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Monotone iterative technique and connectedness of the set of solutions


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Let $B$ be an ordered Banach space with an order cone $P$, $U \subset B$ and let $A: U \to U$ be a continuous operator.

The operator $A$ is order preserving if $x \leq y \Rightarrow A(x) \leq A(y)$, strictly order preserving if $x < y \Rightarrow A(x) < A(y)$, strongly order preserving if $x < y \Rightarrow A(x) \ll A(y)$.

$x \ll y$ means $y - x \in \text{int}P$.

An element $x \in U$ is called a subequilibrium (superequilibrium) provided $x \leq Ax$ ($x \geq Ax$). A sub- or superequilibrium is called strict if the strict inequality holds.

We denote $[v, w] = \{ u \in B; v \leq u \leq w \}$.

**Theorem 1.** [1] Assume that $v, w, v < w$ are a sub- and superequilibrium, respectively, of an order preserving operator $A$, the interval $V = [v, w] \subset U$ and $A(V)$ is a relatively compact set.

Then $A: V \to V$ and there are a minimal and maximal fixed points, say $q, r$, such that the set $F$ of all fixed points of $A$ in $V$ is a subset of the interval $[q, r]$.

A sequence $(x_n)_{n \in \mathbb{Z}}$ in $U$ such that $x_{n+1} = Ax_n, x_n \to x^-$ for $n \to -\infty, x_n \to x^+$ for $n \to \infty, x^- \in U$ is called an entire orbit connecting $x^-$ with $x^+$.

**Theorem 2.** [5, Proposition 2.1] Let $u_1 < u_2$ be fixed points of a strictly order preserving continuous operator $A: U \to U$. Let $W = [u_1, u_2] \subset U$ and $A(W)$ be relatively compact.

Then precisely one of the following three cases occurs:

(a) there is another fixed point of $A$ in $W$,

(b) there is an entire orbit consisting of strict subequilibria, connecting $u_1$ with $u_2$,

(c) there is an entire orbit consisting of strict superequilibria, connecting $u_2$ with $u_1$.

Let us consider again the situation when $A: V \to V, V = [v, w], v < w$ are sub- and superequilibria.

The fixed point $u \in V$ is called stable with respect to $V$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that $A^n(x) \in O(u, \varepsilon)$ for each $x \in O(u, \delta) \cap V$ and each $n \in \mathbb{N}$.

Under the assumptions of stability of each fixed point $u \in V$ and the relative compactness of the set $A(V)$ it is proved that the set of fixed points of a strongly order preserving operator $A: V \to V$ is a continuous totally ordered curve [5, Theorem 3.3].

Under weaker assumptions we obtain the following result.

**Theorem 3.** Let $A: V \to V$ be a strictly order preserving continuous mapping, $A(V)$ be a relatively compact set. Assume that all fixed points of $A$ are stable with respect to $V.$
Then the set $F \subset V$ of fixed points of $A$ is connected.

The proof is based on the following lemma.

**Lemma.** Let the assumptions of Theorem 3 be satisfied and let $y_1, y_2 \in V$, $y_1 < y_2$ be fixed points of $A$. Then there is a continuous totally ordered curve of fixed points of $A$ connecting $y_1$ with $y_2$.

**Proof.** As the set $A(V)$ is relatively compact, the set $F$ of fixed points of $A$ is compact and there exists a countable dense subset $F_0$, $F_0 = F$. We denote by $\text{span}(F_0)$ the spanning set of $F_0$ and $E_1 = \text{span}(F_0)$. Obviously $E_1$ is a separable closed subspace of $E$. We denote by $P_1 = P \cap E_1$ the positive cone in $E_1$. As $y_1, y_2 \in F$, $y_1 < y_2$ we have $y_2 - y_1 \in P \cap E_1$. Moreover, the cones $P, P_1$ induce the same ordering on the set $F$.

As $E_1$ is a separable Banach space, there is a strictly positive linear functional $x^* \in P_1^*$. Obviously for each $u_1, u_2 \in F$ we have $u_1 < u_2 \Rightarrow x^*(u_1) < x^*(u_2)$. Let $F_1$ be the set of fixed points of $A$ in $[y_1, y_2]$.

