \) with the central element \( s^0 \).

The condition (ii) yields that if we are interested only in considerations “up to isomorphisms”, then we need not distinguish between (1) and (2).

We denote by \( F(L, s^0) \) the collection of all internal direct factors of \( L \) with the central element \( s^0 \). Then in view of (i), \( F(L, s^0) \) is a set. On the other hand, it is obvious that the collection of all direct factors of \( L \) is a proper class.

### 3. Auxiliary Results

Assume that the relation (2) is valid. Let \( I_1 \) and \( I_2 \) be nonempty subsets of \( I \) such that \( I_1 \cap I_2 = \emptyset \) and \( I_1 \cup I_2 = I \). Denote

\[ L(I_1) = \{ x \in L : x(L^s_i, \varphi^s) = s^0 \text{ for each } i \in I_2 \} \]

\[ L(I_2) = \{ x \in L : x(L^s_i, \varphi^s) = s^0 \text{ for each } i \in I_1 \} \]

Consider the mapping

\[ \psi : L \rightarrow L(I_1) \times L(I_2) \]

defined by \( \psi(x) = (x_1, x_2) \), where

\[ x_1 = (\ldots, x(L^s_i, \varphi^s), \ldots)_{i \in I_1}, \quad x_2 = (\ldots, x(L^s_i, \varphi^s), \ldots)_{i \in I_2} \]

Then (3) is also an internal direct product decomposition of \( L \) with the central element \( s^0 \).

Further suppose that we have another internal direct product decomposition of \( L \) with the central element \( s^0 \),

\[ \psi^{s^0} : L \rightarrow \prod_{j \in J} P_j^{s^0} \]
3.1. Proposition. Let (2) and (4) be valid. Suppose that there are \( i(l) \in I \) and \( j(l) \in J \) such that \( L_f(i(l)) = P^\varphi_{j(l)} \). Then for each \( x \in L \) the components of \( x \) in \( L_f(i(l)) \) and \( P^\varphi_{j(l)} \) are equal, i.e.,

\[
x(L_f(i(l)), \varphi^\varphi) = x(P^\varphi_{j(l)}, \psi^\varphi).
\]

Proof. This is a consequence of Theorem (A) in [10]. \( \square \)

We denote by \( \text{Con} L \) the set of all congruence relations on \( L \); this set is partially ordered in the usual way. \( R_{\min} \) and \( R_{\max} \) denote the least element of \( \text{Con} L \) or the greatest element of \( \text{Con} L \), respectively. For \( x \in L \) and \( R \in \text{Con} L \) we put \( x_R = \{ y \in L : yRx \} \).

From the well-known theorem concerning direct products and congruence relations of universal algebras and from the definition of the internal direct product decomposition of a lattice we immediately obtain:

3.2. Proposition. Let \( R(1) \) and \( R(2) \) be elements of \( \text{Con} L \) such that they are permutable, \( R(1) \land R(2) = R_{\min} \), \( R(1) \lor R(2) = R_{\max} \). Then the mapping

\[
\varphi : L \rightarrow s^\varphi_{R(1)} \times s^\varphi_{R(2)}
\]

defined by

\[
\varphi(x) = (x^1, x^2), \quad \text{where} \quad \{ x^1 \} = x_{R(2)} \cap s^\varphi_{R(1)}, \{ x^2 \} = x_{R(1)} \cap s^\varphi_{R(2)}
\]

is an internal direct product decomposition of \( L \) with the central element \( s^\varphi \).

3.3. Definition. Congruence relations \( R(1) \) and \( R(2) \) on \( L \) are called interval permutable if, whenever \( [a, b] \) is an interval in \( L \), then there are \( x_1, x_2 \in [a, b] \) such that \( aR(1)x_1 \land R(2)b \) and \( aR(2)x_2 \land R(1)b \).

The following assertion is easy to verify (cf. also [1], p. 15, Exercise 13).

3.4. Lemma. Let \( R(1) \) and \( R(2) \) be interval permutable congruence relations on \( L \). Then

(i) \( R(1) \lor R(2) = R_{\max} \);
(ii) \( R(1) \) and \( R(2) \) are permutable.

If the relation (2) from Section 2 above is valid, then in view of 2.1, it suffices to express this fact by writing

\[
L = (s^\varphi) \prod_{i \in I} L_i,
\]

where \( L_i \) has the same meaning as \( L_f(i) \) in (2) of Section 2.
Also, if $x \in L$, then instead of $x(L^+, s^A)$ we write simply $x(L_i)$.

