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Mathematica Bohemica, Vol. 118 (1993), No. 4, 337–342

Persistent URL: http://dml.cz/dmlcz/126156

ONE-SIDED PRINCIPAL IDEALS IN THE DIRECT PRODUCT OF TWO SEMIGROUPS

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(Received December 2, 1991)

Summary. A necessary and sufficient condition is given for

a) a principal left ideal $L(s,t)$ in $S \times T$ to be equal to the direct product of the corresponding principal left ideals $L(s) \times L(t)$,

b) an $\mathcal{L}$-class $L_{(s,t)}$ to be equal to the direct product of the corresponding $\mathcal{L}$-classes $L_s \times L_t$.

Keywords: direct product of two semigroups, principal left ideal, $\mathcal{L}$-class, maximal $\mathcal{L}$-class

AMS classification: 20M10, 20M12

It is well known that if L_1 is a left ideal of a semigroups S, L_2 is a left ideal of a semigroup T, then the direct product $L_1 \times L_2$ is a left ideal of the direct product of two semigroups $S \times T$. If $s \in S$, $t \in T$, then by $L(s)$, $L(t)$ we denote the principal left ideal of S and of T, respectively, and by $L(s,t)$ the principal left ideal of $S \times T$. $L(s) \times L(t)$ is a left ideal of $S \times T$, but it need not be the principal left ideal of $S \times T$.

Let L_s be an $\mathcal{L}$-class of S containing $s \in S$, let L_t be an $\mathcal{L}$-class of T containing $t \in T$, and let $L_{(s,t)}$ be an $\mathcal{L}$-class of $S \times T$ containing $(s,t) \in (S \times T)$.

The aim of the note is

a) to investigate the mutual relation between $L(s,t)$ and $L(s) \times L(t)$ and to find conditions under which $L(s,t) = L(s) \times L(t)$,

b) to investigate the mutual relation between $L_{(s,t)}$ and $L_s \times L_t$ and to find conditions under which $L_{(s,t)} = L_s \times L_t$.

All results are given for principal left ideals and $\mathcal{L}$-classes, because for principal right ideals and $\mathcal{R}$-classes they are similar. For all notions and notation, which we use and do not define, we refer to [2].

Lemma 1. *Let $(s,t) \in S \times T$. Then $L(s,t) \subset L(s) \times L(t)$.*

Proof. $L(s,t) = (s,t) \cup (Ss \times Tt) \subset (s,t) \cup (s \times Tt) \cup (Ss \times t) \cup (Ss \times Tt) = (s \cup Ss) \times (t \cup Tt) = L(s) \times L(t)$. □

Theorem 1. *$L(s,t) = L(s) \times L(t)$ iff at least one of the following conditions is satisfied:*

1) $Ss = \{s\}$,
2) $Tt = \{t\}$,
3) $s \in Ss$ and $t \in Tt$.

Proof. a) If 1) holds, then $L(s) = \{s\}$ and $L(s) \times L(t) = \{s\} \times (t \cup Tt) = (s,t) \cup (s \times Tt) = (s,t) \cup (Ss \times Tt) = L(s,t)$.

If 2) holds, we proceed analogously.

If 3) holds, then $L(s) = Ss$, $L(t) = Tt$. Hence $L(s) \times L(t) = (Ss \times Tt) = (s,t) \cup (Ss \times Tt) = L(s,t)$.

b) Let none of the conditions hold. This is possible only in two cases:

α) $s \notin Ss$ and $Tt \neq \{t\}$;

β) $\{s\} \neq Ss$ and $t \notin Tt$.

If α) holds then there exists $t_1 \neq t$ such that $t_1 \in Tt$. Then $(s,t_1) \in L(s) \times L(t)$, but $(s,t_1) \neq (s,t)$, so $(s,t_1) \notin (Ss \times Tt)$, since $s \notin Ss$. Then $(s,t_1) \notin (s,t) \cup (Ss \times Tt) = L(s,t)$. Therefore, $L(s,t) \neq L(s) \times L(t)$.

The notion of a projection is used in the usual way ([5]): The function $\Pi_S : S \times T \to S$ defined by $(s,t)\Pi_S = s$ for all $(s,t) \in (S \times T)$ is the projection of $S \times T$ onto S, similarly Π is onto T. □

Remark 1. It is easy to see that $L(s,t)\Pi_S = L(s)$ in S, $L(s,t)\Pi_T = L(t)$ in T.

