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AN AVERAGING PRINCIPLE FOR STOCHASTIC EVOLUTION EQUATIONS II

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Summary. In the present paper integral continuity theorems for solutions of stochastic evolution equations of parabolic type on unbounded time intervals are established. For this purpose, the asymptotic stability of stochastic partial differential equations is investigated, the results obtained being of independent interest. Stochastic evolution equations are treated as equations in Hilbert spaces within the framework of the semigroup approach.

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Key words: stochastic evolution equations, integral continuity theorems, asymptotic stability.

INTRODUCTION

The present paper is intended as an immediate, but in principle self-contained, continuation of our paper [10].

First, let us recall some notation. For Banach spaces V, Z we denote by $\mathscr{L}(V, Z)$ the space of all bounded linear operators from V to Z; $L^p(\Omega; V)$ ($p \in [1, \infty)$) denotes the space of all V-valued Bochner measurable functions on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, for which $\mathsf{E}\|f\|_V^p \equiv \int_\Omega \|f\|_V^p \, d\mathsf{P} < \infty$. We set $\|f\|_{p,V} \equiv (\mathsf{E}\|f\|_V^p)^{1/p}$; we will omit the subscript V if there is no danger of confusion. The norm of the space $L^p(\Omega)$ will be denoted by $|\cdot|_p$. $\mathscr{C}(I; V)$ stands for the space of all V-valued continuous functions on the interval I. If I is compact, we endow this space with the norm $\|f\|_{\mathscr{C}} \equiv$ $\equiv \sup\{\|f(t)\|_V, t \in I\}$; the same norm is considered in the space $\mathscr{C}_b(I; V)$ of bounded functions from $\mathscr{C}(I; V)$.

Given a Hilbert space V, then $\mathrm{J}_2(V)$ will denote the space of all Hilbert-Schmidt operators in V, endowed with the norm $\|A\|_{\mathrm{HS}} = (\mathrm{tr}(A^*A))^{1/2}$.

In the sequel we will adopt the following assumption (the assumptions are denoted in accordance with [10]):

(I) H, Y are real separable Hilbert spaces; $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathsf{P})$ is a stochastic basis, $w(t)$ an $(\mathscr{F}_t)$-adapted Wiener process in Y with a nuclear covariance operator W, $B(t)$ an $(\mathscr{F}_t)$-adapted cylindrical Wiener process in Y; $p \geqq 2$.

In [10] we established integral continuity theorems for mild solutions of stochatic differential equations in H with a small parameter $\alpha \geqq 0$:

$$(1) \qquad dx_\alpha(t) = (A\, x_\alpha(t) + a_\alpha(t, x_\alpha(t)))\, dt + b_\alpha(t, x_\alpha(t))\, dw(t)\,, \quad x_\alpha(0) = \varphi\,.$$

The operator A is assumed to be an infinitesimal generator of a (C_0)-semigroup $S(t)$ on H. Let the coefficients a_α, b_α be Lipschitzian. Under some assumptions we have shown that $x_\alpha \to x_0$ in $\mathscr{C}([0, T]; L^p(\Omega; H))$ for all $T > 0$. In the finite-dimensional case it is known that

$$\sup \{\|x_\alpha(t) - x_0(t)\|_p\,, \quad t \geqq 0\} \to 0$$

holds provided the solution x_0 is asymptotically stable (in a sense which will be made precise later), cf. [11], Th. 3.

Our aim is to derive analogous results on an infinite time interval for some classes of stochastic evolution equations. The main result reads as follows: Assume the coefficients of the equation (1) to be uniformly integral continuous in α, i.e. suppose that if $0 \leqq t_1 \leqq t_2$, then

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} S(t_2 - s)\, [a_\alpha(s + t_0, x) - a_0(s + t_0, x)]\, ds = 0\,,$$

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} (\operatorname{tr} \{\tilde{b}_\alpha(s + t_0, x)\, W(\tilde{b}_\alpha(s + t_0, x))^*\})^{p/2}\, ds = 0$$

uniformly in $t_0 \in \mathbb{R}_+$ and $x \in H$; we have set $\tilde{b}_\alpha(r, x) \equiv b_\alpha(r, x) - b_0(r, x)$. Then we have:

Theorem. *Let $S(t)$ be continuous in the norm topology of $\mathscr{L}(H)$ for $t > 0$. Then*

$$x_\alpha \to x_0 \quad in \quad \mathscr{C}_b([t_0, \infty); L^p(\Omega; H))\,, \quad \alpha \to 0+$$

provided $x_\alpha(t_0) \to x_0(t_0)$ in $L^p(\Omega; H)$, and the limit solution is bounded and asymptotically stable in $L^p(\Omega; H)$. (Here $t_0 \geqq 0$ is arbitrary and x_α denotes the mild solution to (1).)

We cannot apply directly the method adopted in [11], since the results obtained in [10] do not imply that $x_\alpha \to x_0$ in $\mathscr{C}([t_0, t_0 + T]; L^p(\Omega; H))$ uniformly with respect to $t_0 \geqq 0$ and to the initial condition, which is needed in the above mentioned method. The difficulties appear when we try to estimate uniformly the term

$$(2) \qquad \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|x_0(s) - x_0(t_{i-1})\|_p\, ds\,,$$

where $\{t_i\}_{i=0}^N$ is a partition of the interval $[t_0, t_0 + T]$. If $\dim H < \infty$ then this problem is solved easily, because (see [5], Th. 5.2.3) $\|x_0(s) - x_0(t_{i-1})\|_p \leqq$ $\leqq C(s - t_{i-1})^{1/2}\, (1 + \|x_0(t_0)\|_p)$, and the constant C depends only on p, T and on the constant in the estimate of linear growth of the coefficients of the equation (1).

The paper is organized as follows. In Section 1 the desired uniform estimate of the term (2) is obtained for a wide class of equations; this estimate is then used to prove a theorem on partial averaging. Section 2 is devoted to the investigation of the asymptotic stability of the equation (1). The results of the first two sections are used in Section 3 to prove theorems on integral continuity on unbounded intervals; to illustrate the theory, three examples are given. In Appendix an example of a simple hyperbolic equation to which our theory is inapplicable is discussed.

Theorems, lemmas and formulae are numbered independently in each section, the sections number is omitted when reference is made to theorems, lemmas or formulae of the same section.

1. UNIFORM AVERAGING ON BOUNDED TIME INTERVALS

Let us consider equations

$$(1)\qquad \mathrm{d}\varphi(t) = (A\,\varphi(t) + \alpha(t, \varphi(t)))\,\mathrm{d}t + \sigma(t, \varphi(t))\,\mathrm{d}w(t),$$

$$(2)\qquad \mathrm{d}\psi(t) = (\tilde{A}\,\psi(t) + \alpha(t, \psi(t)))\,\mathrm{d}t + \sigma(t, \psi(t))\,\mathrm{d}B(t)$$

in the space H, assuming:

(U1) $\alpha\colon \mathbb{R}_+ \times H \to H$, $\sigma\colon \mathbb{R}_+ \times H \to \mathscr{L}(Y, H)$ are measurable functions such that there exist constants K_1, K_2 satisfying: for every $t \in \mathbb{R}_+$, $x, y \in H$ we have

$$\|\alpha(t, x)\| + \|\sigma(t, x)\| \leqq K_1(1 + \|x\|),$$
$$\|\alpha(t, x) - \alpha(t, y)\| + \|\sigma(t, x) - \sigma(t, y)\| \leqq K_2\|x - y\|.$$

(U2) $A\colon \mathsf{D}(A) \to H$ generates a (C_0)-semigroup $S(t)$ on H such that $S(\cdot) \in \mathscr{C}((0, +\infty); \mathscr{L}(H))$ (i.e. $S(t)$ is continuous in the uniform operator topology for $t > 0$).

(U3) $\tilde{A}\colon \mathsf{D}(\tilde{A}) \to H$ generates a (C_0)-semigroup $S(t)$ on H such that

$$\int_0^T \|S(t)\|_{\mathrm{HS}}^2\,\mathrm{d}s < +\infty \quad \text{for all} \quad T \geqq 0.$$

Remark 1. (i) The assumption (U2) is satisfied if $S(t)$ is a semigroup such that $\operatorname{Rng} S(t) \subseteqq \mathsf{D}(A)$ for each $t > 0$ (i.e. if the function $S(\cdot)\,x$ is differentiable on $(0, +\infty)$ for every $x \in H$), cf. [2], Prop. 1.1.10. In particular, (U2) holds for holomorphic semigroups. Let us note that the hypothesis (U2) implies

$$(3)\qquad \lim_{v \to 0+} \int_0^T \|S(s + v) - S(s)\|_{\mathscr{L}(H)}^{\beta}\,\mathrm{d}s = 0$$

for every $T \geqq 0$, $\beta > 0$, by the dominated convergence theorem.

(ii) The assumption (U3) implies (U2), see e.g. [1], Th. 4.4.1. Moreover, we can show that $S(\cdot) \in \mathscr{C}((0, +\infty); \mathrm{J}_2(H))$ and

$$(4)\qquad \lim_{v \to 0+} \int_0^T \|S(s + v) - S(s)\|_{\mathrm{HS}}^2\,\mathrm{d}s = 0$$

for every $T \geqq 0$.

Indeed, let us choose $\varepsilon > 0$ arbitrarily, let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of H, $0 < \delta \leqq s, t \leqq T$. Then

$$\|S(t) - S(s)\|_{\mathrm{HS}}^2 = \sum_{i=1}^{\infty} \|[S(t) - S(s)]\, e_i\|^2 = \sum_{i=1}^{J} \|[S(t) - S(s)]\, e_i\|^2 +$$

$$+ \sum_{i=J+1}^{\infty} \|[S(t-\delta) - S(s-\delta)]\, S(\delta)\, e_i\|^2 \leqq$$

$$\leqq \sum_{i=1}^{J} \|[S(t) - S(s)]\, e_i\|^2 + Q \sum_{i=J+1}^{\infty} \|S(\delta)\, e_i\|^2 ,$$

where we have set $Q \equiv 2 \sup \{\|S(r)\|^2;\ 0 \leqq r \leqq T\}$. The second term on the right-hand side of the inequality tends to 0 as $J \to +\infty$. For every $J \in \mathbb{N}$, using the strong continuity of the semigroup $S(t)$, we can find $\eta > 0$ such that $|t - s| < \eta$ implies $\|[S(t) - S(s)]\, e_i\|^2 \leqq (2J)^{-1}\, \varepsilon$, $i = 1, \ldots, J$. This shows that for arbitrary δ, $0 < \delta < T$, we have $S(\cdot) \in \mathscr{C}([\delta, T]; \mathrm{J}_2(H))$.

The proof of the formula (4) is analogous, based on the estimate

$$\int_0^T \|S(s+v) - S(s)\|_{\mathrm{HS}}^2\, \mathrm{d}s \leqq \sum_{i=1}^{J} \int_0^T \|S(r)\, [S(v) - I]\, e_i\|^2\, \mathrm{d}r +$$

$$+ \sum_{i=J+1}^{\infty} \int_0^T \|[S(v) - I]\, S(r)\, e_i\|^2\, \mathrm{d}r \leqq QT \sum_{i=1}^{J} \|[S(v) - I]\, e_i\|^2 +$$

$$+ (Q+2) \int_0^T \sum_{i=J+1}^{\infty} \|S(r)\, e_i\|^2\, \mathrm{d}r .$$

The following easy lemma plays a key role in the present section.

Lemma 1. (i) *Let the hypotheses* (I), (U1), (U2) *be satisfied. Then for every* $T > 0$, $\eta > 0$, $\tau_1 > 0$ *there exists* $\delta > 0$ *such that for all* $t_0 \in \mathbb{R}_+$, $s, t \in [t_0 + \tau_1, t_0 + T]$ *and every solution* $\varphi(t)$ *of the equation* (1) *satisfying* $\varphi(t_0) \in L^p(\Omega; H)$ *we have: if* $|t - s| < \delta$, *then*

$$\|\varphi(t) - \varphi(s)\|_p \leqq (1 + \|\varphi(t_0)\|_p)\, \eta .$$

(ii) *Moreover, if the assumption* (U3) *is fulfilled, then the same assertion holds for the equation* (2) *as well.*

Corollary 1. (i) *Under the assumptions* (I), (U1), (U2) *we have: for every* $T > 0$ *and* $\eta > 0$ *there exists a partition* $\{\tau_i\}_{i=0}^{N}$ *of the interval* $[0, T]$ *such that for all* $t_0 \in \mathbb{R}_+$ *and any solution* $\varphi(t)$ *of the equation* (1) *satisfying* $\varphi(t_0) \in L^p(\Omega; H)$ *the following estimate holds:*

$$\sum_{i=0}^{N-1} \int_{t_0+\tau_i}^{t_0+\tau_{i+1}} \|\varphi(t) - \varphi(t_0 + \tau_i)\|_p\, \mathrm{d}t \leqq (1 + \|\varphi(t_0)\|_p)\, \eta .$$

(ii) *Moreover, if the assumption* (U3) *is fulfilled, then the same assertion holds also for the equation* (2), *furthermore*

$$\sum_{i=0}^{N-1} \int_{t_0+\tau_i}^{t_0+\tau_{i+1}} \|S(T-t)\|_{HS}^2 \, \|\psi(t) - \psi(t_0+\tau_i)\|_p^2 \, dt \leqq (1 + \|\psi(t_0)\|_p) \, \eta \, .$$

Remark 2. It will be obvious from the proof that δ depends only on T, τ_1, η, p, K_1, tr$W$ and on the function $S(\cdot)$: $[0, T] \to \mathscr{L}(H)$ (and on the function $\|S(\cdot)\|_{HS}$: $(0, T] \to \mathbb{R}$ if the equation (2) is treated), thus it is independent of the particular form of the coefficients α, σ and of the process $w(t)$; so the derived estimates hold simultaneously for appropriate families of equations.

Remark 3. In Appendix we show that Lemma 1 is no longer valid if the semigroup $S(t)$ is assumed to be only strongly continuous.

Remark 4. Let us notice that, in the situation of Lemma 1, there exists a constant C^* depending only on K_1, T, p, tr$W$ and on $M \equiv \sup \{\|S(r)\|; 0 \leqq r \leqq T\}$ and such that for all $t_0 \in \mathbb{R}_+$ and any solution φ of the equation (1) satisfying $\varphi(t_0) \in L^p(\Omega; H)$ we have

$$(5) \qquad \sup_{t_0 \leqq t \leqq t_0+T} \|\varphi(t)\|_p \leqq C^*(1 + \|\varphi(t_0)\|_p) \, .$$

The estimate (5) holds also for the solutions of the equation (2) if (U3) is fulfilled; in this case the constant C^* depends on K_1, p, T, M and on the function $\|S(\cdot)\|_{HS}$.