If $A, B$ are elements of $F(L, s^0)$ and $x \in L$, then the symbol $x(A)(B)$ means $(x(A))(B)$.

Let the system $(F, L, s^0)$ be partially ordered by the set-theoretical inclusion.

3.5. Lemma. $F(L, s^0)$ is a Boolean algebra.

Proof. This is a consequence of Proposition 3.14 in [9].

It is obvious that if $L$ is bounded, then $F(L, s^0)$ is isomorphic to the center of $L$.

Further, it is easy to verify that if $A, B \in F(L, s^0)$ and $L = (s^0)A \times B$, then $B$ is the complement of $A$ in the Boolean algebra $F(L, s^0)$; we denote $B = A'$.

4. $\alpha$-COMPLETENESS AND INFINITE DISTRIBUTIVITY

Let $\alpha$ be an infinite cardinal. In this section we suppose that $L$ is a lattice belonging to $\tau_\alpha$ and that $s^0$ is an element of $L$.

Let $I$ be a set with card $I = \alpha$ and for each $i \in I$ let $L_i$ be an element of $F(L, s^0)$. Thus for each $i \in I$ we have

$$L = (s^0)L_i \times L'_i.$$

For each $x \in L$ and each $i \in L$ we denote

$$x_i = x(L_i), \quad x'_i = x(L'_i).$$

Let $x, y \in L$ and $i \in I$. We put $xR_ia$ if $x_i = y_i$, similarly we set $xR_iy$ if $x_i = y_i$. Then $R_i$ and $R'_i$ belong to $\text{Con} L$, $R_i \land R'_i = R_{\text{min}}$ and $R_i \lor R'_i = R_{\text{max}}$. Moreover, $R_i$ and $R'_i$ are permutable.

4.1. Lemma. Let $a, b \in L$, $a \leq b$. There exist elements $x, y, x^i$ $(i \in I)$ in $[a, b]$ such that

(i) $x^iR_ia$ for each $i \in I$;
(ii) $yR'_ia$ for each $i \in I$;
(iii) $x = \bigvee_{i \in I} x^i$, $x \land y = a$ and $x \lor y = b$.

Proof. Let $i \in I$. There exist uniquely determined elements $x^i$ and $y^i$ in $L$ such that

$$x^i \in aR_i \cap bR'_i, \quad y^i \in aR'_i \cap bR_i.$$
Hence
\[(x')_i = a_i, \quad (x')_i = b_i,\]
\[(y')_i = b_i, \quad (y')_i = a_i.\]

Then clearly
\[(2) \quad x^i \land y^i = a,\]
\[(3) \quad x^i \lor y^i = b.\]

Denote
\[x = \bigvee_{i \in I} x^i, \quad y = \bigwedge_{i \in I} y^i;\]
these elements exist in \(L\) since \(L\) is \(\alpha\)-complete. By applying the infinite distributivity of \(L\) we get
\[y \land x = y \land \left( \bigvee_{i \in I} x^i \right) = \bigvee_{i \in I} (y \land x^i) = \bigwedge_{i \in I} (y^i \land x^i).\]

For \(j = i\) we have \(y^i \land x^i = a\) (cf. (2)). Hence for each \(i \in I\) the relation
\[\bigwedge_{j \in I} (y^j \land x^j) = a\]
is valid. Thus
\[(4) \quad y \land x = a.\]

Further we obtain
\[x \lor y = x \lor \left( \bigwedge_{i \in I} y^i \right) = \bigwedge_{i \in I} (x \lor y^i) = \bigvee_{i \in I} (x^i \lor y^i).\]

For \(j = i\) we have \(x^i \lor y^i = b\) (cf. (3)). Hence
\[\bigvee_{j \in I} (x^j \lor y^j) = b\]
for each \(i \in I\). Therefore
\[(5) \quad x \lor y = b.\]

The definition of \(x\) and the relations (4), (5) yield that (iii) is valid. Now, in view of the definition of \(x^i\), the condition (i) is satisfied. Let \(i \in I\); then \(y^i \land a\). Since \(y \in [a, y']\), we obtain \(y^i \land a\). Thus (ii) holds. \(\square\)
By an argument dual to that applied in the proof of 4.1 we obtain:

4.2. Lemma. Let \( a, b \in L, a \leq b \). There exist elements \( z, t, z' \ (i \in I) \) in \([a,b]\) such that

(i) \( z'R_i b \) for each \( i \in I \);
(ii) \( tR_i' b \) for each \( i \in I \);
(iii) \( z = \bigwedge_{i \in I} z', z \vee t = b \) and \( z \wedge t = a \).