Theorem 2. *Let $(s,t) \in S \times T$ be any element. Then*

1) $L_{(s,t)} \subseteq L_s \times L_t$.
2) *If $L_{(s,t)} \subset L_s \times L_t$, then $L_s \times L_t$ is the union of at least two $\mathcal{L}$-classes in $S \times T$.*

Proof. 1) Let $(u,v) \in L_{(s,t)}$. Then $L(u,v) = L(s,t)$ and $L(u) = L(s)$ in S, $L(v) = L(t)$ in T, hence $u \in L_s$, $v \in L_t$ and therefore $(u,v) \in L_s \times L_t$, so $L_{(s,t)} \subseteq L_s \times L_t$.

2) Let $(u,v) \in L_s \times L_t - L_{(s,t)}$. Then $u \in L_s$, $v \in L_t$, $L_u = L_s$, $L_v = L_t$. Then $L_{(u,v)} \subseteq L_u \times L_v = L_s \times L_t$. □

Corollary. *If $L_s = \{s\}$, $L_t = \{t\}$, then $L_{(s,t)} = L_s \times L_t$.*

Lemma 2. *If $(s,t) \notin (Ss \times Tt)$, then $L_{(s,t)} - \{(s,t)\}$.*

Proof. $L(s,t) = (s,t) \cup (Ss \times Tt)$ and for any $(u,v) \in L(s,t)$, $(u,v) \neq (s,t)$, $(u,v) \in (Ss \times Tt) \subset L(s,t)$. Then $L(u,v) \subseteq (Ss \times Tt) \subset L(s,t)$, hence $L(u,v) \neq L(s,t)$, therefore $L_{(s,t)} = \{(s,t)\}$. □

Theorem 3. *$L_{(s,t)} = L_s \times L_t$ in $S \times T$ iff at least one of the following conditions holds:*

1) $L_s = \{s\}$ in S, and $L_t = \{t\}$ in T.

2) $s \in Ss$ and $t \in Tt$.

Proof. a) Let $L_{(s,t)} = L_s \times L_t$. We shall consider two possibilities:

(i) $L_{(s,t)} = \{(s,t)\}$,

(ii) $|L_{(s,t)}| > 1$.

If (i) holds, then $L_{(s,t)} = \{(s,t)\}$ implies $L_s = \{s\}$, $L_t = \{t\}$, therefore 1) holds.

If (ii) holds, then there is $(u,v) \neq (s,t)$ such that $(u,v) \in L_{(s,t)}$. Then $(u,v) \cup (Su \times Tv) = (s,t) \cup (Ss \times Tt)$, thence $(u,v) \in (Ss \times Tt)$ and $(s,t) \in (Su \times Tv)$. Hence we have $(Ss \times Tt) = (Su \times Tv)$ and $(s,t) \in (Ss \times Tt)$; therefore, $s \in Ss$ and $t \in Tt$, so 2) holds.

b) Now, if 1) holds, the $L_{(s,t)} = L_s \times L_t$ by Corollary of Theorem 2.

If 2) holds, then $s \in Ss$ and $t \in Tt$, then $(s,t) \in (Ss \times Tt)$. Let $(u,v) \in L_s \times L_t$ so $u \in L_s$, $v \in L_t$. It is easy to show that $Su = Ss$, and $Tv = Tt$ and then $Su \times Tv = Ss \times Tt$. Then $L(s,t) = Ss \times Tt = Su \times Tv = L(u,v)$, therefore $(u,v) \in L_{(s,t)}$. It implies that $L_s \times L_t \subseteq L_{(s,t)}$. Since by Theorem 2 $L_{(s,t)} \subseteq L_s \times L_t$, we conclude $L_{(s,t)} = L_s \times L_t$. □

Theorem 4. *If $|L_s| > 1$ in S and $|L_t| > 1$ in T, then*

1) $s \in Ss$ and $t \in Tt$,

2) $L_{(s,t)} = L_s \times L_t$ in $S \times T$.