Proof of Lemma 1. Choose $T > 0$, $\tau_1 \in (0, T)$, $t_0 \in \mathbb{R}_+$ arbitrarily. Let $\eta > 0$, $t_0 < t_1 \equiv \tau_1 + t_0 \leqq s \leqq t \leqq t_0 + T$. Let us first consider the equation (1). By the definition of the mild solution we obtain

$$\begin{aligned} \varphi(t) - \varphi(s) &= [S(t-t_0) - S(s-t_0)] \, \varphi(t_0) + \\ &+ \int_{t_0}^{s} [S(t-r) - S(s-r)] \, \alpha(r, \varphi(r)) \, dr + \\ &+ \int_{t_0}^{s} [S(t-r) - S(s-r)] \, \sigma(r, \varphi(r)) \, dw(r) + \\ &+ \int_{s}^{t} S(t-r) \, \alpha(r, \varphi(r)) \, dr + \int_{s}^{t} S(t-r) \, \sigma(r, \varphi(r)) \, dw(r) \equiv \\ &\equiv I_1 + \ldots + I_5 \, . \end{aligned}$$

Using the uniform continuity of $S(\cdot)$ on $[\tau_1, T]$ in the uniform operator topology and the formula (3) we find $\tilde{\delta} > 0$ such that for $s, t \in [t_1, t_0 + T]$, $s \leqq t \leqq s + \tilde{\delta}$ we have

$$(6) \qquad \|S(t-t_0) - S(s-t_0)\|_{\mathscr{L}(H)} \leqq \eta \, ,$$

$$(7) \qquad \int_0^T \|S(v+t-s) - S(v)\|_{\mathscr{L}(H)}^p \, dv \leqq \eta^p \, .$$

Let $|t - s| < \delta$, then

$$\|I_1\|_p \leqq \|S(t - t_0) - S(s - t_0)\| \|\varphi(t_0)\|_p \leqq \|\varphi(t_0)\|_p \eta ,$$

$$\|I_2\|_p \leqq \int_{t_0}^s \|S(t - r) - S(s - r)\| \|\alpha(r, \varphi(r))\|_p \mathrm{d}r \leqq$$

$$\leqq K_1 \int_0^{s-t_0} \|S(v + t - s) - S(v)\| (1 + \|\varphi(s - v)\|_p) \mathrm{d}v \leqq$$

$$\leqq K_1(1 + C^*)(1 + \|\varphi(t_0)\|_p)(s - t_0)^{(p-1)/p} .$$

$$. (\int_0^T \|S(v + t - s) - S(v)\|^p \mathrm{d}v)^{1/p} \leqq$$

$$\leqq K_1(1 + C^*)(1 + \|\varphi(t_0)\|_p) T^{(p-1)/p} \eta ,$$

$$\|I_4\|_p \leqq \int_s^t \|S(t - r) \alpha(r, \varphi(r))\|_p \mathrm{d}r \leqq$$

$$\leqq MK_1(1 + C^*)(1 + \|\varphi(t_0)\|_p)(t - s) .$$

Using Prop. 1.9. in [7] we obtain

$$\|I_3\|_p \leqq C(p)(\mathrm{tr} W)^{1/2} (s - t_0)^{1/2 - 1/p} .$$

$$. (\int_{t_0}^s \|[S(t - r) - S(s - r)] \sigma(r, \varphi(r))\|_p^p \mathrm{d}r)^{1/p} \leqq$$

$$\leqq C(p)(\mathrm{tr} W)^{1/2} T^{1/2 - 1/p} K_1(1 + C^*)(1 + \|\varphi(t_0)\|_p) .$$

$$. (\int_0^{s-t_0} \|S(v + t - s) - S(v)\|^p \mathrm{d}v)^{1/p} \leqq$$

$$\leqq C(p)(1 + C^*) K_1 T^{1/2 - 1/p} (\mathrm{tr} W)^{1/2} (1 + \|\varphi(t_0)\|_p) \eta ,$$

$$\|I_5\|_p \leqq C(p)(\mathrm{tr} W)^{1/2} (t - s)^{1/2 - 1/p} .$$

$$. (\int_s^t \|S(t - r)\|^p \|\sigma(r, \varphi(r))\|_p^p \mathrm{d}r)^{1/p} \leqq$$

$$\leqq C(p)(\mathrm{tr} W)^{1/2} MK_1(1 + C^*)(1 + \|\varphi(t_0)\|_p)(t - s)^{1/2} .$$

Combining all the estimates we see that $\|\varphi(t) - \varphi(s)\|_p \leqq Q(\eta + \delta + \delta^{1/2})$. $. (1 + \|\varphi(t_0)\|_p)$, where Q depends only on K_1, T, p, M, $\mathrm{tr} W$. Hence it is obvious how to find δ with the desired properties.

Now, let us consider the equation (2). Notice that the estimates of the terms I_1, I_2, I_4 do not depend on the type of the Wiener process, thus we have again $\|I_1 + I_2 + I_4\|_p \leqq Q(\eta + \delta)(1 + \|\psi(t_0)\|_p)$. Further, acording to (4) we choose $\delta > 0$ so that $|t - s| < \delta$ implies not only (6), (7) but also

$$\int_0^T \|S(v + t - s) - S(v)\|_{\mathrm{HS}}^2 \mathrm{d}v \leqq \eta^2 . \tag{8}$$

Relying on Prop. 1.3 in [6] we can estimate

$$\|I_5\|_p \leqq C(p)(\int_s^t | \|S(t - r) \sigma(r, \psi(r))\|_{\mathrm{HS}}^2|_{p/2} \mathrm{d}r)^{1/2} \leqq$$

$$\leqq C(p)(\int_s^t \|S(t - r)\|_{\mathrm{HS}}^2 \|\sigma(r, \psi(r))\|_p^2 \mathrm{d}r)^{1/2} \leqq$$

$$\leqq K_1 C(p)(1 + C^*)(1 + \|\psi(t_0)\|_p)(\int_0^\delta \|S(v)\|_{\mathrm{HS}}^2 \mathrm{d}v)^{1/2} ;$$

using (8) and the proposition quoted above we obtain

$$\|I_3\|_p \leqq C(p) (\int_{t_0}^s | \|[S(t-r) - S(s-r)]\, \sigma(r, \psi(r))\|_{HS}^2|_{p/2}\, dr)^{1/2} \leqq$$
$$\leqq C(p) (\int_{t_0}^s \|S(t-r) - S(s-r)\|_{HS}^2 \|\sigma(r, \psi(r))\|_p^2\, dr)^{1/2} \leqq$$
$$\leqq K_1\, C(p) (1 + C^*) (1 + \|\psi(t_0)\|_p) \,.$$
$$. (\int_0^{s-t_0} \|S(v + t - s) - S(v)\|_{HS}^2\, dv)^{1/2} \leqq$$
$$\leqq K_1\, C(p) (1 + C^*) (1 + \|\psi(t_0)\|_p)\, \eta \,.$$

The proof of Lemma is complete, we proceed to prove the statement (i) of Corollary. Set $\tau_0 = 0$ and $\tau_1 = (2(1 + C^*))^{-1}\, \eta$, then we have

$$\int_{t_0}^{t_0+\tau_1} \|\varphi(t) - \varphi(t_0)\|_p\, dt \leqq (1 + C^*) (1 + \|\varphi(t_0)\|_p)\, \tau_1 \leqq$$
$$\leqq \tfrac{1}{2}(1 + \|\varphi(t_0)\|_p)\, \eta \,.$$

Next we choose an arbitrary partition $\{\tau_i\}_{i=1}^N$ of the interval $[\tau_1, T]$ with the mesh $\delta > 0$, where δ is found by Lemma 1 so that $|t - s| < \delta$, $s, t \in [\tau_1, T]$, implies

$$\|\varphi(t) - \varphi(s)\|_p \leqq (1 + \|\varphi(t_0)\|_p (2T)^{-1}\, \eta \,.$$

The statement (ii) can be proved analogously. Q.E.D.

We use Lemma 1 to establish a uniform version of Theorems 3, 5 in [10]. Such a result will be needed in the course of the proof of the averaging theorem on the infinite time interval. Let us adopt the following assumptions:

(III) Let $a_\alpha: \mathbb{R}_+ \times H \to H$, $b_\alpha: \mathbb{R}_+ \times H \to \mathscr{L}(Y, H)$, $\alpha \in [0, 1]$,

be measurable functions satisfying: there exists a constant K such that for all $t \in \mathbb{R}_+$, $x, y \in H$, $\alpha \in [0, 1]$ we have

$$\|a_\alpha(t, 0)\| + \|b_\alpha(t, 0)\| \leqq K \,,$$
$$\|a_\alpha(t, x) - a_\alpha(t, y)\| + \|b_\alpha(t, x) - b_\alpha(t, y)\| \leqq K\|x - y\| \,.$$

(Vu) Suppose there exists $\Delta_0 > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$ we have: if $0 \leqq t_1 \leqq \leqq t_2 \leqq t_1 + \Delta_0$, then

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} S(t_2 - s) [a_\alpha(s + t_0, x) - a_0(s + t_0, x)]\, ds = 0 \,, \tag{9}$$

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} (\mathrm{tr}\, \{\tilde{b}_\alpha(s + t_0, x)\, W(\tilde{b}_\alpha(s + t_0, x))^*\})^{p/2}\, ds = 0 \tag{10}$$

uniformly in $t_0 \in \mathbb{R}_+$ and $x \in H$; we have set $\tilde{b}_\alpha(r, x) \equiv b_\alpha(r, x) - b_0(r, x)$.

(Vcu) The same hypothesis as (Vu), only (10) is replaced by

$$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} \|\tilde{b}_\alpha(s + t_0, x)\|^p\, ds = 0$$

uniformly in $t_0 \in \mathbb{R}_+$ and $x \in H$.

Proposition 1. *Let the assumptions* (I), (III), (Vu), (U2) *be fulfilled. Then for every* $T > 0$ *and* $\eta > 0$ *there exists* $\alpha_0 > 0$ *such that for all* $t_0 \in \mathbb{R}_+$ *we have: if* $x_\alpha(t)$, $\alpha \in [0, 1]$, *are mild solutions of the equations*

$$\mathrm{d}x_\alpha(t) = (Ax_\alpha(t) + a_\alpha(t, x_\alpha(t)))\,\mathrm{d}t + b_\alpha(t, x_\alpha(t))\,\mathrm{d}w(t) \tag{11}$$

with initial conditions $x_\alpha(t_0) = x_0(t_0) \in L^p(\Omega, \mathscr{F}_{t_0}, \mathsf{P}; H)$ *and if* $\alpha \in (0, \alpha_0]$ *then*

$$\sup_{t \in [t_0, t_0 + T]} \|x_\alpha(t) - x_0(t)\|_p \leqq \eta(1 + \|x_0(t_0)\|_p)\,.$$

If the hypotheses (I), (III), (Vcu) *and* (U3) *are satisfied then the same assertion is valid also for the mild solutions of the equations*

$$\mathrm{d}x_\alpha(t) = (\tilde{A}x_\alpha(t) + a_\alpha(t, x_\alpha(t)))\,\mathrm{d}t + b_\alpha(t, x_\alpha(t))\,\mathrm{d}B(t)\,. \tag{12}$$

Proof. Under the present strengthened assumptions the proofs of Theorems 3,5 in [10] can be carried out as uniformly as we need. Let us demonstrate this fact by estimating the term

$$R \equiv \int_0^t S(t - s)\,[a_\alpha(s, x_\alpha(s)) - a_0(s, x_0(s))]\,\mathrm{d}s\,.$$

Fix $\eta > 0$, $T > 0$, $t_0 \in \mathbb{R}_+$ arbitrarily. Let $\{\tau_i\}_{i=0}^N$ be the partition the existence of which is ensured by Corollary 1. Set $t_i = t_0 + \tau_i$, $i = 0, \dots, N$, $\tau(t) = \max\{i;\, t_i \leqq t\}$, $\sigma(t) = \max\{t_i;\, t_i \leqq t\}$. In the same way as in [10] we split

$$\begin{aligned} R = {} & \int_{\sigma(t)}^t S(t - s)\,[a_\alpha(s, x_\alpha(s)) - a_0(s, x_0(s))]\,\mathrm{d}s \,+ \\ & + \int_0^{\sigma(t)} S(t - s)\,[a_\alpha(s, x_\alpha(s)) - a_\alpha(s, x_0(s))]\,\mathrm{d}s \,+ \\ & + \sum_{i=1}^{\tau(t)} \int_{t_{i-1}}^{t_i} S(t - s)\,[a_\alpha(s, x_0(s)) - a_\alpha(s, x_0(t_{i-1}))]\,\mathrm{d}s \,+ \\ & + \sum_{i=1}^{\tau(t)} \int_{t_{i-1}}^{t_i} S(t - s)\,[a_\alpha(s, x_0(t_{i-1})) - a_0(s, x_0(t_{i-1}))]\,\mathrm{d}s \,+ \\ & + \sum_{i=1}^{\tau(t)} \int_{t_{i-1}}^{t_i} S(t - s)\,[a_0(s, x_0(t_{i-1})) - a_0(s, x_0(s))]\,\mathrm{d}s \equiv I_1 + \dots + I_5\,. \end{aligned}$$

The estimate of the terms I_1, I_2 requires no change; further,

$$\begin{aligned} \|I_3\|_p & \leqq \sum_{i=1}^{\tau(t)} \int_{t_{i-1}}^{t_i} \|S(t - s)\|\, \|a_\alpha(s, x_0(s)) - a_\alpha(s, x_0(t_{i-1}))\|_p\,\mathrm{d}s \leqq \\ & \leqq MK \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|x_0(s) - x_0(t_{i-1})\|_p \leqq MK(1 + \|x_0(t_0)\|_p)\,\eta\,. \end{aligned}$$

The same estimate holds for $\|I_5\|_p$. By the assumption (Vu) we can find $\alpha_1 > 0$ such that for $\alpha \in (0, \alpha_1]$, $i = 1, \dots, N$ and for every $x \in H$

$$\begin{aligned} & \left\|\int_{t_{i-1}}^{t_i} S(t_i - s)\,[a_\alpha(s, x) - a_0(s, x)]\,\mathrm{d}s\right\| = \\ & = \left\|\int_{\tau_{i-1}}^{\tau_i} S(\tau_i - s)\,[a_\alpha(s + t_0, x) - a_0(s + t_0, x)]\,\mathrm{d}s\right\| \leqq \eta/N\,, \end{aligned}$$

so

$$\left\|\int_{t_{i-1}}^{t_i} S(t_i - s)\left[a_\alpha(s, x_0(t_{i-1})) - a_0(s, x_0(t_{i-1}))\right] \mathrm{d}s\right\| \leqq \eta/N$$

almost surely, thus $\|I_4\|_p \leqq M\eta$.

The estimates of the stochastic integrals can be modified in an analogous way.

Q.E.D.

To assume the convergence in (9), (10) to be uniform with respect to $x \in H$ is rather restrictive. Let us try to use instead of (Vu) only the assumption

(Vlu) There exists $\Delta_0 > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$ and every $L > 0$ we have: if $0 \leqq t_1 \leqq t_2 \leqq t_1 + \Delta_0$ then (9), (10) hold uniformly in $t_0 \in \mathbb{R}_+$ and in $x \in \mathscr{B}_L \equiv \equiv \{d \in H;\ \|d\| \leqq L\}$.

In the same way we derive an assumption (Vlcu) from (Vcu).

Proposition 2. (i) *Let the assumptions* (I), (III), (U2), (Vlu) *be fulfilled. Suppose* $K \subseteqq L^p(\Omega; H)$ *is such that the set* $\mathfrak{M} = \{\|\varphi(t)\|^p;\ t \geqq t_0 \geqq 0,\ (\varphi(t))_{t \geqq t_0}$ *is a mild solution of the problem*

$$\mathrm{d}\varphi(t) = (A\,\varphi(t) + a_0(t, \varphi(t)))\,\mathrm{d}t + b_0(t, \varphi(t))\,\mathrm{d}w(t) \tag{13}$$

with $\varphi(t_0) \in K\}$
is uniformly integrable.

Then for all $T > 0$, $\eta > 0$ *there exists* $\alpha_0 > 0$ *such that for any* $t_0 \in \mathbb{R}_+$ *and for every mild solution* $x_\alpha(t)$, $\alpha \in [0, 1]$, *of the problem* (11) *we have: if* $\alpha \in (0, \alpha_0]$ *and if* $x_\alpha(t_0) = x_0(t_0) \in K$ *then*

$$\sup_{t \in [t_0, t_0 + T]} \|x_\alpha(t) - x_0(t)\|_p \leqq \eta\,.$$

(ii) *Let the hypotheses* (I), (III), (U3), (Vlcu) *be satisfied. Suppose* $K \subseteqq L^p(\Omega; H)$ *is such that the set* $\mathfrak{M} = \{\|\psi(t)\|^p;\ t \geqq t_0 \geqq 0,\ (\psi(t))_{t \geqq t_0}$ *is a mild solution of the problem*

$$\mathrm{d}\psi(t) = (\tilde{A}\,\psi(t) + a_0(t, \psi(t)))\,\mathrm{d}t + b_0(t, \psi(t))\,\mathrm{d}B(t)$$

with $\psi(t_0) \in K\}$
is uniformly integrable. Then the same assertion as in (i) *holds for the mild solutions of the equation* (12).

