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APPLICATIONS OF THE HADAMARD PRODUCT IN GEOMETRIC FUNCTION THEORY

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(Received October 31, 1988)

Summary. Let \( \mathcal{A} \) denote the set of functions \( F \) holomorphic in the unit disc, normalized classically: \( F(0) = 0, F'(0) = 1 \), whereas \( A \subset \mathcal{A} \) is an arbitrarily fixed subset. In this paper various properties of the classes \( A_\alpha, \alpha \in C \setminus \{-1, -\frac{1}{2}, \ldots\} \), of functions of the form \( f = F \ast k_\alpha \) are studied, where

\[
F \in A, \quad k_\alpha(z) = k(z, \alpha) = z + \frac{1}{1 + \alpha} z^2 + \ldots + \frac{1}{1 + (n - 1) \alpha} z^n + ..., \]

and \( F \ast k_\alpha \) denotes the Hadamard product of the functions \( F \) and \( k_\alpha \). Some special cases of the set \( A \) were considered by other authors (see, for example, [15], [6], [3]).

Keywords: Hadamard product, class of type \( A_\alpha \), typically real functions.

1. Let \( \mathcal{A} \) denote the set of functions \( F \) of the form

\[
F(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

holomorphic in the unit disc \( \mathcal{A} = \{ z \in C : |z| < 1 \} \), whereas \( T \) is a subset of \( \mathcal{A} \) consisting of typically-real functions in \( \mathcal{A} \) (see [12]).

In paper [6], for an arbitrarily fixed \( \alpha \in \mathbb{R} \setminus \{-1, -\frac{1}{2}, \ldots\} \), the class

\[
T_\alpha = \{ f \in \mathcal{A} : f = F \ast k_\alpha, F \in T \}
\]

was considered, where

\[
k_\alpha(z) = k(z, \alpha) = \sum_{n=1}^{\infty} \frac{1}{1 + (n - 1) \alpha} z^n, \quad z \in \mathcal{A},
\]

and \( F \ast k_\alpha \) denotes the Hadamard product of the functions \( F \) and \( k_\alpha \) (see, for example, [14], p. 27; [13]).

For nonnegative values of \( \alpha \), the family \( T_\alpha \) was introduced earlier by K. Skalska ([15]) in another way.

The aim of this paper is to study various properties of the class

\[
A_\alpha = \{ f \in \mathcal{A} : f = F \ast k_\alpha, F \in A \}
\]

where \( A \neq \emptyset \) is an arbitrarily fixed subset of the set \( \mathcal{A} \), and \( \alpha \in C \setminus \{-1, -\frac{1}{2}, \ldots\} \).
In the subsequent considerations we shall always assume, if not stated otherwise, that \( \alpha \) is an arbitrarily fixed complex number different from the numbers \(-1, -\frac{1}{2}, \ldots\).

2. It follows directly from the definitions of the family \( A_\alpha \) and the Hadamard product that the function \( f \) of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_{n,f} z^n, \quad z \in \Delta,
\]

belongs to the family \( A_\alpha \) if and only if there exists \( F \in A \) of the form (1) such that

\[
a_{n,f} = \frac{a_{n,F}}{1 + (n - 1) \alpha}, \quad n = 2, 3, \ldots.
\]

So, if the exact estimate \( |a_{n,F}| \leq d_n \) takes place in the class \( A \) \((F \in A)\), then (3) yields the exact estimate \( |a_{n,f}| \leq d_n/|1 + (n - 1) \alpha|, \quad f \in A_\alpha \).

Moreover, from formula (3) we obtain that \( A_0 = A \).

Also, in a simple way, from (2) we obtain the following properties of the classes \( A_\alpha \).

**Theorem 1.** Let \( r \in (0, 1) \). If, for each function \( F \in A \), the function

\[
F_r(z) = \frac{1}{r} F(r z), \quad z \in \Delta,
\]

belongs to the family \( A \), then, for each function \( f \in A_\alpha \), the function

\[
f_r(z) = \frac{1}{r} f(r z), \quad z \in \Delta,
\]

belongs to the family \( A_\alpha \).