Proof. 1) Since $|L_s| > 1$ and $|L_t| > 1$, there is $u \in L_s$, $u \neq s$ and $v \in L_t$, $v \neq t$, such that $L(u) = L(s)$ in S and $L(v) = L(t)$ in T. Then $u \cup Su = s \cup Ss$ and $v \cup Tv = t \cup Tt$. It implies $u \in Ss$ and $s \in Su$ and similarly $v \in Tt$ and $t \in Tv$. Thus we have $Su \subseteq Ss$ and $Ss \subseteq Su$, which gives $Su = Ss$ and $Tv = Tt$ and it implies $s \in Ss$, $t \in Tt$.

2) It implies from Theorem 3. □

Corollary. *If $L_s \times L_t$ in $S \times T$ is a union of at least two $\mathscr{L}$-classes, then necessarily either $|L_s| > 1$ and $L_t = \{t\}$, or $L_s = \{s\}$ and $|L_t| > 1$.*

Theorem 5. *$L_s \times L_t$ is the union of at least two $\mathscr{L}$-classes in $S \times T$ iff either $|L_s| > 1$ and $L_t = \{t\}$, $t \notin Tt$, or $L_s = \{s\}$, $s \notin Ss$ and $|L_t| > 1$.*

Proof. a) If $L_s \times L_t$ is the union of at least two $\mathscr{L}$-classes, then by Corollary of Theorem 4 either $|L_s| > 1$ and $L_t = \{t\}$ or $L_s = \{s\}$ and $|L_t| > 1$. If $|L_s| > 1$, then by Theorem 4 $s \in Ss$, $L_t = \{t\}$ and $t \notin Tt$, because otherwise $s \in Ss$ and $t \in Tt$ implies $L_{(s,t)} = L_s \times L_t$ by Theorem 3, which contradicts the hypothesis, so $t \neq Tt$.

In the case $L_s = \{s\}$ and $|L_t| > 1$ we proceed analogously.

b) Conversely, let $|L_s| > 1$ and $L_t = \{t\}$, $t \notin Tt$. Let $u \in L_s$, $u \neq s$, then $(s,t) \in L_s \times L_t$ as well as $(u,t) \in L_s \times L_t$. Moreover, $(s,t) \notin (Ss \times Tt)$ and $(u,t) \notin (Su \times Tt)$ as $t \notin Tt$, therefore by Lemma 2 $L_{(s,t)} = \{(s,t)\}$, $L_{(u,t)} = \{(u,t)\}$ and both $L_{(s,t)} \subseteq L_s \times L_t$ and $L_{(u,t)} \subseteq L_s \times L_t$.

In the case $L_s = \{s\}$, $s \notin Ss$ and $|L_t| > 1$ we proceed analogously.

In the next part we want to characterize maximal $\mathcal{L}$-classes in $S \times T$ and their mutual relation to maximal $\mathcal{L}$-classes in S and in T, respectively.

An $\mathcal{L}$-class $L_s(L_{(s,t)})$ in S $(S \times T)$ is maximal, if there is no element $u \in S$ $((u,v) \in S \times T)$ such that $L(s) \subset L(u)$ $(L(s,t) \subset L(u,v))$.

An element $s \in S$ is indecomposable if $s \in S - S^2$. □

Remark 2. It is evident that

1) If $s \in S$ is indecomposable, then $s \notin Ss$ and $L_s = \{s\}$.

2) An element $(s,t) \in S \times T$ is indecomposable iff either $s \in S$ or $t \in T$ is indecomposable.

Lemma 3. 1) *If $(S \times T)^2 \subset S \times T$, then for any $(s,t) \in S \times T - (S \times T)^2$, $L_{(s,t)} = \{(s,t)\}$ is maximal $\mathcal{L}$-class in S.*

2) *If $L_{(s,t)} = \{(s,t)\}$ is a maximal $\mathcal{L}$-class of $S \times T$ and $(s,t) \notin (Ss \times Tt)$, then (s,t) is indecomposable.*

Proof. 1) Let $(s,t) \in (S \times T) - (S \times T)^2$. If $L(s,t) \subset L(u,v)$ for some $(u,v) \in S \times T$, then $(s,t) \in (Su \times Tv) \subseteq (S^2 \times T^2)$, which contradicts the hypothesis.