Proof. First, let us notice that every $\mathscr{F}_{t_0}$-measurable function $f \in K$ is an initial condition of some solution of (13), thus $\|f\|^p \in \mathfrak{M}$. $\mathfrak{M}$ is a bounded subset of $L^1(\Omega)$, so there exists a constant F such that for all $t_0 \in \mathbb{R}_+$ and every $\mathscr{F}_{t_0}$-measurable $f \in K$ we have $\|f\|_p^p \leqq F$.

We can easily see that the only step to be modified in the above proof is the estimate of the term I_4. Setting $\tilde{a}_\alpha(s, x) = a_\alpha(s, x) - a_0(s, x)$ and denoting by A_L^i the set $\{\omega \in \Omega;\ \|x_0(t_0 + \tau_{i-1})(\omega)\| \leqq L\}$ we obtain

$$\|I_4\|_p \leqq M \sum_{i=1}^{N} \|\int_{\tau_{i-1}}^{\tau_i} S(\tau_i - s)\, \tilde{a}(s + t_0, x_0(t_0 + \tau_{i-1}))\, ds\|_p \leqq$$

$$\leqq M \sum_{i=1}^{N} \{\|\chi(A_L^i) \int_{\tau_{i-1}}^{\tau_i} \dots ds\|_p + \|\chi(\Omega \setminus A_L^i) \int_{\tau_{i-1}}^{\tau_i} \dots ds\|_p\} \equiv$$

$$\equiv M \sum_{i=1}^{N} \{J_1^i + J_2^i\},$$

where $M = \sup\{\|S(t)\|;\ t \in [0, T]\}$ and $\chi(B) \equiv \chi_B$ denotes the characteristic function of a set B. Now,

$$J_2^i \leqq 2MK|\chi(\Omega \setminus A_L^i) \int_{\tau_{i-1}}^{\tau_i} (1 + \|x_0(t_0 + \tau_{i-1})\|)\, ds|_p \leqq$$
$$\leqq 2^p MKT(\mathsf{E}\chi(\{\|x_0(t_0 + \tau_{i-1})\| > L\})(1 + \|x_0(t_0 + \tau_{i-1})\|^p))^{1/p}.$$

By virtue of the uniform integrability of the set $\mathfrak{M}$ we can find $L > 0$ such that for every $t_0 \in \mathbb{R}_+$ and all $i = 1, \dots, N$ we have $J_2^i \leqq (2N)^{-1} \eta$. Let us fix this L. Then by (Vlu) there exists $\alpha_0 > 0$ such that for every $t_0 \in \mathbb{R}_+$ and $i = 1, \dots, N$ and for almost all $\omega \in \{\|x_0(t_0 + \tau_{i-1})\| \leqq L\}$ the inequality

$$\|\int_{\tau_{i-1}}^{\tau_i} S(\tau_i - s)\, \tilde{a}_\alpha(s + t_0, x_0(t_0 + \tau_{i-1})(\omega))\, ds\| < \eta/2N$$

holds, hence also $J_1^i \leqq (2N)^{-1} \eta$. Q.E.D.

Remark 5. If the semigroup $S(t)$ is holomorphic then the assumption (Vu) can be weakened in accordance with the finite dimensional case. Proceeding as in the proof of Lemma 3 in [10] we can derive the following result:

Let $a_\alpha: \mathbb{R}_+ \times H \to H$, $\alpha \in [0, 1]$, be measurable functions such that

(i) $$\sup_\alpha \sup_{x \in H} \sup_{t \geqq 0} \|a_\alpha(t, x)\| < +\infty,$$

(ii) for every $0 \leqq t_1 \leqq t_2 < +\infty$ we have

(14) $$\lim_{\alpha \to 0+} \int_{t_1}^{t_2} [a_\alpha(t + t_0, x) - a_0(t + t_0, x)]\, dt = 0$$

uniformly in $t_0 \geqq 0$ and in $x \in H$. Let the semigroup $S(t)$ be holomorphic. Then (9) holds uniformly in $t_0 \geqq 0$ and in $x \in H$. If we assume (instead of (i)) that for every $L \geqq 0$ the estimate

$$\sup_\alpha \sup_{x \in \mathscr{B}_L} \sup_{t \geqq 0} \|a_\alpha(t, x)\| < +\infty$$

holds and that for every $L \geqq 0$ the limit passage in (14) is uniform with respect to $t_0 \geqq 0$ and $x \in \mathscr{B}_L$, then (9) holds uniformly in $t_0 \in \mathbb{R}_+$ and $x \in \mathscr{B}_L$. (Recall that we have denoted $\mathscr{B}_L = \{d \in H;\ \|d\| \leqq L\}$.)

As the last topic in this section we consider the method of partial averaging. As in the case of ordinary differential equations (cf. [4]) many schemes of partial averaging, with proofs only slightly different, can be formulated. We content ourselves with one of the simplest cases.

Let H_i, $i = 1, 2$, be separable real Hilbert spaces, then the space $\mathscr{H} \equiv H_1 \oplus H_2$ endowed with the norm $\|(f, g)\|_{\mathscr{H}} = (\|f\|_{H_1}^2 + \|g\|_{H_2}^2)^{1/2}$ is also Hilbert. Let us consider a system of equations ($\alpha > 0$)

$$
\begin{aligned}
(15)\qquad & \mathrm{d}x_\alpha^1(t) = (A_1\, x_\alpha^1(t) + a_\alpha^1(t, x_\alpha^1(t), x_\alpha^2(t)))\, \mathrm{d}t + b_\alpha^1(t, x_\alpha^1(t), x_\alpha^2(t))\, \mathrm{d}w(t)\,, \\
& \mathrm{d}x_\alpha^2(t) = (A_2\, x_\alpha^2(t) + a_\alpha^2(t, x_\alpha^1(t), x_\alpha^2(t)))\, \mathrm{d}t + b_\alpha^2(t, x_\alpha^1(t), x_\alpha^2(t))\, \mathrm{d}w(t)\,, \\
& x_\alpha^1(0) = \varphi_0^1\,, \\
& x_\alpha^2(0) = \varphi_0^2\,.
\end{aligned}
$$

We assume that A_i: $\mathsf{D}(A_i) \to H_i$, $i = 1, 2$, are infinitesimal generators of (C_0)-semigroups $S_i(t)$ on H_i; a_α^i: $\mathbb{R}_+ \times \mathscr{H} \to H_i$, b_α^i: $\mathbb{R}_+ \times \mathscr{H} \to \mathscr{L}(Y, H_i)$, are measurable and satisfy the usual Lipschitz type conditions: there exists $K > 0$ such that for all $t \in \mathbb{R}_+$, (x, y), $(u, v) \in \mathscr{H}$, $\alpha \in (0, 1]$, $i = 1, 2$ we have

$$
\begin{aligned}
(16)\qquad & \|a_\alpha^i(t, x, y) - a_\alpha^i(t, u, v)\| + \|b_\alpha^i(t, x, y) - b_\alpha^i(t, u, v)\| \leqq \\
& \leqq K\|(x, y) - (u, v)\|\,, \\
& \|a_\alpha^i(t, 0, 0)\| + \|b_\alpha^i(t, 0, 0)\| \leqq K\,.
\end{aligned}
$$

As before, $w(t)$ is a Wiener process with the nuclear covariance operator W in the Hilbert space Y.

(A_1, A_2): $\mathsf{D}(A_1) \oplus \mathsf{D}(A_2) \to \mathscr{H}$ generates a (C_0)-semigroup on $\mathscr{H}$, so if we treat (15) as an equation in $\mathscr{H}$, then for every initial condition $\varphi_0 = (\varphi_0^1, \varphi_0^2) \in L^p(\Omega; \mathscr{H})$ there exists (by Ichikawa's theorem, [7], Th. 2.1) a unique mild solution $x_\alpha \in \mathscr{C}([0, \infty); L^p(\Omega; \mathscr{H}))$.

Proposition 3. *Let $S_2 \in \mathscr{C}((0, \infty); \mathscr{L}(H_2))$. Suppose that there exists $\Delta_0 > 0$ such that for all $t_1, t_2 \in \mathbb{R}_+$ we have: if $0 \leqq t_1 \leqq t_2 \leqq t_1 + \Delta_0$, then*

$$
(17)\qquad \lim_{\alpha \to 0+} \int_{t_1}^{t_2} S_1(t_2 - s)\,[a_\alpha^1(s, x, y) - a_0(s, x)]\, \mathrm{d}s = 0\,,
$$

$$
\begin{aligned}
(18)\qquad & \lim_{\alpha \to 0+} \int_{t_1}^{t_2} (\operatorname{tr}\{[b_\alpha^1(s, x, y) - b_0(s, x)]\,. \\
& .\, W[b_\alpha^1(s, x, y) - b_0(s, x)]^*\})^{p/2}\, \mathrm{d}s = 0
\end{aligned}
$$

uniformly for $(x, y) \in \mathscr{H}$, where a_0: $\mathbb{R}_+ \times H_1 \to H_1$, b_0: $\mathbb{R}_+ \times H_1 \to \mathscr{L}(Y, H_1)$ are measurable functions satisfying

$$
\begin{aligned}
& \|a_0(t, x) - a_0(t, u)\| + \|b_0(t, x) - b_0(t, u)\| \leqq K\|x - u\|\,, \\
& \|a_0(t, 0)\| + \|b_0(t, 0)\| \leqq K
\end{aligned}
$$

for all $t \geqq 0$; $x, u \in H_1$.

Let $x_\alpha = (x_\alpha^1, x_\alpha^2)$ be mild solutions of the equation (15) with an initial condition $\varphi_0 = (\varphi_0^1, \varphi_0^2) \in L^p(\Omega; \mathscr{H})$, $p \geqq 2$. Let $y(t)$ be the mild solution of the equation

$$\begin{aligned} dy(t) &= (A_1\, y(t) + a_0(t, y(t)))\, dt + b_0(t, y(t))\, dw(t)\,, \\ y(0) &= \varphi_0^1\,. \end{aligned}$$

Then for all $T > 0$ we have

$$\lim_{\alpha \to 0+} \sup_{t \in [0,T]} \|x_\alpha^1(t) - y(t)\|_p = 0\,.$$

Proof. We will sketch the proof very briefly, because it differs only in technical details from the considerations we have done before.

First, proceeding as in the proof of Lemma 1, we find a partition $\{\tau_i\}_{i=0}^N$ of the interval $[0, T]$ such that for all $\alpha \in (0, 1]$ we have

$$\sum_{i=1}^{N} \int_{\tau_{i-1}}^{\tau_i} \|x_\alpha^2(r) - x_\alpha^2(\tau_{i-1})\|_{p,H_2}\, dr \leqq (1 + \|\varphi_0\|_{p,\mathscr{H}})\, \eta\,.$$

The partition $\{\tau_i\}$ can be chosen fine enough to ensure also

$$\sum_{i=1}^{N} \int_{\tau_{i-1}}^{\tau_i} \|y(r) - y(\tau_{i-1})\|_{p,H_1}\, dr \leqq \eta\,.$$

($T, \eta > 0$ are arbitrary but fixed a priori.) By the definition of a mild solution it follows that

$$\begin{aligned} x_\alpha^1(t) - y(t) &= \int_0^t S_1(t - r)\,[a_\alpha^1(r, x_\alpha^1(r), x_\alpha^2(r)) - a_0(r, y(r))]\, dr + \\ &+ \int_0^t S_1(t - r)\,[b_\alpha^1(r, x_\alpha^1(r), x_\alpha^2(r)) - b_0(r, y(r))]\, dw(r) \equiv R_1 + R_2\,. \end{aligned}$$

Let us split R_1 into the sum

$$\begin{aligned} R_1 &= \int_{\sigma(t)}^t S_1(t - r)\,[a_\alpha^1(r, x_\alpha^1(r), x_\alpha^2(r)) - a_0(r, y(r))]\, dr + \\ &+ \int_0^{\sigma(t)} S_1(t - r)\,[a_\alpha^1(r, x_\alpha^1(r), x_\alpha^2(r)) - a_\alpha^1(r, y(r), x_\alpha^2(r))]\, dr + \\ &+ \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} S_1(t - r)\,[a_\alpha^1(r, y(r), x_\alpha^2(r)) - a_\alpha^1(r, y_{i-1}, x_{\alpha,i-1}^2)]\, dr + \\ &+ \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} S_1(t - r)\,[a_\alpha^1(r, y_{i-1}, x_{\alpha,i-1}^2) - a_0(r, y_{i-1})]\, dr + \\ &+ \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} S_1(t - r)\,[a_0(r, y_{i-1}) - a_0(r, y(r))]\, dr \equiv I_1 + \ldots + I_5\,. \end{aligned}$$

Here $\tau(t)$, $\sigma(t)$ have the same meaning as in the proof of Proposition 1 and we have set $x_{\alpha,i-1}^2 = x_\alpha^2(\tau_{i-1})$, $y_{i-1} = y(\tau_{i-1})$.

Using the inequalities (16) we may derive in a well-known way the estimates

$$\|I_1\|_p \leqq C(t - \sigma(t)),$$

$$\|I_2\|_p \leqq C \int_0^t \|x_\alpha^1(r) - y(r)\|_p \, dr,$$

$$\|I_3\|_p \leqq C \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} \|(y(r), x_\alpha^2(r)) - (y_{i-1}, x_{\alpha,i-1}^2)\|_{p,\mathscr{H}} \, dr \leqq$$

$$\leqq C \sum_{i=1}^{\tau(t)} \int_{\tau_{i-1}}^{\tau_i} (\|y(r) - y_{i-1}\|_{p,H_1} + \|x_\alpha^2(r) - x_{\alpha,i-1}^2\|_{p,H_2}) \, dr \leqq 2C\eta,$$

$$\|I_5\|_p \leqq C\eta.$$

(We have denoted by C some constant depending only on K, T, $\operatorname{tr} W$, $\|\varphi_0\|_p$ and on the function $S_1: [0, T] \to \mathscr{L}(H_1)$.)

By (17) we may find $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0]$, $i = 1, \ldots, N$ and almost all $\omega \in \Omega$ we have

$$(19) \qquad \left\|\int_{\tau_{i-1}}^{\tau_i} S_1(\tau_i - r) \left[a_\alpha^1(r, y_{i-1}, x_{\alpha,i-1}) - a_0(r, y_{i-1})\right] dr\right\|_{H_1} \leqq \eta/N,$$

hence also $\|I_4\|_p \leqq C\eta$. The estimate of the term R_2 can be obtained analogously.

Q.E.D.

Remark 6. We may establish the above results under the assumption that the convergence in (17), (18) is uniform only in $y \in H_2$, if we suppose that

$$\|a_\alpha^1(t, x, y)\| + \|b_\alpha^1(t, x, y)\| \leqq K(1 + \|x\|_{H_1})$$

for $t \geqq 0$, $(x, y) \in \mathscr{H}$, $\alpha \in (0, 1]$. In this case we handle the integrals in terms like (19) using the Lebesgue dominated convergence theorem with the majorant const. $\|y_{i-1}\|^p$ independent of α.

