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The cross-ratio in Hjelmslev planes


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1. INTRODUCTION

A special local ring is a finite commutative local ring $R$ the ideal $I$ of divisors of zero of which is principal. Suppose that $R$ is not a field and that the characteristic of $R$ is odd. Denote the factor ring $R/I$ by the symbol $\overline{R}$. Further denote the set of all regular elements of $R$ by the symbol $R^*$, thus $R^* = R - I$.

Definition 1.1. A projective Hjelmslev plane (we will denote it by $H(R)$) over $R$ is an incidence structure $H(R) = (B; \mathcal{V}; T)$ defined in the following way:

- the elements of $B$—the points of $H(R)$ are classes of ordered triples $(\lambda x_1; \lambda x_2; \lambda x_3)$ where $\lambda \in R^*$, $x_1, x_2, x_3 \in R$ and at least one $x_i$ is regular;
- the elements of $\mathcal{V}$—the lines of $H(R)$ are classes of ordered triples $(\alpha a_1; \alpha a_2; \alpha a_3)$ where $\alpha \in R^*$, $a_1, a_2, a_3 \in R$ and at least one $a_i$ is regular.

A point $X = [x_1; x_2; x_3]$ is incident to the line $\alpha = [a_1; a_2; a_3]$ if and only if

\[ \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0. \]  

Remark 1.1. The canonical homomorphism $\Phi: R \to R/I = \overline{R}$ induces a homomorphism of $H(R)$ onto the projective plane $\pi(\overline{R})$. 

\[ \text{Summary.} \] 

The cross-ratio in Hjelmslev planes is defined. The cross-ratio in the Hjelmslev plane $H(R)$ is independent of the choice of a coordinate system on a line.

Keywords: Hjelmslev plane over a special local ring, cross-ratio in Hjelmslev plane

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We will call the points \( X, Y \in H(R) \) neighbouring if \( X = Y \) where \( \Phi(X) = \bar{X} \) \( \Phi(Y) = \bar{Y} \). Similarly we will call points \( X, Y \in H(R) \) substantially different if \( \bar{X} \neq \bar{Y} \). Two lines are neighbouring if there are points \( A_1, A_2 \in B, A_1 \neq A_2 \) such that \( A_1z_a, b \) and \( A_2z_a, b \). Let \( X \) be a subset of the \( R \)-modul \( M \) and let \( j : X \to M \) be an insertion of the subset \( X \) into \( M \). Then \( M(R) \) is called the free modul over \( R \) with a factorization defined in the following way: triples \((x_1; x_2; x_3)\) and \((x'_1; x'_2; x'_3)\) are identical if there is \( \lambda \in R^* \) such that \( x'_i = \lambda x_i \) for \( i = 1, 2, 3 \) and we do not consider the zero triple.

2. The construction and proof of theorem

Definition 2.1. A coordinate system in \( H(R) \) is an ordered quadruple of points \( E_1, E_2, E_3, E_4 \) such that the points \( E_1, E_2, E_3, E_4 \) generate a coordinate system in \( \pi(R) \).

If a point \( X = [x_1; x_2; x_3] \) is given by the vector \( x = (x_1; x_2; x_3) \), we write \( X = (x) \).

Lemma 2.1. Let \( M(R) \) be a free modul over \( R \) and let \( c_1, c_2, c_3 \) be a basis of \( M(R) \). Then the points \( E_1 = \langle c_1 \rangle, E_2 = \langle c_2 \rangle, E_3 = \langle c_3 \rangle, E_4 = \langle c_1 + c_2 + c_3 \rangle \) generate the coordinate system in the Hjelmslev plane \( H(R) \) corresponding to the modul \( M(R) \).

Proof. It is necessary to prove that the points \( E_1, E_2, E_3, E_4 \) generate a coordinate system in \( \pi(R) \). Obviously \( \bar{c}_1, \bar{c}_2, \bar{c}_3 \) form a basis of a vector space over \( \bar{R} \) and thus the vectors \( \bar{c}_1, \bar{c}_2, \bar{c}_3 \) are linearly independent. It follows that the points \( \bar{E}_1 = \langle \bar{c}_1 \rangle, \bar{E}_2 = \langle \bar{c}_2 \rangle, \bar{E}_3 = \langle \bar{c}_3 \rangle \) and \( \bar{E}_4 = \langle \bar{c}_1 + \bar{c}_2 + \bar{c}_3 \rangle \) are on a unique line. \( \Box \)

Conversely, we have

Lemma 2.2. Let \( E_1, E_2, E_3, E_4 \) be a coordinate system in \( H(R) \). Then there is a basis of the modul \( M(R) \) such that \( \langle c_1 \rangle = E_1, \langle c_2 \rangle = E_2, \langle c_3 \rangle = E_3, \langle c_1 + c_2 + c_3 \rangle = E_4 \).