2) Let $L_{(s,t)} = \{(s,t)\}$ be a maximal $\mathcal{L}$-class of $S \times T$ and $(s,t) \notin (Ss \times Tt)$. If $(s,t) \in (Su \times Tv)$ for $(u,v) \in S \times T$, $(u,v)\#(s,t)$, then $L(s,t) \subseteq L(u,v)$ in $S \times T$. $L(s,t) = L(u,v)$ cannot be satisfied, since $L_{(s,t)} = \{(s,t)\}$, hence $L(s,t) \subset L(u,v)$ and this contradicts the fact that $L_{(s,t)}$ is a maximal $\mathcal{L}$-class in $S \times T$. Consequently for any $(u,v) \in (S \times T)$ we have $(s,t) \notin (Su \times Tv)$, therefore either $s \notin S^2$ or $t \notin T^2$, or both $s \notin S^2$ and $t \notin T^2$. Hence $(s,t) \in (S \times T) - (S \times t)^2$. □

Theorem 6. *Let $(s,t) \in (Ss \times Tt)$. Then $L_{(s,t)} = L_s \times L_t$ is a maximal $\mathcal{L}$-class iff L_s is a maximal $\mathcal{L}$-class in S and at the same time L_t is a maximal $\mathcal{L}$-class in T.*

Proof. a) The equality $L_{(s,t)} = L_s \times L_t$ follows from Theorem 3. Let e.g. L_s be no maximal $\mathcal{L}$-class. Then there is $u \in S$ such that $L(s) \subset L(u)$. If $u \in Su$, then from the relation $L(s) \subset L(u)$ we have $L(s) \subset Su$ and $u \notin L(s)$. Moreover, $(u,t) \notin L(s) \times L(t) = L(s,t)$. However, $u \in Su$, $t \in Tt$ implies $L(u,t) = L(u) \times L(t) = Su \times Tt \supset L(s) \times L(t) = L(s,t)$, since $(u,t) \notin L(s) \times L(t)$. It means that $L_{(s,t)}$ is not a maximal $\mathcal{L}$-class in $S \times T$.

If $u \notin Su$, then $L(s) \subset L(u)$ implies that $L(s) \subseteq Su$ and $u \notin L(s)$. Moreover $(u,t) \notin L(s) \times L(t) = L(s,t)$. But $u \notin Su$, $t \in Tt$ implies that $L(u,t) = (u,t) \cup [Su \times$

$L(t)] \supseteq (u,t) \cup L(s) \times L(t) \supset L(s) \times L(t) = L(s,t)$, since $(u,t) \notin L(s) \times L(t)$. We get again that $L_{(s,t)}$ is no maximal $\mathcal{L}$-class in $S \times T$.

b) Conversely, let $L_{(s,t)} = L_s \times L_t$ be no maximal $\mathcal{L}$-class in $S \times T$. Then there is $(u,v) \in (S \times T) - L_{(s,t)}$ such that $L(s,t) = L(s) \times L(t) = Ss \times Tt \subset L(u,v) \subseteq L(u) \times L(v)$. It implies $L(s) \subseteq L(u)$ in S, $L(t) \subseteq L(v)$ in T. However, $(u,v) \notin L(s,t)$, hence either $u \notin L(s)$ or $v \notin L(t)$. Therefore, either $L(s) \subset L(u)$ in S, or $L(t) \subset L(v)$ in T. It means that either L_s is no maximal $\mathcal{L}$-class in S, or L_t is no maximal $\mathcal{L}$-class in T. □

Theorem 7. *Let $(s,t) \notin (Ss \times Tt)$. Then $L_{(s,t)}$ is a maximal $\mathcal{L}$-class in $S \times T$ iff either $s \in S - S^2$, or $t \in T - T^2$ or both of them.*

Proof. a) Let $(s,t) \notin (Ss \times Tt)$ and let $L_{(s,t)}$ be a maximal $\mathcal{L}$-class in $S \times T$. Then by Lemma 3 and Remark 2 we have $(s,t) \in (S \times T) - (S^2 \times T^2)$, hence either $s \in S - S^2$ or $t \in T - T^2$, or both $s \in S - S^2$ and $t \in T - T^2$.

b) If $s \in S - S^2$, $t \in T$, then $(s,t) \in S \times T$ and $(s,t) \notin S^2 \times T^2$ since $s \notin S^2$, hence $(s,t) \in (S \times T) - (S^2 \times T^2)$ and by Lemma 3 $L_{(s,t)} = \{(s,t)\}$ is a maximal $\mathcal{L}$-class in $S \times T$.