2. ASYMPTOTICAL STABILITY

In this section we will study the asymptotic behaviour of the equation

$$(1) \qquad d\varphi(t) = A\,\varphi(t)\,dt + a(t, \varphi(t))\,dt + b(t, \varphi(t))\,dw(t),$$

where A stands for an infinitesimal generator of a strongly continuous semigroup $S(t)$ on H, $a: \mathbb{R}_+ \times H \to H$, $b: \mathbb{R}_+ \times H \to \mathscr{L}(Y, H)$ satisfy the inequalities from the assumption (U1), and $w(t)$ is a Y-valued Wiener process with a nuclear covariance operator W. We will denote by $\varphi^{s,x}$ the solution to the equation (1) with the initial condition $\varphi^{s,x}(s) = x$. We give some sufficient conditions for the stability properties (in the terms of Liapunov functionals) required in the next section to justify the averaging on an infinite time interval.

Throughout the section we assume

$$(2) \qquad \langle Ax, x \rangle \leqq \beta\|x\|^2, \quad x \in D(A),$$

for some $\beta \in \mathbb{R}$. Note that (2) is satisfied for a large class of equations (including, e.g., parabolic and hyperbolic problems). It also implies a.s. continuity of trajectories of solutions of (1) (cf. [8], Prop. 3.8). Let $\mathscr{C}_s^{1,2}(\mathbb{R}_+ \times H) \equiv \mathscr{C}_s^{1,2}$ be the class of real valued continuous functions on $\mathbb{R}_+ \times H$ with the following properties:

(3) $v(t, y)$ is differentiable in t for each $y \in D(A)$, and $v_t(t, y)$ is continuous on $\mathbb{R}_+ \times D(A)$ provided $D(A)$ is equipped with the graph norm;

(4) $v(t, y)$ is twice Fréchet differentiable in y for each t, $v_y(t, y)$ and $v_{yy}(t, y)\, h$ are continuous on $\mathbb{R}_+ \times H$ for any $h \in H$.

For $v \in \mathscr{C}_s^{1,2}$ we define

$$[Lv](t, x) = \langle v_x(t, x), Ax + a(t, x)\rangle + \\ + \tfrac{1}{2} \operatorname{tr}\{b^*(t, x)\, v_{xx}(t, x)\, b(t, x)\, W\},$$

$x \in D(A)$, $t > 0$. We will use the following useful result by Ichikawa ([8], Corollary 3.4).

Lemma 1. *Let $v \in \mathscr{C}_s^{1,2}$ satisfy*

$$(5) \qquad |v(t, y)| + \|v_y(t, y)\| + \|v_{yy}(t, y)\| \leqq K_T(1 + \|y\|^q)$$

for some $K_T > 0$, $q > 0$ and all $t \in [0, T]$, $T > 0$, $y \in H$. Assume

$$(6) \qquad \left[\left(\frac{\partial}{\partial t} + L\right) v\right](t, x) \leqq u(t, x)\,, \quad x \in D(A)\,, \quad t > 0$$

for a function u continuous on $\mathbb{R}_+ \times H$ such that $|u(t, x)| \leqq K_T(1 + \|x\|^q)$. Then

$$v(t, \varphi^{s,x}(t)) - v(s, x) \leqq \int_s^t u(r, \varphi^{s,x}(r))\, dr + \\ + \int_s^t \langle v_y(r, \varphi^{s,x}(r)), b(r, \varphi^{s,x}(r))\, dw(r)\rangle\,.$$

In particular, if $u \equiv 0$, then $v(t, \varphi^{s,x}(t))$ is a supermartingale.

For $v \in \mathscr{C}_s^{1,2}$ set

$$[L_d v](t, x, y) = \langle v_x(t, x - y), Ax - Ay + a(t, x) - a(t, y)\rangle + \\ + \tfrac{1}{2} \operatorname{tr}\{[b(t, x) - b(t, y)]^*\, v_{xx}(t, x - y)\, [b(t, x) - b(t, y)]\, W\}\,, \\ t > 0\,, \quad x, y \in D(A)\,.$$

Lemma 2. *Let $\xi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a measurable function satisfying*

$$(7) \qquad |\xi(t, u) - \xi(t, v)| \leqq K_T^1 |u - v|\,, \quad u, v \in \mathbb{R}_+\,, \quad t \in [0, T]\,, \quad T \geqq 0$$

for some $K_T^1 > 0$, $\xi(t, 0) = 0$, and let $\xi(t, \cdot)$ be concave for all t. Assume

$$(8) \quad [(\partial/\partial t + L_d)\, v](t, x, y) \leqq \xi(t, v(t, x - y))\,, \quad t > 0\,, \quad x, y \in D(A)$$

for some nonnegative $v \in \mathscr{C}_s^{1,2}$ *satisfying* (5). *Then*

$$\mathrm{E}v(t, \varphi^{s,x}(t) - \varphi^{s,y}(t)) \leqq \psi^{s,v(s,x-y)}(t) \tag{9}$$

holds for all $0 \leqq s \leqq t$, $x, y \in H$, *where* $\psi^{s,v}$ *stands for the solution of the equation* $\dot{\psi} = \xi(t, \psi)$, $\psi^{s,v}(s) = v$.

Proof. By (5), (7) we have

$$|\xi(t, v(t, x - y))| \leqq \tilde{K}_T(1 + \|x\|^q + \|y\|^q), \quad x, y \in H, \quad t \in [0, T]$$

for some $\tilde{K}_T > 0$. Hence we may apply Lemma 1 to the functions $v(t,x - y) \in \mathscr{C}_s^{1,2}$ $(\mathbb{R} \times H \times H), u(t, x, y) = \xi(t, v(t x - y))$ and to the $H \times H$-valued process $(\varphi^{s,x}(t), \varphi^{s,y}(t))$. We obtain

$$h(t) \equiv \mathrm{E}v(t, \varphi^{s,x}(t) - \varphi^{s,y}(t)) \leqq \mathrm{E}v(\sigma, \varphi^{s,x}(\sigma) - \varphi^{s,y}(\sigma)) + \\ + \mathrm{E} \int_\sigma^t \xi(r, v(r, \varphi^{s,x}(r) - \varphi^{s,y}(r)))\, \mathrm{d}r \tag{10}$$

for $0 \leqq s \leqq \sigma \leqq t$. By Jensen's inequality it follows that

$$h(t) \leqq h(\sigma) + \int_\sigma^t \xi(r, h(r))\, \mathrm{d}r\,. \tag{11}$$

Assume that (9) is false, i.e. $h(t_1) > \psi^{s,v(s,x-y)}(t_1)$ for some $t_1 > s$. Since $\varphi^{s,x}, \varphi^{s,y} \in \mathscr{C}([s, T]; L^1(\Omega; H))$, $s \leqq T$, (see [7], [8]), we obtain (using (5)) the continuity of $h(t)$ on $[s, +\infty)$. Hence we can find $t^* \in [s, t_1)$ such that $h(t^*) = \psi^{s,v(s,x-y)}(t^*)$ and $h(r) > \psi^{s,v(s,x-y)}(r)$ for $t^* < r \leqq t_1$. It follows that

$$h(r) - \psi^{s,v(s,x-y)}(r) \leqq \int_{t^*}^r |\xi(\tau, h(\tau)) - \xi(\tau, \psi^{s,v(s,x-y)}(\tau))|\, \mathrm{d}\tau \leqq \\ \leqq K_{t_1}^1 \int_{t^*}^r (h(\tau) - \psi^{s,v(s,x-y)}(\tau))\, \mathrm{d}\tau\,, \quad t^* \leqq r \leqq t_1\,,$$

and thus by Gronwall's lemma $h(t_1) \leqq \psi^{s,v(s,x-y)}(t_1)$, which is a contradiction.

Q.E.D.

In Liapunov type statements on stability it is sometimes useful to relax the condition on differentiability of v at zero. Set

$$\tau_\delta = \tau_\delta^{s,x,y} = \inf\{t \geqq s, \|\varphi^{s,x}(t) - \varphi^{s,y}(t)\| < \delta\}\,, \quad \delta > 0\,, \quad s \geqq 0\,, \\ x, y \in H\,.$$

Lemma 3. *Let* ξ *be the same as in Lemma 2, let* $v \geqq 0$ *be a continuous function on* $\mathbb{R}_+ \times H$ *satisfying the differentiability conditions* (3), (4) *on* $\mathbb{R}_+ \times (D(A) \setminus \{0\})$, $\mathbb{R}_+ \times (H \setminus \{0\})$, *respectively, and let*

$$|v(t, y)| + \|v_y(t, y)\| + \|v_{yy}(t, y)\| \leqq K_{T,\varepsilon}(1 + \|y\|^q), \tag{12}$$

$\|y\| \geqq \varepsilon$, $0 \leqq t \leqq T$, $\varepsilon > 0$, *for some* $K_{T,\varepsilon} > 0$ *and* $q > 0$.

Assume (8) *for* $x, y \in D(A)$, $x \neq y$, *and*

$$\tau_\delta \to +\infty \quad \textit{almost surely for} \quad \delta \to 0+\,, \quad x \neq y\,. \tag{13}$$

Then (9) *is valid.*

Proof. Let $\eta_\delta: \mathbb{R}_+ \to [0, 1]$, $\delta > 0$, be nondecreasing functions with continuous derivatives $\eta'_\delta, \eta''_\delta$, $\eta_\delta(r) = 0$ for $r \leqq \delta/2$, $\eta_\delta(r) = 1$ for $r \geqq \delta$. Set $v_\delta(t, x) = \eta_\delta(\|x\|^2) v(t, x)$. Obviously $v_\delta \in \mathscr{C}_s^{1,2}(\mathbb{R}_+ \times H)$ and the estimate (5) is fulfilled with some $K_T > 0$, $q > 0$. Furthermore, by (2) and (8)

$$(14) \qquad \left[\left(\frac{\partial}{\partial t} + L\right) v_\delta\right](t, x, y) \leqq u^\delta(t, x - y), \quad t \in \mathbb{R}_+, \quad x, y \in D(A),$$

where u^δ is continuous and such that $|u^\delta(t, x)| \leqq \hat{K}_{T,\delta}(1 + \|x\|^p)$ for some $\hat{K}_{T,\delta} > 0$, $p > 0$, $u^\delta(t, x) = \xi(t, v_\delta(t, x))$ for $\|x\| > \delta$. Consequently, applying Lemma 1 similarly as in the proof of Lemma 2, we obtain

$$\begin{aligned} &\mathsf{E} v_\delta(t \wedge \tau_\delta, \varphi^{s,x}(t \wedge \tau_\delta) - \varphi^{s,y}(t \wedge \tau_\delta)) \leqq \\ &\leqq \mathsf{E} v_\delta(\sigma \wedge \tau_\delta, \varphi^{s,x}(\sigma \wedge \tau_\delta) - \varphi^{s,y}(\sigma \wedge \tau_\delta)) + \\ &+ \mathsf{E} \int_{\sigma \wedge \tau_\delta}^{t \wedge \tau_\delta} u^\delta(r, \varphi^{s,x}(r) - \varphi^{s,y}(r))\, \mathrm{d}r \end{aligned}$$

for $0 \leqq s \leqq \sigma \leqq t$, $\tau_\delta = \tau_\delta^{s,x,y}$, $\|x - y\| > \delta$, and hence

$$\begin{aligned} &\mathsf{E} v(t \wedge \tau_\delta, \varphi^{s,x}(t \wedge \tau_\delta) - \varphi^{s,y}(t \wedge \tau_\delta)) \leqq \\ &\leqq \mathsf{E} v(\sigma \wedge \tau_\delta, \varphi^{s,x}(\sigma \wedge \tau_\delta) - \varphi^{s,y}(\sigma \wedge \tau_\delta)) + \\ &+ \mathsf{E} \int_{\sigma \wedge \tau_\delta}^{t \wedge \tau_\delta} \xi(r, v(r, \varphi^{s,x}(r) - \varphi^{s,y}(r)))\, \mathrm{d}r\,. \end{aligned}$$

Taking $\delta \to 0+$ we obtain (11) by the dominated convergence theorem. Further we can proceed identically as in the proof of Lemma 2 provided we show that the function $h(t)$ is continuous. For arbitrary $R > 0$ set $\Omega_{t,R} = \{\omega \in \Omega;\ \|\varphi^{s,x}(t) - \varphi^{s,y}(t)\| \geqq R\}$, $\Omega_{t,R}^c = \Omega \setminus \Omega_{t,R}$. Let $s < T$, $\lambda, t \in [s, T]$, $\varepsilon \in (0, R)$ be arbitrary. Then for some $\tilde{K}_{T,\varepsilon} > 0$ we have

$$\begin{aligned} &|h(t) - h(\lambda)| \leqq \\ &\leqq \tilde{K}_{T,\varepsilon}(1 + R^q) \{\mathsf{E}\|\varphi^{s,x}(t) - \varphi^{s,x}(\lambda)\| + \mathsf{E}\|\varphi^{s,y}(t) - \varphi^{s,y}(\lambda)\|\} + \\ &+ \mathsf{E}((v(t, \varphi^{s,x}(t) - \varphi^{s,y}(t)) + v(\lambda, \varphi^{s,x}(\lambda) - \varphi^{s,y}(\lambda)))\,. \\ &.\, \chi(\Omega_{t,R} \cup \Omega_{t,\varepsilon}^c \cup \Omega_{\lambda,R} \cup \Omega_{\lambda,\varepsilon}^c))\,. \end{aligned}$$

The first summand on the right-hand side tends to 0 whenever $\lambda \to t$ for any R, ε fixed because $\varphi^{s,x}, \varphi^{s,y} \in \mathscr{C}([s, T]; L^1(\Omega; H))$. So it is sufficient to prove that R, ε can be chosen such that the second summand may be arbitrarily small. But we have

$$\lim_{\varepsilon \to 0+} \mathsf{P}(\Omega_{t,\varepsilon}^c) = 0\,, \quad \lim_{R \to \infty} \mathsf{P}(\Omega_{t,R}) = 0$$

by (13) and (1.5), respectively, the limit being uniform for $t \in [s, T]$ in both cases. Furthermore, the continuity of v at $(t, 0)$ together with (12) implies

$$v(t, x) \leqq C_T(1 + \|x\|^q)\,, \quad x \in H\,, \quad t \in [0, T]$$

for some $C_T > 0$. Our assertion follows, as the second summand may be now majorized by the term

$$\{\mathsf{E}(C_T(2 + \|\varphi^{s,x}(t) - \varphi^{s,y}(t)\|^q + \|\varphi^{s,x}(\lambda) - \varphi^{s,y}(\lambda)\|^q))^2\}^{1/2} \cdot$$
$$\cdot \{\mathsf{P}(\Omega_{t,R}) + \mathsf{P}(\Omega_{\lambda,R}) + \mathsf{P}(\Omega^c_{t,\varepsilon}) + \mathsf{P}(\Omega^c_{\lambda,\varepsilon})\}^{1/2}$$

and (1.5) may be used. Q.E.D.

Remark 1. If $\dim H < \infty$, then (13) holds automatically (see [9], Lemma 2.2). This, in general, is not true for infinite dimensional H; as an example we can take the equation $\dot{x} = Ax$, where A is the infinitesimal generator of the semigroup $S(t)\, x(\varrho) \equiv x(t + \varrho)$, $t \geqq 0$, $\varrho > 0$, in the space $H = L^2((0, +\infty))$.

If the equation (1) is linear, i.e. $a(t, x) = a(t)$, $b(t, x) = b(t)$, then (13) is equivalent to

(15) $\qquad S(t)\, x \neq 0 \quad \text{for all} \quad x \neq 0\,, \quad t \geqq 0\,.$

The condition (15) is obviously satisfied for A self-adjoint with a compact resolvent, in which case

$$S(t)\, x = \sum_i \exp(\alpha_i t) \langle x, e_i \rangle\, e_i\,,$$

where $\{e_i\}$ is an orthonormal basis in H, α_i are reals. Furthermore, (15) clearly holds if $S(t)$ is a group ($t \in \mathbb{R}$). These two cases cover the most usual stochastic (self-adjoint) linear parabolic and hyperbolic equations. In the example below we establish (13) for more general hyperbolic equations.