**Theorem 2.** Let \( \theta \in (0, 2\pi) \). If, for each function \( F \in A \), the function

\[
F_\theta(z) = e^{-i\theta} F(ze^{i\theta}), \quad z \in \Delta,
\]

belongs to the family \( A \), then, for each function \( f \in A_\alpha \), the function

\[
f_\theta(z) = e^{-i\theta} f(ze^{i\theta}), \quad z \in \Delta,
\]

belongs to the family \( A_\alpha \).

**Theorem 3.** Let \( \alpha \in \mathbb{R} \setminus \{-1, -\frac{1}{2}, \ldots\} \). If, for each function \( F \in A \), the function

\[
G(z) = \overline{F(\bar{z})} = \sum_{n=1}^{\infty} \bar{a}_{n,f} z^n, \quad z \in \Delta,
\]

belongs to the family \( A \), then, for each function \( f \in A_\alpha \), the function

\[
g(z) = \overline{f(\bar{z})} = \sum_{n=1}^{\infty} \bar{a}_{n,f} z^n, \quad z \in \Delta,
\]

belongs to the family \( A_\alpha \).
Similarly as in the case $A = T$ (see [15], [6]), the following properties of the families $A_\alpha$ may be proved.

**Theorem 4.** A function $f$ belongs to $A_\alpha$ if and only if $f$ is a solution of the differential equation

$$az f'(z) + (1 - \alpha) f(z) = F(z)$$

where $F \in A$.

**Theorem 5.** If $f \in A_\alpha$, then

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta| = \rho < 1} k\left(\frac{z}{\zeta}, \alpha\right) F(\zeta) \frac{d\zeta}{\zeta}, \quad |z| < \rho < 1,$$

where $F \in A$, and vice versa.

**Theorem 6.** If $f \in A_\alpha$, $\Re \alpha > 0$, then

$$f(z) = \frac{1}{\alpha} \int_0^1 t^{1/\alpha - 2} F(zt) \, dt, \quad z \in A,$$

where $F \in A$, and vice versa.

**Theorem 7.** Let $A$ and $B$ be two fixed subsets of $\mathcal{A}$. If, for any functions $F \in A$, $G \in B$, the function $F \ast G \in A$, then, for each $f \in A_\alpha$, the function $f \ast G \in A_\alpha$.

The above theorems can be used in various problems concerning classes of type $A_\alpha$. In particular, the properties of solutions of equations of the form (4) were considered in several cases of the classes $A \subset \mathcal{A}$ (for example, in [15], [6], [8], [2]). From Theorems 5 and 6 one often gets structure formulae for the classes $A_\alpha$ (for example, in [15], [6]; see also [10], [2]). On the other hand the properties of the Hadamard product of functions of the form (1) of the classes frequently considered are well-known: $CV$ (the class of convex functions), $ST(1/2)$ (the class of starlike functions of order 1/2), $CC$ (the class of close-to-convex functions) (see [4], vol. 1, p. 115; vol. 2, p. 2). So, from Theorem 7 and the results of the paper [13] we obtain:

1) for any functions $f \in (CV)_\alpha$, $G \in CV$, the Hadamard product $f \ast G$ belongs to $(CV)_\alpha$;

2) for any functions $f \in (ST(1/2))_\alpha$, $G \in ST(1/2)$, the Hadamard product $f \ast G$ belongs to $(ST(1/2))_\alpha$;

3) for any functions $f \in (CC)_\alpha$, $G \in CV$, the Hadamard product $f \ast G$ belongs to $(CC)_\alpha$.

3. Let $H$ denote the family of all functions holomorphic in the unit disc $A$. The set $H$ with the topology of almost uniform convergence is, of course, a linear topological space.
As is known, certain problems of the geometric theory of analytic functions consist in determining the set $Q$ of values of a complex continuous functional defined on a given family $A \subset H$. If the set $Q$ is bounded, closed, and connected, then we determine it effectively by characterizing its boundary. To ensure that the set $Q$ has the above properties, the family $A$ considered should be compact and connected.

In other extremal problems, support points and extreme points of the families play an essential part (see, for example, [14], pp. 3, 99; [1]).

Let us recall: a function $F \in A$ is called a support point of a compact subset $A$ of $H$ if and only if there exists a continuous linear functional $x^*$ on $H$ such that, $\Re x^*$ is non-constant on $A$ and for each function $G \in A$,

$$\Re x^*(G) \leq \Re x^*(F).$$

So, the problem of characterizing the set of the support points of the class $A_a \subset A \subset H$ seems to be interesting when the characterization of the support points of the family $A \subset A \subset H$ is known.