Denote $Z = \{U \subset F_1, U$ is a totally ordered set, $y_1 \in U, y_2 \in U\}$. The set $Z$ is inductively ordered by the set inclusion. Denote by $U^+$ the maximal element of $Z$. As $U^+$ is a totally ordered set, $x^+$ is a homeomorphism of $U^+$ onto a closed set $x^+(U^+)$. We claim $x^+(U^+)$ is connected. Supposing the contrary there are $u_\alpha, u_\beta \in U^+$ such that $x^+(u_\alpha) = \alpha, x^+(u_\beta) = \beta, \alpha, \beta \in x^+(U^+)$ and $(\alpha, \beta) \subset R \setminus x^+(U^+)$. That means $[u_\alpha, u_\beta]$ contains no fixed point.

The assumption of stability of fixed points implies that there is no strict superequilibrium in $O(u_\alpha, \delta) \cap [u_\alpha, u_\beta]$ and no strict subequilibrium in $O(u_\beta, \delta) \cap [u_\alpha, u_\beta]$ for $\delta$ sufficiently small.

Thus we have obtained a contradiction with Theorem 2 as neither case (a), nor cases (b), (c) occur.

**Proof of Theorem 3.** Theorem 1 implies there are a minimal and a maximal fixed point $g, r$ and that $F \subset [g, r]$.

If $r = g$, the set $F$ is a singleton.

If $r > g$ then for each $u \in F$ there are continuous totally ordered curves of fixed points of $A$ connecting $g$ with $u$ and $u$ with $r$. That means $F$ is connected.

**Corollary.** Assuming $g \neq r$ the set $F \subset V$ is a union of continuous totally ordered curves of fixed points of $A$ connecting $g$ with $r$. 325
The above Corollary completes the cascade of results of Krasnosel'skiĭ and Lusnikov [6] concerning the relations between the type of monotony and the structure of the set of fixed points. The authors in [6] use another assumption instead of the stability of each fixed point. They assume the interval \([g, r]\) is degenerated, i.e. there is no strict sub- or superequilibrium inside.

The following theorem presents a summary of results of Krasnosel'skiĭ and Lusnikov (parts (a), (b) and (d)) and ours (part (c)).

**Theorem 4.** Let an interval \(V = [g, r]\) be degenerated for a completely continuous operator \(A: V \to V\). Then

(a) the set \(F\) of fixed points of \(A\) forms a continuous branch in \(V\) (i.e. \(F\) has a nonzero intersection with the boundary \(\partial V\) of each bounded open set \(\Omega\) such that \(r \in \Omega\) and \(g \notin \Omega\));

(b) if the operator \(A\) is order preserving then \(F\) contains a continuous curve;

(c) if the operator \(A\) is strictly order preserving then \(F\) is a union of continuous curves;

(d) if the operator \(A\) is strongly order preserving then \(F\) is a continuous curve.

**Remark.** The assumption of degeneracy of the interval \([g, r]\) can be slightly weakened by assuming that for each fixed point \(x \in [g, r]\) there is \(0 < \delta\) such that there is no strict sub- or superequilibrium in the interval \([x, x + \delta] \cap [g, r]\).

**Application**

We are interested in the structure of the set of solutions of the periodic boundary value problem

(1) \(u'' + f(t, u) = 0,\)

(2) \(u(0) = u(2\pi), u'(0) = u'(2\pi),\)

where \(f: I \times \mathbb{R} \to \mathbb{R}\) is a continuous function.

We assume that there are constants \(a, b \in \mathbb{R}\) such that

(i) there is a lower solution \(\alpha\) and an upper solution \(\beta\), \(\alpha, \beta \in C^2(I)\), of the problem (1), (2) such that \(a \leq \alpha(t) \leq \beta(t) \leq b\);

(ii) there is a constant \(M > 0\) such that for each \(u, v \in [a, b], t \in I,\) if \(u \leq v\) then \(f(t, v) - f(t, u) \geq -M^2(v - u)\);

(iii) the function \(f(t, \cdot)\) is nonincreasing in the variable \(x\) for \(a \leq x \leq b\).