Example 1. Consider the second order stochastic equation

(16) $\qquad z_{tt} + \alpha z_t + A_0 z = f(t, z) + g(t, z)\, \dot{w}(t)\,,$

where $\alpha \geqq 0$ and A_0 is a positive self-adjoint operator on a real Hilbert space H_2 with domain $\mathsf{D}(A_0)$, such that

$$\langle A_0 z, z \rangle_{H_2} \geqq k \|z\|^2_{H_2}\,, \quad z \in \mathsf{D}(A_0)$$

for some $k > 0$. We rewrite (16) in the form (1) in an obvious way, putting $H = \mathsf{D}(A_0^{1/2}) \times H_2$,

$$\langle x, y \rangle = \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = \langle A_0^{1/2} x_1, A_0^{1/2} y_1 \rangle_{H_2} + \langle x_2, y_2 \rangle_{H_2}\,,$$

$$A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0, & I \\ -A_0, & -\alpha I \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathsf{D}(A) = \mathsf{D}(A_0) \times \mathsf{D}(A_0^{1/2})\,.$$

Note that

(17) $\qquad \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, A \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = -\alpha \|y\|^2_{H_2}\,, \quad (x, y) \in \mathsf{D}(A)\,.$

Assuming Lipschitz continuity of f and g we get by (17)

$$\mathbf{L}_d(\|x - y\|^{-1}) \leqq c\|x - y\|^{-1}, \quad x, y \in \mathsf{D}(A), \quad x \neq y$$

for some $c > 0$, and hence

$$\left(\frac{\partial}{\partial t} + \mathbf{L}_d\right)(e^{-ct}\|x - y\|^{-1}) \leqq 0 .$$

Similarly as in the proof of Lemma 3 we obtain

$$\mathsf{E}(\exp(-c(t \wedge \tau_\delta)) \|\varphi^{s,x}(t \wedge \tau_\delta) - \varphi^{s,y}(t \wedge \tau_\delta)\|^{-1}) \leqq e^{-cs}\|x - y\|^{-1},$$

$0 \leqq s \leqq t$, $\|x - y\| > \delta$, $\tau_\delta = \tau_\delta^{s,x,y}$. Thus

$$\mathsf{E}\|\varphi^{s,x}(t \wedge \tau_\delta) - \varphi^{s,y}(t \wedge \tau_\delta)\|^{-1} \leqq e^{c(t-s)}\|x - y\|^{-1},$$

consequently

$$\frac{1}{\delta} \mathsf{P}[\tau_\delta^{s,x,y} \leqq t] = \mathsf{E}\chi([\tau_\delta^{s,x,y} \leqq t]\| \varphi^{s,x}(\tau_\delta) - \varphi^{s,y}(\tau_\delta)\|^{-1} \leqq$$
$$\leqq e^{c(t-s)}\|x - y\|^{-1},$$

and

$$\lim_{\delta \to 0+} \mathsf{P}[\tau_\delta^{s,x,y} \leqq t] = 0 .$$

Definition 1. *A solution φ of the equation (1) is said to be*

(i) *p-stable ($p > 0$), if for any $\varepsilon > 0$ we find $\delta > 0$ such that for every $t_0 \geqq 0$ and for all solutions $\tilde{\varphi}$ of the equation (1) we have: if $\|\varphi(t_0) - \tilde{\varphi}(t_0)\|_p \leqq \delta$ then $\sup_{t \geqq t_0} \|\varphi(t) - \tilde{\varphi}(t)\|_p \leqq \varepsilon$;*

(ii) *asymptotically p-stable, if it is p-stable and there exists $\Pi > 0$ such that for all $\varepsilon > 0$, $\delta \in (0, \Pi]$ there exists $T(\varepsilon, \delta) > 0$ such that for all $t_0 \in \mathbb{R}_+$ and any solution $\tilde{\varphi}$ of the equation (1) satisfying $\|\varphi(t_0) - \tilde{\varphi}(t_0)\|_p \leqq \delta$ we have*

$$\sup_{t \geqq t_0 + T(\varepsilon, \delta)} \|\varphi(t) - \tilde{\varphi}(t)\|_p \leqq \varepsilon ;$$

(iii) *stable in probability, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $t_0 \geqq 0$ and any solution $\tilde{\varphi}$ of (1) satisfying $\mathsf{P}[\|\varphi(t_0) - \tilde{\varphi}(t_0)\| \geqq \delta] \leqq \delta$ we have $\mathsf{P}[\sup_{t \geqq t_0} \|\varphi(t) - \tilde{\varphi}(t)\| \geqq \varepsilon] \leqq \varepsilon$.*

Definition 2. (i) *The equation (1) is said to be p-stable (asymptotically p-stable, stable in probability), if each of its solutions is p-stable (asymptotically p-stable, stable in probability).*

(ii) *We say that the equation (1) is asymptotically stable in probability provided it is stable in probability and for all $\varepsilon > 0$, $R > 0$ there exists $T(\varepsilon, R) > 0$ such* *that for all* $t_0 \geqq 0$, $x, y \in H$, $\|x - y\| \leqq R$, *we have*

$$\sup_{t \geqq t_0 + T(\varepsilon, R)} \mathsf{P}[\|\varphi^{t_0,x}(t) - \varphi^{t_0,y}(t)\| \geqq \varepsilon] \leqq \varepsilon .$$

The notions of stability introduced above are in fact rather strong; uniformity with respect to initial conditions is required. However, this kind of stability is exactly what we need to prove the averaging properties below.

Recall that a trivial solution $x \equiv 0$ of an ordinary differential equation $\dot{x} = \xi(t, x)$ is said to be uniformly asymptotically stable (in the Liapunov sense) if it is uniformly (Liapunov) stable (this means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x(t_0)| \leqq \delta$, $t_0 \geqq 0$ implies $|x(t)| \leqq \varepsilon$ for $t \geqq t_0$) and there exists $D \in (0, \infty]$ such that for $\varepsilon > 0$, $D > \delta > 0$ we can find $T(\varepsilon, \delta) > 0$ such that $|x(t_0)| \leqq \delta$, $t_0 \geqq 0$ implies $|x(t)| \leqq \varepsilon$ for $t \geqq t_0 + T(\varepsilon, \delta)$. If $D = \infty$ then the solution $x \equiv 0$ is called globally uniformly asymptotically stable.

Proposition 1. *Suppose that the assumptions of either Lemma 2 or Lemma 3 are fulfilled with some* v, ξ. *Let the trivial solution* $x_0 \equiv 0$ *to the equation* $\dot{x} = \xi(t, x)$ *be uniformly Liapunov stable.*

(α) *Assume*

$$c_1 \|x\|^p \leqq v(t, x) \leqq c_2 \|x\|^p , \quad t \geqq 0 , \quad x \in H$$

for some $c_1, c_2, p > 0$. *Then the equation* (1) *is p-stable. Moreover, if* x_0 *is uniformly asymptotically stable then* (1) *is asymptotically p-stable.*

(β) *Assume* $\xi \leqq 0$ *and*

$$\limsup_{x \to 0} \sup_{t \geqq 0} v(t, x) = 0 , \tag{18}$$

$$b(r) \equiv \inf \{v(t, x), (t, x) \in \mathbb{R}_+ \times \{\|x\| \geqq r\}\} > 0 , \quad r > 0 . \tag{19}$$

Then the equation (1) *is stable in probability. Moreover, if* x_0 *is globally uniformly asymptotically stable, then* (1) *is asymptotically stable in probability.*

Proof. (α) If $\mathsf{E}\|\tilde{\varphi}(t_0) - \varphi(t_0)\|^p \leqq \delta$ for some $t_0 \geqq 0$, $\delta > 0$, then $\mathsf{E}v(t_0, \tilde{\varphi}(t_0) - \varphi(t_0)) \leqq c_2\delta$. On the other hand, $\mathsf{E}v(t, \tilde{\varphi}(t) - \varphi(t)) \leqq \varepsilon$, $\varepsilon > 0$, $t \geqq t_0 \geqq 0$ implies $\mathsf{E}\|\tilde{\varphi}(t) - \varphi(t)\|^p \leqq c_1^{-1}\varepsilon$. Thus ($\alpha$) follows from Lemma 2 (or Lemma 3).

(β) Let $\varepsilon > 0$, $\|x - y\| \leqq \varepsilon$, $t_0 \geqq 0$,

$$\tau_\varepsilon = \tau_\varepsilon^{t_0,x,y} = \inf\{t \geqq t_0, \|\varphi^{t_0,x}(t) - \varphi^{t_0,y}(t)\| > \varepsilon\} .$$

By Lemma 2 (or Lemma 3) $(v(t, \varphi^{t_0,x}(t) - \varphi^{t_0,y}(t)))_{t \geqq t_0}$ is a supermartingale and hence by the optional sampling theorem

$$\mathsf{E}v(t \wedge \tau_\varepsilon, \varphi^{t_0,x}(t \wedge \tau_\varepsilon) - \varphi^{t_0,y}(t \wedge \tau_\varepsilon)) \leqq v(t_0, x - y) , \quad t \geqq t_0 .$$

Setting $\Theta = \{\omega; \sup_{s \in [t_0, t]} \| \varphi^{t_0,x}(s) - \varphi^{t_0,y}(s)\| > \varepsilon\}$, we obtain the inequality

$$v(t_0, x - y) \geqq \mathsf{E}\chi(\Theta)\, v(t \wedge \tau_\varepsilon, \varphi^{t_0,x}(t \wedge \tau_\varepsilon) - \varphi^{t_0,y}(t \wedge \tau_\varepsilon)) \geqq$$
$$\geqq \mathsf{E}\chi(\Theta)\, v(\tau_\varepsilon, \varphi^{t_0,x}(\tau_\varepsilon) - \varphi^{t_0,y}(\tau_\varepsilon)) \geqq b(\varepsilon)\, \mathsf{P}(\Theta) ,$$

and hence

$$\mathsf{P}[\sup_{s\in[t_0,t]} \| \varphi^{t_0,x}(s) - \varphi^{t_0,y}(s)\| > \varepsilon] \leqq b(\varepsilon)^{-1}\, v(t_0, x - y)$$

for any $t \geqq t_0$, so that we have

$$\mathsf{P}[\sup_{s\geqq t_0} \|\varphi^{t_0,x}(s) - \varphi^{t_0,y}(s)\| > \varepsilon] \leqq b(\varepsilon)^{-1}\, v(t_0, x - y)\,,$$

which together with (18), (19) implies stability in probability, since for arbitrary solutions $\varphi(t)$, $\tilde{\varphi}(t)$ of the equation (1) the identity

$$\mathsf{P}[\sup_{s\geqq t_0} \|\varphi(s) - \tilde{\varphi}(s)\| > \varepsilon] =$$
$$= \int_{H\times H} \mathsf{P}[\sup_{s\geqq t_0} \|\varphi^{t_0,x}(s) - \varphi^{t_0,y}(s)\| > \varepsilon]\ \mathsf{P}[\varphi(t_0)\in \mathrm{d}x,\ \tilde{\varphi}(t_0)\in \mathrm{d}y]$$

holds. Asymptotical stability in probability easily follows since we have

$$\mathsf{P}[\|\varphi^{t_0,x}(t) - \varphi^{t_0,y}(t)\| > \varepsilon] \leqq b(\varepsilon)^{-1}\, \mathsf{E}v(t, \varphi^{t_0,x}(t) - \varphi^{t_0,y}(t))\,. \quad \text{Q.E.D.}$$

3. AVERAGING ON INFINITE TIME INTERVALS

In the previous sections we have prepared all tools needed for treating the averaging problem on unbounded time intervals for the equations (1.11), (1.12). So, let us consider the equations

(1) $$\mathrm{d}x_\alpha(t) = (A\, x_\alpha(t) + a_\alpha(t, x_\alpha(t)))\,\mathrm{d}t + b_\alpha(t, x_\alpha(t))\,\mathrm{d}w(t)\,,$$

(2) $$\mathrm{d}x_\alpha(t) = (\tilde{A}\, x_\alpha(t) + a_\alpha(t, x_\alpha(t)))\,\mathrm{d}t + b_\alpha(t, x_\alpha(t))\,\mathrm{d}B(t)\,.$$

First, we will prove the theorem announced in Introduction.

Theorem 1. (i) *Let the assumptions* (I), (III), (Vu), (U2) *be fulfilled. Let* $x_0(t)$ *be a mild solution to the equation*

(3) $$\mathrm{d}x_0(t) = (A\, x_0(t) + a_0(t, x_0(t)))\,\mathrm{d}t + b_0(t, x_0(t))\,\mathrm{d}w(t)\,,$$

which is bounded in $L^p(\Omega; H)$ (*i.e.* $\sup_{t\geqq 0} \|x_0(t)\|_p \equiv \Gamma < \infty$) *and asymptotically p-stable. Then for every* $\eta > 0$ *we can find* $\alpha_0 > 0$, $\delta > 0$ *such that for all* $t_0 \in \mathbb{R}_+$ *and any mild solution* $x_\alpha(t)$ *to* (1) *we have: if* $\alpha \in (0, \alpha_0]$ *and* $\|x_\alpha(t_0) - x_0(t_0)\|_p \leqq \delta$, *then*

$$\sup_{t\geqq t_0} \|x_\alpha(t) - x_0(t)\|_p \leqq \eta\,.$$

(ii) *Let the assumptions* (I), (III), (Vcu) *and* (U3) *be fulfilled. Let* x_0 *be a mild solution to the equation*

(4) $$\mathrm{d}x_0(t) = (\tilde{A}\, x_0(t) + a_0(t, x_0(t)))\,\mathrm{d}t + b_0(t, x_0(t))\,\mathrm{d}B(t)\,,$$

which is bounded in $L^p(\Omega; H)$ and asymptotically p-stable. Then the assertion in (i) *is valid also for mild solutions to* (2).

(iii) *Let the assumptions* (I), (III), (VIu), (U2) *be fulfilled. Suppose* $K \subseteqq L^p(\Omega; H)$ *is such that the set*

$$\mathfrak{M} = \{\|\varphi(t)\|^p;\ t \geqq t_0 \geqq 0,\ \varphi \text{ is a mild solution to } (3),\ \varphi(t_0) \in K\}$$

is uniformly integrable. Let there exist $\hat{\alpha} > 0$ *and* $\hat{K} \subseteqq K$ *such that for any mild solution* $x_\alpha(t)$ *of* (1) *we have: if* $\alpha \in (0, \hat{\alpha}]$ *and* $x_\alpha(t_0) \in \hat{K}$ *then* $x_\alpha(t) \in K$ *for all* $t \geqq t_0$. *Let* $x_0(t)$ *be a solution of* (3) *which is asymptotically p-stable. Then for every* $\eta > 0$ *there exist* $\alpha_0 > 0$, $\delta > 0$ *such that for every mild solution* $x_\alpha(t)$ *of* (1) *and for any* $t_0 \in \mathbb{R}_+$ *we have: if* $\alpha \in (0, \alpha_0]$, $x_\alpha(t_0) \in \hat{K}$, *and* $\|x_\alpha(t_0) - x_0(t_0)\|_p \leqq \delta$ *then*

$$\sup_{t \geqq t_0} \|x_\alpha(t) - x_0(t)\|_p \leqq \eta .$$

(iv) *Let the assumptions* (I), (III), (VIcu), (U3) *be fulfilled. Suppose* $K \subseteqq L^p(\Omega; H)$ *is such that the set*

$$\mathfrak{M} = \{\|\varphi(t)\|^p;\ t \geqq t_0 \geqq 0,\ \varphi \text{ is a mild solution to } (4),\ \varphi(t_0) \in K\}$$

is uniformly integrable. Let $\hat{\alpha} > 0$ *and* $\hat{K}$ *have the same properties as in* (iii), *but with respect to mild solutions of the problem* (2). *Let* x_0 *be a solution to* (4) *which is asymptotically p-stable. Then the conclusion in* (iii) *holds also for mild solutions of the equation* (2).