4.3. Lemma. Let \( a, b, x \) and \( x', i \in I \) be as in 4.1. Suppose that \( u, v \in \[a,x]\), \( u \leq v \) and \( uR_i'v \) for each \( i \in I \). Then \( u = v \).

Proof. By way of contradiction, assume that \( u < v \). From the definition of \( x \) we conclude that

\[
u = u \vee (v \wedge x) = u \vee \left( \bigvee_{i \in I} x' \right) = \bigvee_{i \in I} (u \vee (v \wedge x')).\]

Hence there exists \( i \in I \) such that \( u < u \vee (v \wedge x') \). From \( aR_i x' \) we obtain

\[
u = u \vee (v \wedge x') = uR_i (u \vee (v \wedge x')).\]

whence \( uR_i (u \vee (v \wedge x')) \). At the same time, since \( u \vee (v \wedge x') \) belongs to the interval \([u,v]\) and \( uR_i'v \), we get \( R'_i (u \vee (v \wedge x')) \). Therefore \( u = u \vee (v \wedge x') \), which is a contradiction. \(\Box\)

Analogously, by applying 4.2 we obtain

4.4. Lemma. Let \( a, b \) and \( z \) be as in 4.2. Suppose that \( u, v \in [z,b], u \leq v \) and \( uR_i'v \) for each \( i \in I \). Then \( u = v \).

4.5. Lemma. Let \( a, b, x, y, z \) and \( t \) be as in 4.1 and 4.2. Then \( t = x \) and \( z = y \).

Proof. a) We have

\[
t = t \wedge b = t \wedge (x \vee y) = (t \wedge x) \vee (t \wedge y).
\]

The interval \([t \wedge x,x]\) is projectable to the interval \([t,t \vee x]\) and \([t,t \vee x] \subseteq [t,b]\).

Hence in view of 4.2, \( (t \wedge x)R'_i x \) for each \( i \in I \). Thus according to 4.3, \( t \wedge x = x \) and therefore \( t \geq x \).

b) Analogously,

\[
y = y \vee a = y \vee (t \wedge z) = (y \vee t) \wedge (y \vee z).
\]
The interval \([y \land z, y]\) is protectable to the interval \([z, z \lor y]\) and \(y \land z, y \subseteq [a, y]\). Hence in view of 4.1, \(z \in R_i(z \lor y)\) for each \(i \in I\). Then by applying 4.4 we get \(y = z \lor y\). whence \(z \geq y\).

\(c)\) Since \(L\) is distributive, if either \(t > x\) or \(z > y\) then \(t \land z > a\), which is impossible in view of 4.2 (iii). Thus \(t = z\) and \(x = y\). \(\square\)

5. THE RELATIONS \(R\) AND \(R'\)

We apply the same assumptions and the same notation as in the previous section. If \(a, b \in L\), \(a \leq b\) and if \(x, y\) are as in 4.1, then we write:

\[x = x(a, b), \quad y = y(a, b)\]

Let \(p, q \in L\). We put \(p \in R q\) if

\[x(p \land q, p \lor q) = p \lor q\]

Further we put \(p \in R' q\) if

\[y(p \land q, p \lor q) = p \lor q\]

Thus \(p \in R q\) if and only if \(p \in R_i q\) for each \(i \in I\). Hence we have

5.1. Lemma. \(R'\) is a congruence relation on \(L\).

In view of the definition, the relation \(R\) is reflexive and symmetric.

5.2. Lemma. Let \(p, q \in L\). Then the following conditions are equivalent:

(i) \(p \in R q\).

(ii) There exists no interval \([u, v]\) \(\subseteq L\) such that \([u, v] \subseteq [p, \land q, p \lor q]\), \(u < v\) and \(u \in R_i v\) for each \(i \in I\).

Proof. Denote \(p \land q = a\), \(p \lor q = b\). Let (i) be valid. Then in view of 4.2, the condition (ii) is satisfied. Conversely, assume that (ii) holds. Put \(x(a, b) = x\), \(y(a, b) = y\). If \(y > a\), then by putting \([u, v] = [a, y]\) we arrive at a contradiction with the condition (ii). Hence \(y = a\). Then 4.1 yields that \(x = b\), whence (i) is valid. \(\square\)

5.2.1. Corollary. Let \(a_1, a_2, b_1, b_2 \in L\), \(a_1 \leq b_1 \leq b_2 \leq a_2\), \(a_1 \in R a_2\). Then \(b_1 \in R b_2\).