In the proof of the theorem solving this problem we shall use the following well-known result of Toeplitz ([16]).

**Lemma.** A functional $x^*$ defined on $H$ is linear and continuous if and only if there exists a sequence of complex numbers $\{b_n\}$ such that, for each function $g \in H$,

$$x^*(g) = \sum_{n=0}^{\infty} a_n g b_n,$$

$$\limsup_{n \to \infty} |b_n|^{1/n} < 1.$$ 

**Theorem 8.** A function $f_0$ is a support point of the set $A_a$ if and only if $f_0 = f_0 \ast k^a$ where $F_0$ is a support point of the set $A$.

**Proof.** Let $F_0$ be a support point of the set $A$. Then there exists a linear and continuous functional $x^*$ on $H$ such that, for each function $F \in A$,

$$\Re x^*(F) \leq \Re x^*(F_0).$$

The above lemma and formula (1) imply that this inequality can be written in the following equivalent form:

$$\Re \left( \sum_{n=2}^{\infty} a_n x b_n \right) \leq \Re \left( \sum_{n=2}^{\infty} a_n F_0 b_n \right), \quad F \in A, \quad (5)$$

where $\{b_n\}$ is a sequence determining the functional $x^*$.

As $\limsup_{n \to \infty} |b_n[1 + (n - 1) \alpha]|^{1/n} < 1$, the sequence $\{b_n[1 + (n - 1) \alpha]\}$ also determines a linear and continuous functional on $H$. Let us denote it by $x^*_a$. Let $f$
be an arbitrarily fixed function of the family $A_a$, whereas $f_0 = F_0 \ast k_a$. Then there exists exactly one function $F \in A$ such that $f = F \ast k_a$. Hence, taking formula (3) and inequality (5) into consideration, we obtain

$$
\Re x_a^*(f) - \Re x_a^*(f_0) = \Re x_a^*(F \ast k_a) - \Re x_a^*(F_0 \ast k_a) =
$$

$$
= \Re \left( \sum_{n=2}^{\infty} \frac{a_n}{1 + (n - 1) \alpha} b_n [1 + (n - 1) \alpha] \right)
$$

which proves that the function $f_0 = F_0 \ast k_a$ is a support point of the set $A_a$. We also note that if $\Re x^*$ is non-constant on $A$ then $\Re x_a^*$ is non-constant on $A_a$.

The proof of the converse theorem proceeds analogously.

From the linearity and the injectivity of the Hadamard product $F \ast k_a$ in the space $H$ the following properties of the classes $A_a$ follow.

**Theorem 9.** A set $A_a$ is convex in the space $H$ if and only if $A$ is convex in this space.

**Theorem 10.** If a set $A$ is a convex set in space $H$, then $f \in A_a$ is an extreme point of the set $A_a$ if and only if $f = F \ast k_a$ where $F$ is extreme point of the set $A$.

Next, let us recall that a topological space $X$ is called arcwise connected if, for any two points $x_1, x_2 \in X$, there exists a continuous mapping $\gamma(t)$ of an interval $\langle a, b \rangle$ into the space $X$ such that $\gamma(a) = x_1, \gamma(b) = x_2$. Such a mapping will be called a path joining the points $x_1$ and $x_2$.

We shall prove the following property of the class $A_a$.

**Theorem 11.** If a set $A$ is arcwise connected, then the set $A_a$ is arcwise connected.

**Proof.** Let $f_1, f_2 \in A_a$. Then there exist functions $F_1, F_2 \in A$ such that $f_1 = F_1 \ast k_a$, $f_2 = F_2 \ast k_a$, and a path $\Gamma(t) = F(z, t), t \in \langle a, b \rangle$, joining $F_1$ and $F_2$. Using the formula given in Theorem 5, we prove in the elementary way that $\gamma(t) = f(z, t) = F(z, t) \ast k_a(z)$ is a path joining $f_1$ and $f_2$, which completes the proof.

Since the arcwise connectedness implies the topological connectedness, Theorem 11 yields that, for the arcwise connected family $A$, the families $A_a$ are connected.

Similarly, the following property of the families $A_a$ may easily be proved.