Remark 1. The statements (iii), (iv) look rather sophisticated, but, unlike (i) and (ii), they can be used for linear problems, in which case we take for K, $\hat{K}$ appropriate balls in $L^q(\Omega; H)$, $q > p$; see also Example 2 below. Note that in (iii), (iv) we need not assume the boundedness of x_0 (which, of course, follows from the assertion).

Proof. The idea of the proof closely resembles that of the proof of Th. 3 in [11] where the case dim $H < \infty$ and $x_0 \equiv 0$ is investigated. For the sake of completeness we repeat here all necessary arguments.

We shall prove the statement (i). Let us choose $\eta > 0$, $t_0 \in \mathbb{R}_+$ arbitrarily. By the p-stability of x_0 we find $\delta > 0$ such that $\|\varphi(t_0) - x_0(t_0)\|_p \leqq \delta$ implies $\sup_{t \geqq t_0} \|x_0(t) - \varphi(t)\|_p \leqq \eta/4$ for any solution φ of (3). Without loss of generality we may choose $\delta \in (0, \min(\eta, \Pi))$, where Π is the constant from the definition of the asymptotic p-stability. According to Prop. 1.1 there exists $\alpha_0 > 0$ such that for $\alpha \in (0, \alpha_0]$ and for any mild solution φ to (3) satisfying $\varphi(t_0) = x_\alpha(t_0)$ and $\|\varphi(t_0)\|_p \leqq \Gamma + \delta$ we have

$$\sup \{\|x_\alpha(t) - \varphi(t)\|_p,\ t_0 \leqq t \leqq t_0 + T(\delta/2, \delta)\} \leqq \delta/2 .$$

Let us prove that these α_0, δ are the desired quantities. Let x_α, $\alpha \leqq \alpha_0$ be a solution to (1) such that $\|x_\alpha(t_0) - x_0(t_0)\|_p \leqq \delta$. Let $\bar{x}_0$ be a solution of the problem (3) satisfying $\bar{x}_0(t_0) = x_\alpha(t_0)$. Then $\|\bar{x}_0(t_0) - x_0(t_0)\|_p \leqq \delta$, hence

$$\sup \{\|\bar{x}_0(t) - x_0(t)\|_p, \ t_0 \leqq t \leqq t_0 + T(\delta/2, \delta)\} \leqq \eta/4 ;$$

further $\|\bar{x}_0(t_0)\|_p \leqq \|x_0(t_0)\|_p + \|\bar{x}_0(t_0) - x_0(t_0)\|_p \leqq \Gamma + \delta$, thus

$$\sup \{\|\bar{x}_0(t) - x_\alpha(t)\|_p, \ t_0 \leqq t \leqq t_0 + T(\delta/2, \delta)\} \leqq \delta/2 \leqq \eta/2 .$$

Combining all these estimates we obtain

$$\sup \{\|x_0(t) - x_\alpha(t)\|_p, \ t_0 \leqq t \leqq t_0 + T(\delta/2, \delta)\} \leqq \eta .$$

Moreover, by the asymptotical p-stability

$$\|\bar{x}_0(t_0 + T(\delta/2, \delta)) - x_0(t_0 + T(\delta/2, \delta))\|_p \leqq \delta/2 ,$$

hence

$$\|x_\alpha(t_0 + T(\delta/2, \delta)) - x_0(t_0 + T(\delta/2, \delta))\|_p \leqq \delta .$$

We see that all the above considerations can be repeated on the interval $[t_0 + T(\delta/2, \delta), t_0 + 2T(\delta/2, \delta)]$ with an auxiliary solution $\tilde{x}_0(t), \tilde{x}_0(t_0 + T(\delta/2, \delta)) = x_\alpha(t_0 + T(\delta/2, \delta))$, and we complete the proof by induction.

The statement (iii) can be proved similarly, if Prop. 1.2 is used instead of Prop. 1.1; the proofs of (ii), (iv) are analogous to those of (i), (iii), respectively. Q.E.D.

To assume the asymptotic p-stability of the process x_0 is quite restrictive. In the sequel we content ourselves with the supposition that the equation (3) is asymptotically stable in probability, and we will prove that $x_\alpha \to x_0$ in probability. In such a case we will not need Prop. 1.1 in its full strength, so we leave out some of the hypotheses of that proposition and rely on the following assumption, which is weaker than the assertion of Prop. 1.1:

(P) Suppose that for every $\eta > 0$, $T > 0$, $R > 0$ there exists $\alpha_1 > 0$ such that for all $\alpha \in (0, \alpha_1]$, $t_0 \in \mathbb{R}_+$, $x \in H$, $\|x\| \leqq R$ we have

$$\sup_{t \in [t_0, t_0 + T]} \mathsf{P}[\|x_\alpha^{t_0,x}(t) - x_0^{t_0,x}(t)\| \geqq \eta] \leqq \eta ,$$

where $x_\alpha^{t_0,x}$ denotes the mild solution to (1) with the initial condition $x_\alpha^{t_0,x}(t_0) = x$.

Recall that a family $\{X_\lambda\}$ of random variables is said to be equibounded in probability if for any $\varepsilon > 0$ there exists $R \geqq 0$ such that $\sup_\lambda \mathsf{P}[|X_\lambda| \geqq R] \leqq \varepsilon$.

Proposition 1. *Let the hypotheses* (I), (III), (P) *be satisfied, let the equation* (3) *be asymptotically stable in probability. Let the set*

$$\mathfrak{K} = \{\|x_\alpha^{t_0,x}(t)\|, \ 0 \leqq \alpha \leqq \alpha_0, \ \|x\| \leqq \delta_0, \ t \geqq t_0 \geqq 0\}$$

be equibounded in probability for some $\alpha_0 > 0$ *and any* $\delta_0 > 0$. *Then for every* $\eta > 0$, $x \in H$ *there exist* $\alpha_1 > 0$, $\delta > 0$ *such that for all* $y \in H$, $\|x - y\| \leqq \delta$, $\alpha \in (0, \alpha_1]$ *and any* $t_0 \in \mathbb{R}_+$ *we have*

$$\sup_{t \geqq t_0} \mathsf{P}[\|x_\alpha^{t_0,y}(t) - x_0^{t_0,x}(t)\| \geqq \eta] \leqq \eta .$$

Proof. Take $\eta \in (0,1)$, $x \in H$. The stability in probability implies that there exists $\delta > 0$ (we can take $\delta < \min(1, \eta/2)$) such that

$$\mathsf{P}[\|x_0^{t_0,X}(t) - x_0^{t_0,Y}(t)\| \geqq \eta/2] \leqq \eta/2\,, \quad t \geqq t_0 \tag{5}$$

for all $t_0 \geqq 0$ and any random initial conditions X, Y such that $\mathsf{P}[\|X - Y\| \geqq \delta] \leqq \leqq \delta$. By the equiboundedness in probability of $\mathfrak{K}$ (with $\delta_0 = \|x\| + 1$) we find $R \geqq 0$ (take $R \geqq \delta_0$) such that

$$\sup\{\mathsf{P}[\|x_\alpha^{t_0,y}(t)\| + \|x_0^{t_0,Y}(t)\| \geqq R];\ t \geqq t_0 \geqq 0,\ \alpha \leqq \alpha_0, \\ \|x - y\| \leqq \delta\} \leqq \delta/4\,. \tag{6}$$

Furthermore, by the asymptotical stability in probability we find $T = T(\delta/4, R)$ such that for all $t_0 \in \mathbb{R}_+$, $y, z \in H$, $\|y - z\| \leqq R$ we have

$$\sup_{t \geqq t_0 + T} \mathsf{P}[\|x_0^{t_0,y}(t) - x_0^{t_0,z}(t)\| \geqq 4^{-1}\delta] \leqq 4^{-1}\delta\,. \tag{7}$$

Finally, by (P) we find $\alpha_2 \in (0, \alpha_0)$ such that

$$\sup_{t \in [t_1, t_1 + T]} \mathsf{P}[\|x_\alpha^{t_1,y}(t) - x_0^{t_1,y}(t)\| \geqq \delta/2] \leqq \delta/2 \tag{8}$$

for all $t_1 \in \mathbb{R}_+$, $y \in H$, $\|y\| \leqq R + 1$, $\alpha \leqq \alpha_2$.

Take $y \in H$, $\|y - x\| \leqq \delta$. By (5) it follows that

$$\sup_{t \geqq t_0} \mathsf{P}[\|x_0^{t_0,y}(t) - x_0^{t_0,x}(t)\| \geqq \eta/2] \leqq \eta/2\,.$$

By (5) and (8) we obtain (note that $\delta < \eta/2$)

$$\sup_{t \in [t_0, t_0 + T]} \mathsf{P}[\|x_\alpha^{t_0,y}(t) - x_0^{t_0,x}(t)\| \geqq \eta] \leqq \eta \tag{9}$$

for $\alpha \leqq \alpha_2$. Similarly, by (7) and (8) we get

$$\mathsf{P}[\|x_\alpha^{t_0,y}(t_0 + T) - x_0^{t_0,x}(t_0 + T)\| \geqq \tfrac{3}{4}\delta] \leqq \tfrac{3}{4}\delta \tag{10}$$

for $\alpha \leqq \alpha_2$. Set $Y_1 = x_\alpha^{t_0,y}(t_0 + T)$, $X_1 = x_0^{t_0,x}(t_0 + T)$. By (5) and (10) we have

$$\sup_{t \geqq t_0 + T} \mathsf{P}[\|x_0^{t_0+T,X_1}(t) - x_0^{t_0+T,Y_1}(t)\| \geqq \eta/2] \leqq \eta/2\,. \tag{11}$$

Since $\mathsf{P}[\|Y_1\| \geqq R] \leqq \delta/4$, we get from (8), (11)

$$\sup_{t \in [t_0 + T, t_0 + 2T]} \mathsf{P}[\|x_0^{t_0,x}(t) - x_\alpha^{t_0,y}(t)\| \geqq \eta] =$$
$$= \sup_{t \in [t_0 + T, t_0 + 2T]} \mathsf{P}[\|x_0^{t_0+T,X_1}(t) - x_\alpha^{t_0+T,Y_1}(t)\| \geqq \eta] \leqq$$
$$\leqq \eta/2 + \delta/2 + \delta/4 \leqq \eta\,.$$

Furthermore, since $\mathsf{P}[\|X_1\| + \|Y_1\| \geqq R] \leqq \delta/4$, we get

$$\mathsf{P}[\|x_0^{t_0+T,X_1}(t_0 + 2T) - x_\alpha^{t_0+T,Y_1}(t_0 + 2T)\| \geqq \tfrac{3}{4}\delta] \leqq$$
$$\leqq \mathsf{P}([x_0^{t_0+T,Y_1}(t_0 + 2T) - x_\alpha^{t_0+T,Y_1}(t_0 + 2T)\| \geqq \delta/2] \cup$$
$$\cup [\|x_0^{t_0+T,Y_1}(t_0 + 2T) - x_0^{t_0+T,X_1}(t_0 + 2T)\| \geqq \delta/4]) \leqq \delta,$$

and we can proceed similarly on $[t_0 + 2T, t_0 + 3T]$. The proof can be easily completed by induction. Q.E.D.

The rest of the section is devoted to the averaging problem

$$dx_\varepsilon(t) = \left(Ax_\varepsilon(t) + \alpha\left(\frac{t}{\varepsilon}, x_\varepsilon(t)\right)\right) dt + \sigma\left(\frac{t}{\varepsilon}, x_\varepsilon(t)\right) dw(t), \tag{12}$$

where $A\colon \mathsf{D}(A) \to H$ is an infinitesimal generator of a strongly continuous semigroup $S(t)$ on H satisfying (U2) and the coefficients α, σ satisfy the estimates of (U1). Assume further that there exist Lipschitz functions $\bar{\alpha}\colon H \to H$, $\bar{\sigma}\colon H \to \mathscr{L}(Y, H)$ such that for some $\Delta_0 > 0$ we have: if $t_1, t_2 \in \mathbb{R}_+$, $0 \leqq t_1 \leqq t_2 \leqq t_1 + \Delta_0$ then

$$\lim_{\varepsilon \to 0+} \int_{t_1}^{t_2} S(t_2 - s)\left(\alpha\left(\frac{s + t_0}{\varepsilon}, x\right) - \bar{\alpha}(x)\right) ds = 0$$

uniformly for $t_0 \in \mathbb{R}_+$, and

$$\lim_{T \to \infty} \frac{1}{T} \int_{\beta T}^{\beta T + T} (\operatorname{tr}\{[\sigma(s, x) - \bar{\sigma}(x)]\, W[\sigma(s, x) - \bar{\sigma}(x)]^*\})^{p/2}\, ds = 0$$

uniformly for $\beta \geqq 0$.

If the convergence in the above formulae is uniform also for $x \in H$ then it is obvious how to apply Theorem 1; however, if we assume the convergence to be only locally uniform in x then in order to obtain effective results on averaging in $L^p(\Omega; H)$ and in probability for the equation (12) we need verifiable conditions guaranteeing boundedness of the q-th moment of the solution to the limit equation

$$d\bar{x}(t) = (A\,\bar{x}(t) + \bar{\alpha}(\bar{x}(t)))\, dt + \bar{\sigma}(\bar{x}(t))\, dw(t), \tag{13}$$

or the equiboundedness in probability of the set $\mathfrak{K}$ defined in Proposition 1.

If $v \in \mathscr{C}^2(H)$ (the set of twice continuously differentiable functions on H) then we set

$$[\boldsymbol{L}v](x) = \langle Ax + \bar{\alpha}(x), v_x(x)\rangle + \tfrac{1}{2}\operatorname{tr}(\bar{\sigma}^*(x)\, v_{xx}(x)\, \bar{\sigma}(x)\, W), \quad x \in \mathsf{D}(A),$$
$$[\boldsymbol{L}^\varepsilon v](t, x) = \langle Ax + \alpha(t/\varepsilon, x), v_x(x)\rangle +$$
$$+ \tfrac{1}{2}\operatorname{tr}(\sigma^*(t/\varepsilon, x)\, v_{xx}(x)\, \sigma(t/\varepsilon, x)\, W), \quad x \in \mathsf{D}(A),$$
$$[\boldsymbol{L}_d v](x, y) = \langle Ax - Ay + \bar{\alpha}(x) - \bar{\alpha}(y), v_x(x - y)\rangle +$$
$$+ \tfrac{1}{2}\operatorname{tr}([\bar{\sigma}(x) - \bar{\sigma}(y)]^*\, v_{xx}(x - y)\, [\bar{\sigma}(x) - \bar{\sigma}(y)]\, W), \quad x, y \in \mathsf{D}(A).$$

Proposition 2. *Let $v \in \mathscr{C}^2(H)$ be a nonnegative function satisfying* (2.5) *and*

$$d_1\|x\|^p + c_1 \leqq v(x) \leqq d_2\|x\|^p + c_2\,, \quad x \in H \tag{14}$$

for some $p > 0$, $d_1, d_2 > 0$, $c_1, c_2 \in \mathbb{R}$. Assume $\bar{L}v(x) \leqq \xi(v(x))$, $x \in D(A)$, where $\xi\colon \mathbb{R}_+ \to \mathbb{R}$ is a concave Lipschitz function such that all solutions of the equation

$$\dot{y}(t) = \xi(y(t))\,, \quad t \geqq t_0\,, \quad y(t_0) = y_0\,,$$

are bounded on their domains uniformly with respect to $t_0 \in \mathbb{R}_+$ and to y_0 from compact intervals. Then for every $K > 0$ there exists $M > 0$ such that $\mathsf{E}\|\bar{x}(t)\|^p \leqq M$, $t \geqq t_0$, provided $\mathsf{E}\|\bar{x}(t_0)\|^p \leqq K$, where $\bar{x}$ stands for a mild solution to the problem (13). *If moreover $[L^1 v](t, x) \leqq \xi(v(x))$, $x \in D(A)$, then also $\mathsf{E}\|x_\varepsilon(t)\|^p \leqq M$, $t \geqq t_0$, $\varepsilon \in (0, 1]$, provided $\mathsf{E}\|x_\varepsilon(t_0)\|^p \leqq K$, where x_ε denotes a mild solution to* (12) *and $M = M(K)$ does not depend on $\varepsilon \in (0, 1]$, $t_0 \in \mathbb{R}_+$.*

Proof. Lemma 2.1 applied to the equation (13) yields

$$\mathsf{E}v(\bar{x}(t)) \leqq \mathsf{E}v(\bar{x}(t_0)) + \mathsf{E}\int_{t_0}^{t} \xi(\bar{x}(s))\,\mathrm{d}s\,.$$

By the same procedure as in the proof of Lemma 2.2 we obtain

$$\mathsf{E}v(\bar{x}(t)) \leqq \sup\{y(t);\ t \geqq t_0,\ 0 \leqq y(t_0) \leqq d_2K + c_2\}\,,$$

if the solution $\bar{x}(t)$ of (13) satisfies $\mathsf{E}v(\bar{x}(t_0)) \leqq d_2K + c_2$. By (14) it follows that

$$\mathsf{E}\|\bar{x}(t)\|^p \leqq \sup\{d_1^{-1}\,y(t) - c_1;\ t \geqq t_0,\ 0 \leqq y(t_0) \leqq d_2K + c_2\} \equiv M\,.$$

The assertion on $x_\varepsilon(t)$ can be proved analogously (note that $[L^\varepsilon v]\,(t, x) = [L_v^1]\,.\,(t/\varepsilon, x)$). Q.E.D.