Theorem 1 presents conditions under which $L(s,t) = L(s) \times L(t)$, Theorem 3 presents conditions under which $L_{(s,t)} = L_s \times L_t$ for a given element $(s,t) \in (S \times T)$.

The next statements express conditions under which $L(s,t) = L(s) \times L(t)$, $L_{(s,t)} = L_s \times L_t$ for any $(s,t) \in (S \times T)$. □

From Theorem 1 we immediately get

Theorem 8. *$L(s,t) = L(s) \times L(t)$ for any $(s,t) \in (S \times T)$ iff at least one of the following conditions holds:*

1) $Ss = \{s\}$ for any $s \in S$;

2) $Tt = \{t\}$ for any $t \in T$;

3) $s \in Ss$ and $t \in Tt$ for any $s \in S$, $t \in T$.

Theorem 9. *$L_{(s,t)} = L_s \times L_t$ for any $(s,t) \in S \times T$ iff at least one of the following conditions holds:*

1) $s \in Ss$ and $t \in Tt$ for any $s \in S$, $t \in T$.

2) Either for any $s \in S$, $s \in Ss$, $L_s = \{s\}$, there is at least one element $t \in T$ such that $t \notin Tt$, or for any $t \in T$, $t \in Tt$, $L_t = \{t\}$, there is at least one element $s \in S$ such that $s \notin Ss$.

3) $L_s = \{s\}$, $L_t = \{t\}$ for any $s \in S$, $t \in T$.

Proof. a) Let $L_{(s,t)} = L_s \times L_t$ for any $(s,t) \in S \times T$. As we know from Theorem 5, $L_{(s,t)} \subset L_s \times L_t$ iff either $s \notin Ss$ and $|L_t| > 1$, or $|L_s| > 1$ and $t \notin Tt$.

If we suppose that $L_{(s,t)} = L_s \times L_t$, then we have to eliminate the conditions under which $L_{(s,t)} \subset L_s \times L_t$.

In our procedure the following cases are considered:

α) Neither S nor T contain elements $s \in S$, $t \in T$ such that $s \notin Ss$, $t \notin Tt$.

β) Just one of the semigroups S, T contains at least one element $s \in S$ or $t \in T$, respectively such that $s \notin Ss$, $t \notin Tt$.

γ) Both S and T contain at least one element $s \in S$, $t \in T$ such that $s \notin Ss$, $t \notin Tt$.

If α) holds, then any $s \in S$, $t \in T$ satisfy $s \in Ss$, $t \in Tt$, and this is 1).

If β) holds and $s \in S$, $s \notin Ss$, then for any element $t \in T$ we have $t \in Tt$ and $L_t = \{t\}$, because if it were $|L_t| > 1$, then for $(s,t) \in L_s \times L_t$ we would have $L_{(s,t)} \subset L_s \times L_t$. Hence, $L_t = \{t\}$ for any $t \in T$. In the case that T contains such element $t \in T$, $t \notin Tt$, we proceed analogously obtaining $L_s = \{s\}$ for any $s \in S$, and this is 2).

γ) Let S contain at least one element $s \in S$ such that $s \notin Ss$, and let T contain at least one element $t \in T$ such that $t \notin Tt$. Then β) implies that $L_t = \{t\}$ for any $t \in T$ and $L_s = \{s\}$ for any $s \in S$, and this is 3).

b) Conversely, if 1) holds, then by Theorem 3 $L_{(s,t)} = L_s \times L_t$.

If 2) holds, then for any $s \in S$, $s \in Ss$, $L_s = \{s\}$ there is at least one $t_1 \in T$ such that $t_1 \notin Tt_1$. Let $t \in T$ be any element. If $t \in Tt$, then the condition 2) of Theorem 3 is satisfied and therefore $L_{(s,t)} = L_s \times L_t$. If $t \notin Tt$, then $L_t = \{t\}$ (Lemma 2), $L_s = \{s\}$ for any $s \in S$, so $L_{(s,t)} = L_s \times L_t$. In the second possibility we proceed analogously.

If 3) holds, then $L_s = \{s\}$, $L_t = \{t\}$ for any $s \in S$, $t \in T$. Then $L_{(s,t)} = L_s \times L_t$. $\square$

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