**Theorem 12.** If $A$ is a compact family, then the families $A_a$ are also compact.

4. K. Skalska in her paper [15] proved that if $A = T$, then the following inclusions hold:

$$
T_\beta \subset T_a \subset T_0 = T, \quad 0 < a < \beta.
$$
In the general case, neither of the inclusions $A_{\beta} \subset A_{\alpha} \subset A$, $0 < \alpha < \beta$, need be true. Indeed, let $A = \{z; z + z^2\}$; then $A_{\alpha} = \{z; z + 1/(1 + \alpha)z^2\}$, so $A_{\alpha} \not\subset A_{\beta}$ for $0 < \alpha < \beta$. Moreover, if $A = \{z + z^2\}$, then $A_{\alpha} = \{z + 1/(1 + \alpha)z^2\}$, thus the above inclusions are not true, either, and furthermore, for $\alpha = 0$, even $A_{\alpha} \cap A = \emptyset$.

Next, let $A = S$ where $S$ is the well-known class of univalent functions $F$ of the form (1) in $A$. D. M. Campbell & V. Singh ([2]) proved that then the classes $S_{\alpha} = A_{\alpha}$, even for $\alpha = \frac{1}{2}$, include infinite-valent functions. So, $S_{\alpha} \not\subset S$ for $\alpha = \frac{1}{2}$. Of course, it is also known that $S_{1} \not\subset S$ (see [7]). On the other hand Z. Lewandowski, S. Miller, E. Ziotkiewicz in their paper [8] proved that if $A = ST$, then $(ST)_{\alpha} \subset ST$ for all $\alpha \in C$ from the disc $|\alpha - \frac{1}{2}| \leq \frac{1}{2}$. Another non-trivial example of a family $A$ for which the inclusion $A_{\alpha} \subset A$ is true for a complex $\alpha$ is the family $B_{1}(M)$, $M > 1$, (see [4], vol. 2, p. 36) of functions of the form (1) satisfying the inequality

$$|F(z)| < M, \quad z \in A.$$ 

Namely, we have the following theorem.

**Theorem 13. If** $M > 1$ and Re $\alpha > 0$, then

$$(B_{1}(M))_{\alpha} \subset B_{1}(M).$$

**Proof.** Let $f \in (B_{1}(M))_{\alpha}$ and suppose that, at the same time, $f \notin B_{1}(M)$. It is easy to verify then that there exists a point $z_0 \in A$ such that

$$\max_{|z| \leq r} |f(z)| = |f(z_0)| = M, \quad r = |z_0|.$$ 

Hence, in view of Jack's lemma ([5]), we obtain that there exists a number $m \geq 1$ such that

$$z_0 f'(z_0) = m f(z_0).$$ 

Consequently, in view of Theorem 4 we obtain

$$|\alpha z_0 f'(z_0) + (1 - \alpha) f(z_0)| = |f(z_0)| |\alpha(m - 1) + 1| \geq |f(z_0)| = M$$

in spite of the assumption that $f \in (B_{1}(M))_{\alpha}$, which completes the proof.

Now, we shall give a construction of the families $A$ for which both the inclusion relations above will be true. For this purpose, let us consider the operator $D: H \to H$ defined by the formula

$$D F(z) = z F'(z), \quad z \in A,$$

and the set $\mathcal{A}' = \{F \in H, F(0) = 1\}$. Let $\mathcal{J}$ denote the class of operators $J: \mathcal{A} \to \mathcal{A}'$ satisfying for all $F \in \mathcal{A}$ the condition

(i) \hspace{1cm} $J(\alpha DF + (1 - \alpha) F) = J(F) + \alpha D J(F), \quad \alpha \in C.$
Let us observe that, for example, the operators $J_k: \mathcal{A} \rightarrow \mathcal{A}'$, $k = 1, 2, 3, 4$, defined by the formulas

\begin{align*}
J_1(F)(z) &= F'(0) = 1, \quad z \in \Delta, \\
J_2(F)(z) &= F'(z), \quad z \in \Delta, \\
J_3(F)(z) &= F(z)/(z), \quad z \in \Delta, \\
J_4(F)(z) &= \frac{1}{z \int_0^1 F(\theta)} d\theta, \quad z \in \Delta,
\end{align*}

belong to the class $\mathcal{J}$.

Let

\begin{equation}(6)\end{equation}

$A = \{ F \in \mathcal{A}, \ Re J(F)(z) > 0, \ z \in \Delta \}$

where $J$ denotes an arbitrarily fixed operator of the class $\mathcal{J}$.