Lemma 1. *Let $\xi\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfy the assumptions of Lemma 2.2. Denote by z_ε the solution to the equation $\dot{z}_\varepsilon(t) = \xi(t/\varepsilon, z_\varepsilon(t))$, $z_\varepsilon(0) = z$, $z > 0$, $0 < \varepsilon \leqq 1$. Let $y(t)$ be the solution to $\dot{y} = \xi(t, y)$, $y(0) = z$. Then*

$$\sup_{t \geqq 0} z_\varepsilon(t) \leqq \sup_{t \geqq 0} y(t)\,.$$

Proof. With no loss of generality we can assume $y(t) > 0$ for $t \geqq 0$. Set $h_\varepsilon(t) = z_\varepsilon(\varepsilon t)$, $M = \sup_{t \geqq 0} y(t)$. Assume that $h_\varepsilon(t) > M$ for some $\varepsilon \in (0, 1)$, $t > 0$. Let $t_0 < t$ be such that $h_\varepsilon(t_0) = y(t_0)$ and $h_\varepsilon(s) > y(s)$ for $s \in (t_0, t]$. Concavity of $\xi(t, \cdot)$ and the identity $\xi(t, 0) = 0$ yield

$$\frac{\dot{h}_\varepsilon(s)}{h_\varepsilon(s)} = \frac{\varepsilon\xi(s, h_\varepsilon(s))}{h_\varepsilon(s)} \leqq \frac{\varepsilon\xi(s, y(s))}{y(s)} = \frac{\varepsilon\,\dot{y}(s)}{y(s)}\,, \quad s \in (t_0, t]\,.$$

It follows that

$$\log\left[h_\varepsilon(t)\,\frac{(y(t_0))^\varepsilon}{h_\varepsilon(t_0)}\right] \leqq \log\,(y(t))^\varepsilon$$

and hence

$$h_\varepsilon(t) \leqq h_\varepsilon(t_0)\left(\frac{y(t)}{y(t_0)}\right)^\varepsilon \leqq (y(t_0))^{1-\varepsilon}(y(t))^\varepsilon \leqq M\,,$$

which is a contradiction. Q.E.D.

Proposition 3. *Let $v \in \mathscr{C}^2(H)$, $v \geqq 0$, satisfy (2.5) and $[L^1 v](t, x) \leqq \xi_1(t, v(x))$, $[\bar{L}v](x) \leqq \xi_2(v(x))$, $x \in D(A)$, where both functions ξ_1, ξ_2 fulfil the assumptions (on ξ) of Lemma 2.2. Assume further that*

$$(15)\qquad \lim_{R\to+\infty} b(R) \equiv \lim_{R\to+\infty} \inf_{\|x\|\geqq R} v(x) = +\infty\,.$$

Then the set

$$\mathfrak{R} = \{\|x_\varepsilon^{t_0,y}(t)\| + \|\bar{x}^{t_0,x}(t)\|,\ \varepsilon\in(0,1],\ t \geqq t_0 \geqq 0,\ \|x\| + \|y\| \leqq \delta_0\}\,,$$

where $x_\varepsilon^{t_0,y}$, $\bar{x}^{t_0,x}$ denote mild solutions to (12), (13), respectively, is equibounded in probability for all $\delta_0 > 0$ provided the solutions to the equations $\dot{z} = \xi_1(t, z)$, $z(t_0) = z_0$, and $\dot{h} = \xi_2(h)$, $h(t_0) = h_0$, are bounded on $[t_0, +\infty)$ uniformly with respect to $t_0 \in \mathbb{R}_+$ and to z_0, h_0 in compact intervals.

Proof. We have $[L^\varepsilon v](t, x) = [L^1 v](t/\varepsilon, x) \leqq \xi_1(t/\varepsilon, v(x))$, $x \in D(A)$, and hence by Lemma 2.1 and Jensen's inequality

$$\mathsf{E}v(x_\varepsilon^{t_0,x}(t)) \leqq \mathsf{E}v(x_\varepsilon^{t_0,x}(s)) + \int_s^t \xi_1(r/\varepsilon, \mathsf{E}v(x_\varepsilon^{t_0,x}(r)))\,\mathrm{d}r$$

for all $0 \leqq t_0 \leqq s \leqq t$, $\varepsilon \in (0, 1]$. By the same argument as in the proof of Lemma 2.2 we get

$$\mathsf{E}v(x_\varepsilon^{t_0,x}(t)) \leqq u_\varepsilon(t)\,, \quad t \geqq t_0\,,$$

where $\dot{u}_\varepsilon(t) = \xi_1(t/\varepsilon, u_\varepsilon(t))$, $u_\varepsilon(t_0) = v(x)$. Hence by Lemma 1 for every $\delta_0 > 0$ there exists a constant $c > 0$ such that $\mathsf{E}v(x_\varepsilon^{t_0,x}(t)) \leqq c$ for all $\varepsilon \in (0, 1)$, $\|x\| \leqq \delta_0$, $0 \leqq t_0 \leqq t$. It follows that

$$\mathsf{P}[\|x_{\varepsilon_j}^{t_0,x}\| > R] \leqq b(R)^{-1}\,c\,, \quad R > 0\,,$$

and thus the family $\{\|x_\varepsilon^{t_0,y}(t)\|,\ \varepsilon \in (0, 1],\ t \geqq t_0 \geqq 0,\ \|y\| \leqq \delta_0\}$ is equibounded in probability. For the process $\bar{x}^{t_0,x}$ we can proceed similarly. Q.E.D.

Example 1. Let the coefficients α, σ of the equation (12) be bounded on $\mathbb{R}_+ \times H$ and assume

$$\langle Ax, x\rangle \leqq -\lambda_0\|x\|^2\,, \quad x \in D(A)\,,$$

for some $\lambda_0 > 0$. Then the conclusion of Proposition 3 is valid, i.e. $\mathfrak{R}$ is equibounded in probability. Furthermore, the p-th moment of $\bar{x}^{t_0,x}$ is bounded on $[t_0, +\infty)$ for any $t_0 \in \mathbb{R}_+$, $x \in H$, $p \geqq 2$. Indeed, we have

$$L^1(\|x\|^p) \leqq -p\lambda_0\|x\|^p + p\|x\|^{p-1} \sup_{\mathbb{R}_+ \times H} \|\alpha\| +$$

$$+ \tfrac{1}{2}p(p-1)\|x\|^{p-2} \operatorname{tr} W \sup_{\mathbb{R}_+ \times H} \|\sigma\|^2 \leqq -\varkappa\|x\|^p + M,$$

$x \in D(A)$, for some $\varkappa > 0$, $M > 0$; similarly

$$\bar{L}(\|x\|^p) \leqq -\varkappa\|x\|^p + M,$$

$x \in D(A)$, and we may apply Propositions 2 and 3.

Example 2. Let $D \subseteqq \mathbb{R}^n$ be a bounded region with a $\mathscr{C}^2$-boundary. Let us consider the stochastic parabolic equation

$$\text{(16)} \qquad \frac{\partial u}{\partial t} = \Delta u(t,x) + r_0(t)\, u(t,x) + \frac{r_1(t)\, u(t,x)}{1 + |u(t,x)|} +$$

$$+ \left(\frac{r_2(t)\, u(t,x)}{1 + |u(t,x)|} + g(t,x)\right) \dot{w}(t,x), \quad (t,x) \in \mathbb{R}_+ \times D,$$

$$u(0,x) = u_0(x), \quad u(t,x)|_{\partial D} = 0,$$

where r_0, r_1, r_2 and g are bounded measurable functions, $\dot{w}(t,x)$ stands symbolically for a space dependent white noise. In order to give a precise meaning to (16) we consider its infinite dimensional version

$$\mathrm{d}\zeta(t) = (A\,\zeta(t) + f(t, \zeta(t)))\,\mathrm{d}t + \Phi(t, \zeta(t))\,\mathrm{d}w(t)$$

in the space $H = L^2(D)$, where $w(t)$ is a $Y \equiv \mathrm{H}^k(D)$-valued Wiener process with a nuclear covariance operator W, $2k > n$, and

$$f\colon \mathbb{R}_+ \times H \to H, \quad f(t,x)(\vartheta) = r_0(t)\, x(\vartheta) + \frac{r_1(t)\, x(\vartheta)}{1 + |x(\vartheta)|}, \vartheta \in D,$$

$$\Phi\colon \mathbb{R}_+ \times H \to \mathscr{L}(Y, H), \quad [\Phi(t,x)\, h](\vartheta) = \left(\frac{r_2(t)\, x(\vartheta)}{1 + |x(\vartheta)|} +\right.$$

$$\left. + g(t,\vartheta)\right) h(\vartheta), \quad \vartheta \in D, \quad h \in Y,$$

$$A = \Delta|_{\mathrm{H}_0^1(D) \cap \mathrm{H}^2(D)}.$$

Recall that $\mathrm{H}^k(D)$ denotes the usual Sobolev space of functions in $L^2(D)$ the distributive derivatives of which up to the k-th order lie in $L^2(D)$, $\mathrm{H}_0^k(D)$ is the subspace of functions with zero trace on the boundary. It is easy to see that the estimates of the assumption (U1) are fulfilled with some $K_1, K_2 > 0$. Also, A gives rise to a holomorphic semigroup $S(t)$ (cf. e.g. [3], Th. XIV. 8.1). We will consider the averaging problem

$$\text{(17)} \qquad d\zeta_\varepsilon(t) = (A\,\zeta_\varepsilon(t) + f(t/\varepsilon, \zeta_\varepsilon(t)))\,dt + \Phi(t/\varepsilon, \zeta_\varepsilon(t))\,dw(t)$$

assuming that

$$\lim_{T\to\infty} \frac{1}{T}\int_{\mu T}^{\mu T+T} r_0(t)\,dt = r_0\,, \quad \lim_{T\to\infty} \frac{1}{T}\int_{\mu T}^{\mu T+T} r_1(t)\,dt = r_1\,,$$

$$\lim_{T\to\infty} \frac{1}{T}\int_{\mu T}^{\mu T+T} |r_2(t) - r_2|^p\,dt = 0\,, \quad \lim_{T\to\infty} \frac{1}{T}\int_{\mu T}^{\mu T+T} \|g(t,\cdot) - \bar{g}\|^p_{L^\infty(D)}\,dt = 0$$

hold uniformly in $\mu \geqq 0$ for some $r_0, r_1, r_2 \in \mathbb{R}$, $\bar{g} \in L^\infty(D)$, and $p \geqq 2$. Then the conditions (1.9), (1.10) are fulfilled uniformly for $t_0 \in \mathbb{R}_+$ and for $x \in H$, $\|x\| \leqq L$, for any $L > 0$ (cf. Remark 1.5) provided we set $a_\alpha(t, x) = f(t/\alpha, x)$, $b_\alpha(t, x) = \Phi(t/\alpha, x)$, $\alpha > 0$, and $a_0 = \bar{f}$, $b_0 = \bar{\Phi}$, where $\bar{f}$, $\bar{\Phi}$ are the limit coefficients

$$\bar{f}(x)(\vartheta) = r_0\,x(\vartheta) + \frac{r_1\,x(\vartheta)}{1 + |x(\vartheta)|}\,, \quad \vartheta \in D\,,$$

$$|\bar{\Phi}(x)\,h|\,(\vartheta) = \left(\frac{r_2\,x(\vartheta)}{1 + |x(\vartheta)|} + \bar{g}(\vartheta)\right) h(\vartheta)\,, \quad \vartheta \in D\,, \quad h \in Y\,.$$

It is well known that

$$\langle Ax, x\rangle \leqq -\lambda_0\|x\|^2\,, \quad x \in D(A)$$

for some $\lambda_0 > 0$. Hence

$$\begin{aligned}
&\mathbf{L}_d(\|x - y\|^p) \leqq \\
&\leqq p\|x - y\|^p\{-\lambda_0 + r_0 + \max(0, r_1) + \tfrac{1}{2}(p-1)K_2^2 \operatorname{tr} W\}\,,\\
&\bar{\mathbf{L}}(\|x\|^q) \leqq q\|x\|^q\{-\lambda_0 + r_0 + \max(0, r_1)\} + \\
&+ \tfrac{1}{2}\|\bar{\Phi}(x)\|^2\,q(q-1)\|x\|^{q-2} \operatorname{tr} W \leqq \\
&\leqq q\|x\|^q\{-\lambda_0 + r_0 + \max(0, r_1) + K_2^2(q-1)\operatorname{tr} W\} + \\
&+ q(q-1)\,\|x\|^{q-2}\,\|\bar{g}\|^2_{L^\infty} \operatorname{tr} W
\end{aligned}$$

for any $q \geqq 2$, $x, y \in D(A)$. Similarly

$$\begin{aligned}
&L^1(\|x\|^q) \leqq q\|x\|^q\{-\lambda_0 + r_0(t) + \max(0, r_1(t)) + K_2^2(q-1)\operatorname{tr} W\} + \\
&+ q(q-1)\,\|x\|^{q-2}\,\|g(t,\cdot)\|^2_{L^\infty} \operatorname{tr} W\,.
\end{aligned}$$

Assume

$$-\lambda_0 + r_0(t) + \max(0, r_1(t)) + K_2^2(q'-1)\operatorname{tr} W < 0\,, \quad t \in \mathbb{R}_+$$

for some $q' > p$. Then $\bar{L}_d(\|x - y\|^p) \leqq -\alpha_1\|x - y\|^p$, $x, y \in D(A)$, for some $\alpha_1 > 0$ and hence the limit equation

$$\text{(18)} \qquad d\bar{\zeta}(t) = (A\,\bar{\zeta}(t) + \bar{f}(\bar{\zeta}(t)))\,dt + \bar{\Phi}(\bar{\zeta}(t))\,dw(t)$$

is asymptotically p-stable (Prop. 2.1 (α)). Take any $q \in (p, q')$. We have

$$\bar{L}(\|x\|^q) \leqq -\alpha_2\|x\|^q + M_2\,, \quad L^1(\|x\|^q) \leqq -\alpha_2\|x\|^q + M_2$$

for some $\alpha_2, M_2 > 0$. Thus by Proposition 2 we get for arbitrary $\hat{R} > 0$

$$R \equiv \sup\{\|\zeta_\varepsilon(t)\|_q;\ 0 < \varepsilon \leqq 1,\ t \geqq t_0 \geqq 0,\ \|\zeta_\varepsilon(t_0)\|_q \leqq \hat{R}\} < \infty$$

and

$$\bar{R} \equiv \sup\{\|\bar{\zeta}(t)\|_q;\ t \geqq t_0 \geqq 0,\ \|\bar{\zeta}(t_0)\|_q \leqq R\} < \infty\,.$$