5.3. Lemma. Let \(a_1, a_2, a_3 \in L\), \(a_1 \leq a_2 \leq a_3\), \(a_1 \in R a_2\), \(a_2 \in R a_3\). Then \(a_1 \in R a_3\).
Proof. Suppose that \([u, v] \subseteq [a_1, a_2]\) and \(uR'v\). Denote

\[
\begin{align*}
    &u_1 = u \land a_2, &v_1 = v \land a_2, &u_2 = u \lor a_2, &v_2 = v \lor a_2, \\
    &s = v_1 \lor u.
\end{align*}
\]

Thus \(u \leq s \leq v\). Hence if \(u < v\), then either \(u < s\) or \(s < v\).

It is easy to verify that \([u, s]\) is projectable to a subinterval of \([a_1, a_2]\) (namely, to the interval \([v_1 \land u, v_1]\)). Hence \((v_1 \land u)R'v_1\) and thus \(v_1 \land u = v_1\). Therefore \(u = s\).

Analogously we obtain the relation \(s = v\). Thus \(u = v\). According to 5.2, \(a_1R_a_2\). \(\square\)

5.4. Lemma. Let \(a_1, a_2 \in L\), \(s \in L\), \(a_1Ra_2\). Then \((a_1 \lor s)R(a_2 \lor s)\) and \((a_1 \land s)R(a_2 \land s)\).

Proof. If \([u, v]\) is a subinterval of \([a_1 \lor s, a_2 \lor s]\), then \([u, v]\) is projectable to the interval \([a_2 \land u, a_2 \land v]\) and this is a subinterval of \([a_1, a_2]\). Hence in view of 5.2, if \(uR'v\), then \(u = v\). Therefore \((a_1 \lor s)R(a_2 \lor s)\). Similarly we verify that \((a_1 \land s)R(a_2 \land s)\). \(\square\)

5.5. Lemma. The relation \(R\) is transitive.

Proof. Let \(p_1, p_2, p_3 \in L\), \(p_1Rp_2, p_2Rp_3\). Denote

\[
\begin{align*}
    &p_1 \land p_2 = u_1, &p_2 \land p_3 = u_2, &u_1 \land u_2 = u_3, \\
    &p_1 \lor p_2 = v_1, &p_2 \lor p_3 = v_2, &v_1 \lor v_2 = v_3.
\end{align*}
\]

In view of 5.4 we have \(p_1R_p_1 \land p_2\), thus \(p_1Ru_1\). Analogously we obtain \(p_2Ru_2\). The interval \([u_3, u_1]\) is projectable to some subinterval of \([u_2, p_2]\), hence \(u_3Ru_1\). Similarly we verify that \(p_1Ru_1\) and \(v_3Ru_1\). Thus \(u_3Ru_3\) by 5.2.1. Since \([p_1 \land p_2, p_1 \lor p_2] \subseteq [u_3, v_3]\), 5.2 yields that \(p_1Rp_3\). \(\square\)

From 5.4 and 5.5 we infer

5.6. Lemma. \(R\) is a congruence relation on \(L\).

5.7. Lemma. \(R \land R' = R_{\min}\), \(R \lor R' = R_{\max}\) and \(R, R'\) are permutable.

Proof. In view of 5.2 we have \(R \land R' = R_{\min}\). Let \(a, b \in L\), \(a \leq b\). Let \(x\) and \(y\) be as in 4.1. Then we have

\[
\begin{align*}
    &aRx, &aR'y, \\
    &x \land y = a \land x \lor y = b. &Thus in view of the projectability we obtain
\end{align*}
\]

\[
\begin{align*}
    &aR'b, &yRb.
\end{align*}
\]

Hence \(a(R \lor R')b\). From this we easily obtain \(R \lor R' = R_{\max}\). Further, from (1), (2) and 3.4 we conclude that \(R\) and \(R'\) are permutable. \(\square\)
Proof of Theorem 1. Let \( L \in \mathcal{L}_a \) and \( s^0 \in L \). Let \( \{L_i\}_{i \in I} \) be a subset of \( F(L, s^0) \) such that \( \text{card} I < \alpha \). First we verify that \( \bigvee_{i \in I} L_i \) exists in the Boolean algebra \( F(L, s^0) \). Let us apply the notation as above.