In the sequel, family (6) will be called a family of type $J$.

Let us observe that the identity function belongs to each family $A$ of type $J$, $(J(I)(z) = 1, z \in \Delta, J \in \mathcal{J})$; moreover, the class $A$ of type $J_1$ coincides with the whole family $\mathcal{A}$. The well-known families (see [4], vol. 1, p. 101; vol. 2, p. 97)

\begin{align*}
(7) & \quad \{ F \in \mathcal{A}: \ Re F'(z) > 0, \ z \in \Delta \}, \\
(8) & \quad \left\{ F \in \mathcal{A}: \ Re \frac{F(z)}{z} > 0, \ z \in \Delta \right\}
\end{align*}

are classes of type $J_2, J_3$, respectively. The family $A$ of type $J_4$, as far as we know, has not been investigated yet.

The families $A_\alpha$ associated with the classes $A$ of type $J$ have the following properties.

**Theorem 14.** If $A$ is a family of type $J$, then for each $\alpha \in C$, $Re \alpha \geq 0$, the inclusion $A_\alpha \subset A$ is true.

**Proof.** Let $f \in A_\alpha$. Then from (6) and (4) we have

$$Re J(\alpha Df + (1 - \alpha)f)(z) > 0, \quad z \in \Delta.$$ 

This inequality, in view of property (i) of the operator $J$, is equivalent to

\begin{equation}(9)\end{equation}

$$Re (p + \alpha Dp)(z) > 0, \quad z \in \Delta,$$

where $p = J(f)$. Using S. Miller's result ([9], Corollary) we get $Re p(z) > 0, z \in \Delta$. Therefore, $Re J(f)(z) > 0, z \in \Delta$, and, consequently, $f \in A$, which completes the proof.

**Theorem 15.** If $A$ is a family of type $J$ and $0 \leq \alpha \leq \beta$, then $A_\beta \subset A_\alpha \subset A_0 = A$. 

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Proof. Of course, it is sufficient to consider the case $0 < \alpha < \beta$. So, let $0 < \alpha < \beta$, $f \in A_\beta$ and $f \notin A_\alpha$. Then, in view of (4), (6) and property (i), there exists $z_0 \in A$ such that

$$\text{Re } J(f)(z_0) + \beta \text{Re } D J(f)(z_0) > 0,$$
$$\text{Re } J(f)(z_0) + \alpha \text{Re } D J(f)(z_0) \leq 0.$$ 

Multiplying the first inequality by $\alpha > 0$ and the second inequality by $(-\beta) < 0$ and adding them, we get

$$(\alpha - \beta) \text{Re } J(f)(z_0) > 0.$$ 

Since $\alpha - \beta < 0$, therefore $\text{Re } J(f)(z_0) < 0$ and, consequently, $f \notin A_\alpha$, which contradicts the relation $A_\beta \subset A$ proved in Theorem 14.

In particular cases, if the family $A$ is of the form (7) or (8), Theorems 14 and 15 give some results from paper [3], (see Sections 4 and 5).

5. Let $A$ be a family of type $J = J_k$, $k = 2, 3, 4$. Then there exists a function $F = F_k$, $k = 2, 3, 4$, of this class, such that

$$(10) \quad J(F)(z) = \frac{1 + z}{1 - z}, \quad z \in A.$$ 

From property (i) of the operator $J$ we get

$$\text{Re } J(\alpha DF + (1 - \alpha) F)(z) = \text{Re } (1 + 2\alpha z - z^2)/(1 - z)^2 \rightarrow -\frac{1}{2} \text{Re } \alpha \leq 0,$$

as $z \rightarrow -1$, $z \in A$, for each $\text{Re } \alpha \geq 0$. So, $F_k$ does not belong to the respective class $A_\alpha$ if $\text{Re } \alpha > 0$. Consequently, the classes $A_\alpha$ associated with the families $A$ of type $J = J_k$, $k = 2, 3, 4$, are essential subclasses of the families $A$.

From the course of the argument carried out we infer that $A_\alpha$ will be an essential subclass of the family $A$ of type $J$ if, for example, we assume in addition that the solution $F$ of equation (10) belongs to $A$. Then the family $A$ will be called a family of type $\bar{J}$. So: if $A$ is a family of type $\bar{J}$, then $A \notin A_\alpha$ for $\text{Re } \alpha > 0$.