Using the existence result and the method of Lakshmikantham and Leela [8] we obtain that under the assumptions (i), (ii) there are maximal and minimal solutions
\( r(t), g(t) \) of the boundary value problem (1), (2), and that for each \( \eta \in [a, b] \subseteq C(I) \) the linear problem

\[
-u'' + M^2 u = f(t, \eta) + M^2 \eta, \\
u(0) = u(2\pi), \quad u'(0) = u'(2\pi),
\]

has the unique solution

\[
u(t) = c_1 e^{Mt} - c_2 e^{-Mt} - \frac{e^{Mt}}{2M} \int_0^t \sigma(s) e^{-Ms} ds + \frac{e^{-Mt}}{2M} \int_0^t \sigma(s) e^{Ms} ds,
\]

where

\[
c_1 = \frac{e^{2Ms}}{2M(e^{2Ms} - 1)} \int_0^{2\pi} \sigma(s) e^{-Ms} ds, \\
c_2 = \frac{1}{2M(e^{2Ms} - 1)} \int_0^{2\pi} \sigma(s) e^{Ms} ds,
\]

and

\[
\sigma(t) = f(t, \eta) + M^2 \eta.
\]

The operator \( A = V \rightarrow V, V = [a, b] \subseteq C(I) \) defined by \( A(\eta) = u, \ u \) being a solution of (3), (2) is relatively compact and strictly monotone [8].

Let \( x(t) \) be a fixed point of \( A \) and let \( \delta \) be a constant. We denote

\[
A(x(t) + \delta) = x(t) + e(t).
\]

From the definition of \( A \) we obtain that \( e(t) \) is a solution of the boundary value problem

\[
-e(t)'' + M^2 e(t) = F(t), \\
e(0) = e(2\pi), \quad e'(0) = e'(2\pi),
\]

where \( F(t) = f(t, x + \delta) - f(t, x) + M^2 \delta. \)

The assumption (iii) implies that \( |e(t)| \leq |\delta| \) and that each fixed point of the operator \( A \) is stable. Theorem 3 implies that the set of solutions of the boundary value problem (1), (2) is connected.

**Remark.** Our example is only an illustrative one. The direct computation yields that \( r(t) - g(t) = c_0, \) where \( c_0 \) is a nonnegative constant and the solution set has the form \( S = \{g(t) + c; \ c \in [0, c_0]\}. \) See [9, Theorem].
As our second application we give the result concerning the structure of the set of solutions \( x(t) \in C(I), I = [0,1] \) of the integral equation

\[
x(t) = \int K(t,s)f(s,x(s))\,ds.
\]

We assume that

(i) the function \( K(t,s): I \times I \to \mathbb{R} \) is continuous and \( 0 \leq K(t,s) \),

(ii) the function \( f(t,\cdot) \) is increasing in the variable \( x \),

(iii) there is a lower solution \( \alpha \) and an upper solution \( \beta, \alpha, \beta \in C(I), \alpha(t) \leq \beta(t) \),

(iv) the function \( f \) is continuous and there is \( \delta > 0 \) such that

\[
f(t,u + \delta) - f(t,u) < \frac{\delta}{m}
\]

for each \( t \in I \) and \( u \in [\alpha(t), \beta(t)] \), where \( m \in \mathbb{R} \) and

\[
m = \int_0^1 K(t,s)\,ds = \max_{t \in I} \int_0^1 K(t,s)\,ds.
\]

(v) \( K(t,s) \) is not identically zero in any subset \( I \times [s_1, s_2], s_1, s_2 \in I \). The operator \( A: V \to V, V = [\alpha, \beta] \subset C(I) \) defined by \( A(u) = u \), where

\[
u(t) = \int K(t,s)f(s,u(s))\,ds,
\]

is relatively compact and strictly monotone.

Let \( x(t) \) be a fixed point of \( A \), and let \( \delta > 0 \) be a constant. Then

\[
A(x(t) + \delta) = x(t) + \varepsilon(t),
\]

where \( \varepsilon(t) \) is given by

\[
\varepsilon(t) = \int K(t,s)F(s)\,ds,
\]

\[
F(t) = f(t,x(t) + \delta) - f(t,x(t)).
\]

The assumption (iv) implies \( \varepsilon(t) \leq \delta \).

Thus the solution \( x(t) \) is stable. Theorem 3 implies that the set of solutions of the integral equation (4) bounded between \( \alpha(t) \) and \( \beta(t) \) is connected.

In the paper [3] it is proved that under assumptions somewhat weaker than (i)–(iii) the set of solutions is a complete lattice. Adding the assumptions (iv), (v) we obtain that this lattice is connected (in topology of \( C(I) \)) and is either a singleton or a union of totally ordered continuous curves.
References


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