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NORMAL SPACES AND THE LUSIN-MENCHOFF PROPERTY

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Summary. We study the relation between the Lusin-Menchoff property and the $F_a$-"semiseparation" property of a fine topology in normal spaces. Three examples of normal topological spaces having the $F_a$-"semiseparation" property without the Lusin-Menchoff property are given. A positive result is obtained in the countable compact space.

Keywords: fine topology, finely separated sets, Lusin-Menchoff property, normal space

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1. INTRODUCTION

All topological spaces considered should be Hausdorff. Let $(X, \varrho)$ be a topological space. Any topology $\tau$ finer than $\varrho$ is called a fine topology. We use the terms finely open, finely closed, ... with respect to a fine topology (similarly for another topology). We say that $A, B \subseteq X$ are finely separated if there are disjoint finely open sets $G_A$ and $G_B$ such that $A \subseteq G_A, B \subseteq G_B$.

An important tool in the study of fine topologies is the Lusin-Menchoff property. We say that a fine topology $\tau$ on $(X, \varrho)$ has the Lusin-Menchoff property (with respect to $\varrho$) if for each pair of disjoint subsets $F$ and $G$ of $X$, $F$ closed, $G$ finely closed, there are disjoint subsets $F$ and $G$ of $X$, $F$ open, $G$ finely open, such that $F \subseteq G, F \subseteq G$ ([2], p. 85).

In [5] we proved the following

Theorem 1.1. Let a fine topology have the Lusin-Menchoff property. Suppose $a$ and $b$ are finely closed sets. Suppose $A$ and $B$ are sets of type $F_a$ with $a \subseteq A,

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\[ b \subset B, A \text{ disjoint with } b, \text{ and } B \text{ disjoint with } a. \text{ Then there are disjoint finely open sets } a \text{ and } \beta \text{ such that } a \subset a \text{ and } b \subset \beta. \]

Let \( a \subset A \subset X \) and \( b \subset B \subset X \) where \( A \) and \( B \) are of type \( F_\alpha \). \( A \) is disjoint with \( b \), and \( B \) is disjoint with \( a \). In this situation we say that \( a \) and \( b \) are \( F_\alpha \)-"semiseparated". Theorem 1.1 says (assuming the Lusin-Menchoff property) that \( F_\alpha \) "semiseparated" finely closed sets are finely separated.

We can formulate a simple corollary.

**Corollary 1.2.** Let a fine topology have the Lusin-Menchoff property and the \( F_\alpha \)-"semiseparation" property (it means that any two finely closed sets can be \( F_\alpha \)-"semiseparated"). Then the fine topology is normal.

A natural question arises:

**Question 1.3.** Let a fine topology be normal and have the \( E_\alpha \)-"separation" property. Does this imply that the fine topology has the Lusin-Menchoff property?

In the following examples we show that the answer is no, even with stronger assumptions (see Propositions 2.3, 3.4 and 4.3). A positive result is obtained in the countable compact space (see Proposition 5.1).

### 2. The train topology

**Definition 2.1.** Let \( X = \mathbb{R}^2 \). We define the train topology by the neighbourhood basis of any point. The origin has the neighbourhood basis consisting of sets of the kind

\[
U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \varepsilon^2\} \cup \{(x, y) \in \mathbb{R}^2 : |y| < 1, x > K\}
\]

(the second set is the "long train") for any \( \varepsilon, K > 0 \). Other points have the neighbourhood basis of Euclidean open sets.

We can easily see the following

**Observation 2.2.** The properties of the train topology:

(i) the Euclidean topology is strongly finer than the train topology;
(ii) the family of \( G_\delta \) sets in the train topology contains all Euclidean open sets;
(iii) the train topology is not normal (the origin and \( \{(x, y) \in \mathbb{R}^2 : y = 1\} \) are train closed sets which are not train separated).
Proposition 2.3. There exists a fine topology which is normal, has the \( F_a \)-"semiseparation" property and has not the Lusin-Menchoff property.

Proof. Let the original topology on \( \mathbb{R}^2 \) be the train topology and let the fine topology be the Euclidean one. Then the \( F_a \)-"semiseparation" property of the fine topology follows from Observation 2.2 (ii). The set \( F = \{(x, y) \in \mathbb{R}^2 : y = 1\} \) is closed in the train topology, \( F = \{(0,0)\} \) is a Euclidean closed set and any train open cover of \( F \) meets any Euclidean open cover of \( F \). The train topology has not the Lusin-Menchoff property with respect to the Euclidean topology on \( \mathbb{R}^2 \). \( \square \)

3. THE CUCKOO TOPOLOGY

Definition 3.1. Let \( e_n \rightarrow 0, c_n \rightarrow \infty \) be disjoint non zero points, \( X = \mathbb{R} \setminus \{e_n\} \). We define the cuckoo topology by the neighbourhood basis of any point. The origin has the neighbourhood basis consisting of sets of the kind \( \{x \in X : |x| < \varepsilon\} \cup \{x \in X : |x| > K\} \) for any \( \varepsilon, K > 0 \). The points \( c_n \) (the cuckoos) have the neighbourhood basis of the form \( \{x \in X : |x - c_n| < \varepsilon\} \cup \{x \in X : |x - e_n| < \varepsilon\} \) (the "home" united with the punctured "egg" given near the origin = "bird") for \( \varepsilon > 0 \). Other points of \( X \) have the neighbourhood basis of all Euclidean open sets.