A family $A$ of type $J_1$ is not a family of type $J_1$, whereas families $A$ of type $J_k$, $k = 2, 3, 4$, are families of type $\bar{J}_k$.

The following property for the families of type $J$ turns out to be true.

**Theorem 16.** If $A$ is a family of type $J$, then

$$A \subset A[\Delta_{r(\alpha)}], \quad \text{for } r(\alpha) = \sqrt{1 + |\alpha|^2} - |\alpha| \leq 1$$

where

$$A[\Delta_\alpha] = \{f \in A: \text{Re } J(f)(z) > 0, z \in \Delta_\alpha\}; \quad \Delta_{r(\alpha)} = \{z \in C: |z| < r(\alpha)\}.$$ 

Moreover, the disc $\Delta_{r(\alpha)}$ for $\alpha \in R$ cannot be enlarged.
**Proof.** Let \( f \in A \). In view of the definitions of the families \( A_x \) and the sets \( A[A_r] \), the assertion will be proved if we determine the largest number \( r(x) \in (0, 1) \) such that
\[
\Re J(\alpha Df + (1 - \alpha)f) (z) > 0 , \quad z \in A_{r(x)} .
\]

By virtue of property (i) of the operator \( J \), it is sufficient to prove that
\[
\Re (p + \alpha Dp) (z) > 0 , \quad z \in A_{r(x)} ,
\]
where \( p = J(f) \). Since \( f \in A \), therefore \( p \) is a Carathéodory function with a positive real part, so ([11], (6.2)) \( |z p'(z)|/\Re p(z) \leq 2|z|/(1 - |z|^2) \). Hence
\[
\Re (p + \alpha Dp) (z) \geq \left( 1 - \frac{2|\alpha|}{1 - r^2} \right) \Re p(z) , \quad |z| = r < 1 .
\]

But \( 1 - 2|\alpha|r - r^2 > 0 \) if and only if \( 0 < r < r(x) = \sqrt{(1 + |\alpha|^2) - |\alpha|} \), therefore relation (11) follows from (12), which accounts for the inclusion announced in the theorem.

As \( A \) is a family of type \( J \), the solution \( F \) of equation (10) belongs to \( A \). This function turns out to belong to the family \( A_x [A_{r(x)}]_x \) and not belong to \( A_x [A_r]_x \) for \( r > r(x), \alpha \in R \). Thus the proof is complete.

6. Let \( A \) be a family of type \( J \) and \( \alpha \geq 0 \). In view of Theorems 14 and 15 and the fact that \( A \not\subset A_x \) for \( \alpha > 0 \), the following considerations seem to be interesting.

Let \( f \in A, \alpha \geq 0 \). Let us put
\[
\alpha_f = \{ \sup \alpha : f \in A_x \} , \quad A(\alpha) = \{ f \in A : \alpha_f = \alpha \} .
\]

**Theorem 17.** If \( A \) is a family of type \( J \), then each class \( A(\alpha) \) is nonempty and the following relations hold:
\[
\begin{align*}
(13) & \quad f \in A(0) \text{ if and only if } f \notin A_x \text{ for each } \alpha > 0 ; \\
(14) & \quad f \in A(\infty) \text{ if and only if } f \in A_x \text{ for each } \alpha \geq 0 ; \\
(15) & \quad f \in A(\alpha), \alpha \in (0, \infty), \text{ if and only if } f \in A_\beta \text{ for any } \beta \in (0, \alpha) \\
& \quad \text{ and } f \notin A_\beta \text{ for each } \beta > \alpha .
\end{align*}
\]

**Proof.** As \( A \) is of type \( J \), then, as we observed earlier, \( A(0) \neq \emptyset \). Let \( \alpha > 0 \) and let \( \mathcal{F} \in A \) be a solution of equation (10). Let us put \( \mathcal{F} = \mathcal{F} * k_\alpha \). Then, by virtue of (2), \( \mathcal{F} \in A_x \), so from (4)
\[
J(\alpha D\mathcal{F} + (1 - \alpha)\mathcal{F}) (z) = J(\mathcal{F}) (z) = \frac{1 + z}{1 - z} , \quad z \in A .
\]