Consider the lattices \( s_R^0 \) and \( s_R^1 \). According to 5.1, 5.6, 5.7 and 3.2 we have

\[
L = (s^0)_R \times s_R^1.
\]

According to the definition of \( R' \) we obviously have

\[
s_R^0 = \bigcap_{i \in I} L_i;
\]

Then (3) and (4) yield

\[
s_R^1 = \bigwedge_{i \in I} L_i;
\]

Further, in view of the definition of \( R \), \( L_i \subseteq s_R^0 \) for each \( i \in I \). Let \( X \in F(L, s^0) \) and suppose that \( L_i \subseteq X \) for each \( i \in I \). Put \( Y = X \cap s_R^0 \). Then \( Y \in F(L, s^0) \) and \( L_i \subseteq Y \) for each \( i \in I \). Moreover, \( Y \) is a closed sublattice of \( L \).

Let \( p \in s_R^0 \). Put \( a = p \wedge s^0 \) and \( b = p \vee s^0 \). Thus \( a, b \in s_R^0 \). Hence \( s_R^0 \subseteq R \). In view of the definition of \( R \) there exist \( x^i \in [s^0, b] \) (\( i \in I \)) such that \( x^i \in L_i \) and \( \bigvee_{i \in I} x^i = b \).

Then all \( x^i \) belong to \( Y \); since \( Y \) is closed, we get \( b \in Y \). By a dual argument (using Lemma 4.2) we obtain the relation \( a \in Y \). Hence, by the convexity of \( Y \), the element \( p \) belongs to \( Y \). Therefore, \( s_R^0 \subseteq Y \). Thus

\[
s_R^0 = \bigvee_{i \in I} L_i.
\]

Further, we have to verify that each subset of \( F(L, s^0) \) having the cardinality \( \leq \alpha \) possesses the infimum. But this is a consequence of the just proved result concerning the existence of suprema and of the fact that each Boolean algebra is self-dual. \( \square \)

5.8. Corollary. Under the assumptions as in Theorem 1 and under the notation as above we have

\[
L = (s^0) \left( \bigvee_{i \in I} L_i \right) \times \left( \bigwedge_{i \in I} L_i \right).
\]

Proof. This is a consequence of (3)—(6). \( \square \)
are elements of the element $A$ in the set $F(L,i)$. We denote $\text{fac} \, C$ is a relative round sum of the element $A$ in the set $F(L,i)$. To use Theorem 1 and apply the method which is analogous to the W"{o}rl of the proof of Cantor-Bernstein Theorem, we denote $A$ in $F(L,i)$. Then $A$ is isomorphic to $L$. TSiene exists a homomorphism $\phi$ onto $D$. Put $1 = i$, by we define

$$\Delta H \equiv \phi(1)$$

each $n \in \mathbb{N}$. Hence,

$$A_{n-1} \equiv l \quad \text{for each } l \in \mathbb{N},$$
e: $\leq$ is the relation of isomorphism between lattices. By induction, we can verify that $A_n \in F(L,i)$ and

$$l_{n+1} \equiv A_{n+1} \quad \text{for each } n \in \mathbb{N}.$$}

Then (2) yields

$$L_{n-2} \equiv l_{n-2} \quad \text{for each } n \in \mathbb{N}$$

$1 \leq i \leq m \neq 2$ are distinct positive integers. Then

$$L_{n-2} \equiv \{s^i\}.$$
If \( L \) is a Boolean algebra, then each interval of \( L \) is isomorphic to a direct factor of \( L \). Further, each Boolean algebra is infinitely distributive and contains the least element. Hence Theorem 2 yields as a corollary the following result:

6.5. Theorem. (Sikorski [13]; cf. also Sikorski [14] and Tarski [15].) Let \( L_1 \) and \( L_2 \) be \( \sigma \)-complete Boolean algebras. Suppose that
(i) there exists \( a_2 \in L_2 \) such that \( L_1 \) is isomorphic to the interval \([0, a_2]\) of \( L_2 \);
(ii) there exists \( a_1 \in L_1 \) such that \( L_2 \) is isomorphic to the interval \([0, a_1]\) of \( L_1 \).

Then \( L_1 \) and \( L_2 \) are isomorphic.

References


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