We can easily see the following:

Observation 3.2. The properties of the cuckoo topology:

(i) the Euclidean topology is strongly finer than the cuckoo topology;
(ii) the family of \( G \) sets in the cuckoo topology contains all Euclidean open sets;
(iii) the cuckoo topology is compact (near infinity and near "eggs" \( e_n \), the situation is simple, due to the definition of the cuckoo topology);
(iv) the Euclidean topology on \( X \) is normal.

Proposition 3.3. The cuckoo topology on \( X \) is normal.

Proof. Let \( F, G \) be disjoint cuckoo closed sets. Then
(i) near the origin and finitely many \( e_n \) the cuckoo topology is topologically like the Euclidean topology near infinity;
(ii) if \( c_n \in F \), then some neighbourhood of \( c_n \) (containing an "egg" near \( e_n \)) is disjoint with \( G \);
(iii) if \( 0 \in F \), then some cuckoo neighbourhood of the origin is disjoint with \( G \).
In all situations we can easily find the cuckoo open sets separating \( F \) and \( G \). \( \square \)
Proposition 3.4. There exists a normal fine topology having the $F_\sigma$-"semiseparation" property with respect to a normal and compact original topology such that the fine topology has not the Lusin-Menchoff property with respect to the original topology.

Proof. Let the fine and the original topologies be the Euclidean and the cuckoo topology on $X$ (Definition 3.1), respectively. Then due to Observation 3.2 and Proposition 3.3 it is enough to show that the Lusin-Menchoff property does not hold. We take a cuckoo closed set $F = \{0\}$ and a Euclidean closed set $F = \{e_n\}_{n=1}^\infty$. Any Euclidean open cover of $F$ meets some "egg" in any cuckoo cover of $F$. The Lusin-Menchoff property does not hold.

4. The Jump Topology

Definition 4.1. Let $a_n \to 0$ be nonzero points of $X = [0,1]$. We define the jump topology on $X$ by the jump metric $d_{\text{jump}}(x,y) = d(\varphi(x),\varphi(y))$, where $\varphi: X \to \mathbb{R}^2$, $\varphi(a_n) = (a_n,1)$, $\varphi(x) = (x,0)$ elsewhere (at $a_n$ the function $\varphi$ "jumps" to 1) and $d$ is the Euclidean metric in $\mathbb{R}^2$.

We can easily see the following

Observation 4.2. The properties of the jump topology:

(i) the jump topology is finer than the Euclidean topology;
(ii) the jump topology is metric;
(iii) the jump closed sets are $F_\sigma$ sets in the Euclidean topology;
(iv) the jump topology has the $F_\sigma$-"semiseparation" property.

Proposition 4.3. There exists a metric fine topology having the $F_\sigma$-"semiseparation" property with respect to a compact metric original topology such that the fine topology has not the Lusin-Menchoff property with respect to the original topology.

Proof. Let the fine and the original topologies be the jump and the Euclidean topology on $X$ (Definition 4.1), respectively. Then due to Observation 4.2 it is enough to show that the Lusin-Menchoff property does not hold. We take a jump closed set $F = \{a_n\}_{n=1}^\infty$ and a Euclidean closed set $F = \{0\}$. Any Euclidean open cover of $F$ meets any jump cover of $F$. The Lusin-Menchoff property does not hold.
5. THE COUNTABLE COMPACT TOPOLOGY

We see that for a compact fine topology both topologies coincide. Hence we weaken the compactness to the following notion. We say that a topological space is countable compact if from any countable open cover we can select a finite subcover. We can easily prove

**Proposition 5.1.** Let a fine topology be countable compact and have the $F_\sigma$-"semiseparation" property with respect to a normal original topology. Then the fine topology has the Lusin-Menchoff property.

**Proof.** Let $F$ be a closed set disjoint with a finely closed $F$. Due to the $F_\sigma$-"semiseparation" property we find $\{F_n\}$ such that $F \subseteq \bigcup F_n$, $F_n$ disjoint with $F$. Due to normality of the original topology, for any couple $F, F_n$ we find a disjoint couple of open sets $G_n$ and $H_n$ such that $F_n \subseteq G_n$ and $F \subseteq H_n$. Due to the countable compactness of the fine topology we find $m$ such that $F \subseteq G = \bigcup_{n=1}^m F_n$.

The set $G = \bigcap_{n=1}^m H_n$ is an open cover of $F$, the set $G$ is an open cover of $F$. The sets $G$ and $G$ show that the Lusin-Menchoff property holds. □

**Remark 5.2.** Other material on this subject can be found in [1], [2], [3], [4], [5], [6].

**References**


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