Proof. Let \( E_1 = \langle b_1 \rangle, E_2 = \langle b_2 \rangle, E_3 = \langle b_3 \rangle \) and \( E_4 = \langle b_4 \rangle \). Because \( \{b_1, b_2, b_3\} \) is a basis of \( M(R) \) the vector \( b_4 \) can be expressed in the form

\[ b_4 = \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3. \]
If we denote \( e_1 = \beta_1 b_1, e_2 = \beta_2 b_2, e_3 = \beta_3 b_3 \) then \( e_1, e_2, e_3 \) are the vectors from the statement of the lemma.

Let \( E_1, E_2, E_3, E_4 \) and \( E_1', E_2', E_3', E_4' \) be coordinate systems in \( H(R) \). If \( e_1, e_2, e_3 \) and \( e_1', e_2', e_3' \) are the corresponding bases of the modul \( M(R) \) then there is a regular matrix \( A = [a_{ij}] \) such that

\[
e'_i = \sum_j a_{ij} e_j, \quad i = 1, 2, 3.
\]

Let \( X_E = [x_1; x_2; x_3] \), \( X'_E = [x'_1; x'_2; x'_3] \). Then

\[
x = \sum_i x_i e_i = \sum_i x'_i e'_i = \sum_i \left( \sum_j x'_i a_{ij} \right) e_j = \sum_j x_j e_j.
\]

Comparing the two identities, we get

\[
(2.1) \quad x_j = \sum_i x'_i a_{ij}.
\]

The relation (2.1) can be written also in the form

\[
(2.2) \quad X_E = X'_E A, \quad X'_E = X_E A^{-1}.
\]

Let an invertible matrix \( A \) and a coordinate system \( E_1, E_2, E_3, E_4 \) be given, then points \( E_1, E_2, E_3, E_4 \) generate a coordinate system and the corresponding vectors of the point \( X \in H(R) \) satisfy

\[
X_E = X'_E A.
\]

Let the special local ring \( R \) be given. We introduce a set \( \Omega \) by

\[
(2.3) \quad \Omega \cap R = \emptyset, \quad |\Omega| = |I|.
\]

Thus there is a bijective mapping \( \omega \) such that

\[
(2.4) \quad \omega: I \to \Omega, \quad \omega : i \to \omega_i = \omega(i), \quad i \in I
\]

where \( \omega_i \) are "inverse" elements of elements \( i \in I \), thus \( \omega_i \sim 1/i \). \( \Omega \) is the set of "infinites" corresponding to singular elements. Define an extension of the canonical homomorphism \( \Phi \) to the set \( R \cup \Omega \), let us put

\[
(2.5) \quad \Phi(\Omega) = \infty.
\]

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Let \( A, B, E \) be three substantially different points generating a coordinate system on a line. Then every point \( X \) of this line can be expressed uniquely (the single-valuedness guarantees the point \( E \)) in the form

\[
X = sA + tB
\]

and hence the point \( X = [s; t] \) is determined by the pair \((s; t)\).

On the line with the coordinate system \( A, B, E \) let us have points \( P_1, P_2, P_3, P_4 \) where \( P_i = s_iA + t_iB \) thus \( P_i[s; t_i] \).

**Definition 2.2.** The cross-ratio of an ordered quadruple of points \( P_1, P_2, P_3, P_4 \) on a line in \( H(R) \), of which at least three are substantially different is an element \((P_1P_2, P_3P_4) \in R \cup \Omega \) which is defined by relations

\[
(P_1P_2, P_3P_4) = \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \\ s_3 & t_3 \\ s_4 & t_4 \end{vmatrix}
\]

if points \( P_1P_4 \) and \( P_2P_3 \) are substantially different,

\[
(P_1P_2, P_3P_4) = \omega(P_1P_2, P_3P_4)
\]

if points \( P_1, P_4 \) and \( P_2, P_3 \) are neighbouring. Suppose that points \( P_1, P_3 \) and \( P_2, P_4 \) are substantially different.

**Remark.** If \( R \) is a field, \( I = \{0\} \) then Definition 2.2 is the same as the definition of the cross-ratio in a projective plane.

**Theorem 2.3.** The cross-ratio introduced by relations 2.7 and 2.8 is independent of the choice of a coordinate system on the line.

**Proof.** Let a line \( p \in H(R) \) be given and on this line let us have coordinate systems \( A, B, E \) and \( A', B', E' \). Let \( P_1, P_2, P_3, P_4 \) be points on the given line \( p \) whose the cross-ratio we want to investigate. There is obviously a linear transformation which maps the points \( A, B \) to the points \( A', B' \) on \( p \). We want to verify that the cross-ratio is independent of the choice of the coordinate points on the line. Thus

\[
(P_1P_2, P_3P_4)_{AB} = (P_1P_2, P_3P_4)_{A'B'}
\]

We have

\[
A' = a_1A + a_2B \\
B' = b_1A + b_2B
\]
and thus
\[ P_1 = s'_1 A' + t'_1 B' \]
and after a substitution we get
\[ P_i = (s'_i a + t'_i b)_i A + (s'_i a + t'_i b)_i B = s_i A + t_i B, \quad i = 1, 2, 3, 4. \]

By direct calculation we obtain \((P_1 P_2, P_3 P_4)_{AB} = (P_1 P_2, P_3 P_4)_{A'B'}\) which was to be proved.

References


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