Hence, in view of (i),
\[
\alpha D J(\mathcal{F}) (z) + J(\mathcal{F}) (z) = \frac{1 + z}{1 - z} , \quad z \in A .
\]
Let us consider $\beta > \alpha$. From (i) we get

$$J(\beta Df + (1 - \beta) f)(z) = \beta D J(f)(z) + J(f)(z) = \frac{\beta 1 + z}{\alpha 1 - z} + \frac{\alpha - \beta}{\alpha} J(f)(z) = \frac{\beta 1 + z}{\alpha 1 - z} + \frac{\alpha - \beta}{\alpha} \int_0^1 t^{1/z-1} \frac{1 + tz}{1 - tz} dt \to \frac{\alpha - \beta}{\alpha} a < 0 ,$$

as $z \to -1, z \in A$. Consequently, $f \in A_{\alpha}$, whence $A(\alpha) \neq \emptyset$. Since the identity function belongs to the family $A$ of type $J$, it belongs to each class $A_{\alpha}$, thus to $A(\infty)$, too. Hence it follows that $A(\infty) \neq \emptyset$.

Now, let us observe that for $\alpha \in (0, \infty)$, conditions (13), (14) and the sufficient condition in (15) follow directly from the definition of the family $A_{\alpha}$ and the properties of the family $A_{\alpha}$. It only remains to prove the necessary condition in (15).

So, let $f \in A(\alpha), \alpha \in (0, \infty)$. Then the definition of the family $A(\alpha)$ and Theorem 15 imply that $f \notin A_{\beta}$ for each $\beta > \alpha$, and $f \in A_{\beta}$ for each $0 \leq \beta < \alpha$. In view of (4), the last fact is equivalent to

$$\text{Re} J(\beta Df + (1 - \beta) f)(z) > 0 , \quad z \in A ,$$

for $\beta \in (0, \alpha)$. Passing to the limit $\beta \to \alpha^-$ in the above inequality, we get

$$\text{Re} J(\alpha Df + (1 - \alpha) f)(z) \geq 0 , \quad z \in A ,$$

which, in view of the extremum principle for harmonic functions, gives

$$\text{Re} J(\alpha Df + (1 - \alpha) f)(z) > 0 , \quad z \in A ,$$

and, consequently, $f \in A_{\alpha}$. Thus the proof is complete.

Theorem 17 evidently yields that

$$A = \bigcup_{\alpha \geq 0} A(\alpha) .$$

Finally, let us observe that the operator $J_g : \mathcal{A} \to \mathcal{A}'$ defined by the formula

$$(J_g(F))(z) = \frac{(F \ast g)(z)}{z} , \quad z \in A ,$$

where $g$ is an arbitrarily fixed function of the family $\mathcal{A}$, belongs to the class $\mathcal{J}$, too. Moreover, putting $g = g_k, k = 1, 2, 3, 4$, where

$$g_1(z) = z , \quad z \in A ;$$

$$g_2(z) = \frac{z}{(1 - z)^2} , \quad z \in A ;$$

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we get $J_k = J_{g_k}$.

There arises a natural question if $J_g$ is the most general form of the operator $J \in \mathcal{J}$.

References

APLIKACE HADAMARDOVA SOUČINU V GEOMETRICKÉ TEORII FUNKCÍ

Zbigniew Jerzy Jakubowski, Piotr Liczberski, Łucja Żywień

Nechť \( \mathcal{A} \) je množina funkcí \( F \) holomorfních v jednotkovém kruhu a normalizovaných klasickým způsobem: \( F(0) = 0, F'(0) = 1 \), a nechť \( A \in \mathcal{A} \) je její libovolná pevně zvolená podmnožina. V článku se studují různé vlastnosti tříd \( A_\alpha, \alpha \in C \setminus \{-1, -\frac{1}{2}, \ldots\} \), funkcí tvaru \( f = F \ast k_\alpha \), kde

\[
F \in A, \quad k_\alpha(z) = k(z, \alpha) = z + \frac{1}{1 + \alpha} z^2 + \ldots + \frac{1}{1 + (n - 1) \alpha} z^n + \ldots,
\]

a \( F \ast k_\alpha(z) \) znamená Hadamardův součin funkcí \( F, k_\alpha \). Některé speciální případy množiny \( A \) byly vyšetřeny dříve jinými autory (viz např. [15], [6], [3]).

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