Now we can apply Theorem 1 (iii) setting $\hat{K} = \{u \in L^q(\Omega; H);\ \|u\|_q \leqq \hat{R}\}$ and $K = \{u \in L^q(\Omega; H);\ \|u\|_q \leqq R\}$, $\hat{\alpha} = 1$. The uniform integrability of $\mathfrak{M}$ follows from the Hölder inequality. Indeed, for any solution $\bar{\zeta}(t)$ of (18) such that $\bar{\zeta}(t_0) \in K$ and for any measurable set $B \subseteqq \Omega$ we obtain

$$\mathsf{E}\chi_B\|\bar{\zeta}(t)\|^p \leqq \|\bar{\zeta}(t)\|_q^p\,(\mathsf{P}(B))^{1-p/q} \leqq \bar{R}^p(\mathsf{P}(B))^{1-p/q}\,.$$

By Theorem 1 (iii) we conclude that for every solution $(\bar{\zeta}(t))_{t\geqq 0}$ to (18) and every $\eta > 0$, $\hat{R} > 0$ there exist $\varepsilon_0 > 0$, $\delta > 0$ such that for all $t_0 \in \mathbb{R}_+$, $\varepsilon \in (0, \varepsilon_0]$ and any solution ζ_ε to (17) satisfying $\zeta_\varepsilon(t_0) \in \hat{K}$, $\|\zeta_\varepsilon(t_0) - \bar{\zeta}(t_0)\|_p \leqq \delta$ we have

$$\sup_{t \geqq t_0} \|\zeta_\varepsilon(t) - \bar{\zeta}(t)\|_p \leqq \eta\,.$$

Example 3. Consider the averaging problem

$$d\xi_\varepsilon(t) = (A\,\xi_\varepsilon(t) + r_1\,\xi_\varepsilon(t) + f(t/\varepsilon, \xi_\varepsilon(t)))\,dt + r_2\,\xi_\varepsilon(t)\,d\beta(t)$$

in a Hilbert space H, where $\beta(t)$ is a scalar Wiener process, $r_1, r_2 \in \mathbb{R}$, A generates a holomorphic semigroup $S(t)$ satisfying (2.15) and $\langle Ax, x\rangle \leqq \gamma\|x\|^2$, $x \in D(A)$, for some $\gamma \in \mathbb{R}$. The function $f(t, \cdot)$ is bounded and Lipschitz continuous uniformly with respect to $t \in \mathbb{R}_+$. Assume

$$\lim_{T\to\infty} \frac{1}{T}\int_{\mu T}^{\mu T + T} f(s, x)\,ds = 0$$

uniformly in $x \in H$ and in $\mu \geqq 0$. Let

$$\text{(19)} \qquad \gamma + r_1 + \sup_{\mathbb{R}_+\times H}\|f\| - \tfrac{1}{2}r_2^2 < 0\,.$$

Using Proposition 1 we show that the averaging in probability is possible, the limit equation being

$$\text{(20)} \qquad d\xi(t) = (A\,\xi(t) + r_1\,\xi(t))\,dt + r_2\,\xi(t)\,d\beta(t)\,.$$

The assumption (P) is fulfilled by Remark 1.5 and Proposition 1.1 (used with any $p \geqq 2$). Furthermore, set $v(x) = \eta_1(\|x\|^2)\,\|x\|^q$ for $q > 0$, where η_1 is the function η_δ defined in the proof of Lemma 2.3 with $\delta = 1$. We have

$$[L^1 v](t, x) \leqq q\|x\|^q (\gamma + r_1 + \sup \|f\| + \tfrac{1}{2} r_2^2 (q - 1)), \quad x \in D(A),$$
$$\|x\| > 1,$$

and

$$[L^1 v](t, x) \leqq M, \quad x \in D(A), \quad \|x\| \leqq 1$$

for some $M > 0$. Takin $q > 0$ sufficiently small we obtain by (19) that

$$[L^1 v](t, x) \leqq \psi(v(x)), \quad x \in D(A),$$

where $\psi(r) = M,\ 0 \leqq r \leqq 1,\ \psi(r) = -\alpha r + M + \alpha,\ r \geqq 1,$ for some $\alpha > 0$. Similarly we get $[\bar{L}v](x) \leqq \psi(v(x))$, $x \in D(A)$. Thus by Proposition 3 (in which we set $\xi_1 = \xi_2 = \psi$) the set $\mathfrak{K}$ is equibounded in probability. It remains to show the asymptotical stability in probability of the limit equation (20). We have

$$\xi^{t_0, x}(t) = \exp\{(r_1 - \tfrac{1}{2} r_2^2)(t - t_0) + r_2(\beta(t) - \beta(t_0))\}\, S(t - t_0)\, x$$

and hence (2.13) is fulfilled. Furthermore,

$$\bar{L}_d(\|x - y\|^q) \leqq (\gamma + r_1 + \tfrac{1}{2} r_2^2 (q - 1))\, q\|x - y\|^q, \quad x, y \in D(A),$$

and by (19) and Proposition 2.1 (β) it follows that the equation (20) is asymptotically stable in probability.

Note that (19) can be satisfied even in the case when the corresponding deterministic limit equation (i.e. (20) with $r_2 = 0$) is unstable. This is the case when the deterministic equation $\dot{x} = Ax + r_1 x + f(t, x)$ is effectively stabilized by a noise in the sense of averaging.

APPENDIX

Now we are going to show that Lemma 1.1 is not applicable for the stochastic wave equation; this means that the assumption $S(\cdot) \in \mathscr{C}((0, \infty); \mathscr{L}(H))$ in the lemma cannot be fulfilled.

Let us consider a hyperbolic equation, formally written as

$$(1) \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial u}{\partial t}\, \dot{w}, \quad (t, x) \in (0, 1] \times \mathbb{R},$$

where $\gamma > 0$ and $\dot{w}$ is a 1-dimensional white noise.

We will treat (1) as an equation in the Hilbert space $\mathscr{H} = E \times L^2(\mathbb{R})$, where E is the completion of the space $\mathscr{D}(\mathbb{R})$ of smooth functions with compact supports with respect to the norm $\|f\|_E \equiv (\int_{-\infty}^{+\infty} |f'|^2 \, dx)^{1/2}$. We endow the space $\mathscr{H}$ with the norm

$$\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathscr{H}}^2 = \|f\|_E^2 + \int_{-\infty}^{+\infty} |g|^2 \, dx .$$

Let A be the closure of the operator

$$\begin{pmatrix} 0, & I \\ d^2/dx^2, & 0 \end{pmatrix}$$

defined on $\mathscr{D}(\mathbb{R}) \times \mathscr{D}(\mathbb{R})$, then A generates a (C_0)-semigroup $S(t)$ on $\mathscr{H}$; $\|S(t)\| = 1$; and for each $(f, g)^* \in \mathscr{H}$ we have

$$2\, S(t)\begin{pmatrix} f \\ g \end{pmatrix}(x) = \begin{pmatrix} f(x+t) + f(x-t) + \int_{x-t}^{x+t} g(v) \, dv \\ f'(x+t) - f'(x-t) + g(x+t) + g(x-t) \end{pmatrix}$$

for almost all $x \in \mathbb{R}$. Let $w(t)$ be a real Wiener process. We interpret (1) as an equation for an $\mathscr{H}$-valued process

$$y(t) = \begin{pmatrix} u(t) \\ u_t(t) \end{pmatrix}:$$

$$dy(t) = A\, y(t) + \begin{pmatrix} 0 \\ \gamma u_t(t) \end{pmatrix} dw(t),$$

$$y(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Hence

$$\mathsf{E}\|y(t)\|_{\mathscr{H}}^2 \leqq 2\mathsf{E}\|y(0)\|_{\mathscr{H}}^2 + 2\int_0^t \mathsf{E}\left\| \begin{pmatrix} 0 \\ \gamma\, u_t(s) \end{pmatrix} \right\|_{\mathscr{H}}^2 ds \leqq$$

$$\leqq 2\mathsf{E}\|y(0)\|_{\mathscr{H}}^2 + 2\gamma^2 \int_0^t \mathsf{E}\|y(s)\|_{\mathscr{H}}^2 \, ds,$$

thus $\mathsf{E}\|y(t)\|_{\mathscr{H}}^2 \leqq 2 \exp(2\gamma^2 t)\, \mathsf{E}\|y(0)\|_{\mathscr{H}}^2$.

We claim that there exists $C > 0$ such that for every partition $\{t_i\}_{i=0}^N$ of the interval $[0, 1]$ there exists an initial condition

$$u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathscr{H},$$

$u \neq 0$, such that

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|y(s) - y(t_i)\|_{2,\mathscr{H}} \, ds \geqq C\|u\|_{\mathscr{H}}.$$

Indeed, let us fix the partition $\{t_i\}_{i=0}^N$ arbitrarily, let for the moment u be an arbitrary element in $\mathscr{H}$. Then

$$\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \|y(s) - y(t_i)\|_2 \, ds = \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \|[S(s) - S(t_i)]\, u +$$
$$+ \int_0^s S(s-r)\begin{pmatrix} 0 \\ \gamma\, u_t(r)\end{pmatrix} dw(r) - \int_0^{t_i} S(t_i - r)\begin{pmatrix} 0 \\ \gamma\, u_t(r)\end{pmatrix} dw(r)\|_2 \, ds \geqq$$
$$\geqq \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \Big\|[S(s) - S(t_i)]\, u\Big\|_{\mathscr{H}} ds -$$
$$-\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \left\|\int_0^s S(s-r)\begin{pmatrix} 0 \\ \gamma\, u_t(r)\end{pmatrix} dw(r) - \int_0^{t_i} S(t_i - r)\begin{pmatrix} 0 \\ \gamma\, u_t(r)\end{pmatrix} dw(r)\right\|_2 ds \equiv$$
$$\equiv I_1 - I_2 \,.$$

Further,

$$I_2 \leqq \sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \left(\left(\int_0^s \|\ldots\|_2^2 \, dr\right)^{1/2} + \left(\int_0^{t_i} \|\ldots\|_2^2 \, dr\right)^{1/2}\right) ds \leqq$$
$$\leqq 2^{3/2} \exp(\gamma^2)\, \gamma \|u\|_{\mathscr{H}} \,.$$

Now, let us specify $u_0(t) = \int_0^t c\chi_{[a,b)}(x)\, dx$, $u_1 = 0$, where $[a, b)$ is such an interval that $b - a < 2t_i$, $i = 1, \ldots, N-1$, $b - a < \min\{t_{i+1} - t_i,\ i = 0, \ldots, N-1\}$. This choice yields

$$I_1 = \frac{1}{2}\sum_{i=0}^{N-1}$$
$$\int_{t_i}^{t_{i+1}} \left\|\begin{pmatrix} u_0(x+t) + u_0(x-t) - u_0(x+t_i) - u_0(x-t_i) \\ u_0'(x+t) - u_0'(x-t) - u_0'(x+t_i) + u_0'(x-t_i)\end{pmatrix}\right\|_{2,\mathscr{H}} dt \geqq$$
$$\geqq \frac{1}{2}\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} \|u_0(x+t) + u_0(x-t) - u_0(x+t_i) -$$
$$- u_0(x - t_i)\|_{2,E}\, dt =$$
$$= \frac{c}{2}\sum_{i=0}^{N-1}\int_0^{T_i}\left(\int_{-\infty}^{+\infty} |\chi_{[a,b)}(x + t_i + \tau) + \chi_{[a,b)}(x - t_i - \tau) -\right.$$
$$\left. - \chi_{[a,b)}(x + t_i) - \chi_{[a,b)}(x - t_i)|^2 \, dx\right)^{1/2} d\tau =$$
$$= \frac{c}{2}\sum_{i=0}^{N-1}\int_0^{T_i}\left(\int_{-\infty}^{+\infty} |\chi_{[a-t_i-\tau, b-t_i-\tau)}(x) - \chi_{[a-t_i, b-t_i)}(x) +\right.$$
$$\left. + \chi_{[a+t_i+\tau, b+t_i+\tau)}(x) - \chi_{[a+t_i, b+t_i)}(x)|^2 \, dx\right)^{1/2} d\tau \equiv J \,,$$

where we have set $T_i = t_{i+1} - t_i$. Note that $[a + t_i + \tau, b + t_i + \tau) \cap \cap [a + t_i, b + t_i) = \emptyset$ if $\tau \geqq b - a$, in particular if $\tau \geqq \frac{1}{2} \min \{t_{i+1} - t_i, i = = 0, \ldots, N - 1\}$, and further $[a - t_i - \tau, b - t_i - \tau) \cap [a - t_i, b - t_i) = \emptyset$ if $\tau \geqq b - a$, and $[a + t_i, b + t_i) \cap [a - t_i, b - t_i) = \emptyset$ if $b - a \leqq 2t_i$. Hence

$$J \geqq \frac{c}{2} \sum_{i=0}^{N-1} \int_{T_i/2}^{T_i} \left(\int_{-\infty}^{+\infty} |\ldots|^2 \, dx \right)^{1/2} d\tau =$$

$$= \frac{c}{2} \sum_{i=1}^{N-1} \int_{T_i/2}^{T_i} \left(\int_{-\infty}^{+\infty} |\ldots|^2 \, dx \right)^{1/2} d\tau + \frac{c}{2} \int_{t_1/2}^{t_1} (|\chi_{[a+\tau, b+\tau)}(x) +$$

$$+ \chi_{[a-\tau, b-\tau)}(x) - 2\chi_{[a,b)}(x)|^2 \, dx)^{1/2} \, d\tau \geqq$$

$$\geqq \frac{c}{2} \sum_{i=1}^{N-1} \int_{T_i/2}^{T_i} (4(b - a))^{1/2} \, d\tau + \frac{c}{2} \int_{t_1/2}^{t_1} (6(b - a))^{1/2} \, d\tau \geqq$$

$$\geqq c(b - a)^{1/2} \sum_{i=0}^{N-1} \frac{t_{i+1} - t_i}{2} = \tfrac{1}{2} \|u_0\|_E = \tfrac{1}{2} \|u\|_{\mathscr{H}} .$$

We have obtained the estimate

$$\sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} \|y(s) - y(t_i)\|_{2,\mathscr{H}} \, ds \geqq \left(\tfrac{1}{2} - 2^{3/2} e^{\gamma^2} \gamma\right) \|u\|_{\mathscr{H}} ,$$

and for $\gamma > 0$ small enough we have $C \equiv \left(\frac{1}{2} - 2^{3/2} \exp(\gamma^2) \gamma\right) > 0$.

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Souhrn

METODA PRŮMĚROVÁNÍ PRO STOCHASTICKÉ EVOLUČNÍ ROVNICE II

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Ve stati jsou vyšetřovány věty o integrální spojitosti pro stochastické evoluční rovnice parabolického typu na neomezeném časovém intervalu. Jako pomocné výsledky nezávislého významu jsou odvozena tvrzení o asymptotické stabilitě stochastických parciálních diferenciálních rovnic. Stochastické evoluční rovnice jsou zkoumány v rámci semigroupového přístupu jako rovnice v Hilbertově prostoru.

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