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Representation of undirected graphs by anticommutative conservative groupoids


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REPRESENTATION OF UNDIRECTED GRAPHS
BY ANTICOMMUTATIVE CONSERVATIVE GROUPOIDS

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Summary. The paper studies tolerances and congruences on anticommutative conservative groupoids. These groupoids can be assigned in a one-to-one way to undirected graphs.

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Various authors have studied graphs by algebraic methods. Among these methods there was also assigning certain algebraic structures to graphs in a one-to-one way. But usually only special classes of graphs were considered, e.g. directed graphs assigned to unary algebras. Representation of trees by certain ternary algebras was done by L. Nebeský [2], G. F. McNulty and C. R. Shallon [1] and R. Pöschel [3] have represented directed graphs by groupoids. In this case the support of the groupoid was equal to the union of the vertex set of the graph with some one-element set and thus not to the vertex set itself. Here we shall study another way of expressing graphs algebraically, namely by anticommutative conservative groupoids.

The multiplication in a groupoid will be denoted by simple juxtaposition and a groupoid will be identified with its support. Graphs will be always undirected, without loops and multiple edges.

A groupoid $\Gamma$ is called anticommutative, if

$$xy = yx \Rightarrow x = y$$

for any $x, y$ of $\Gamma$.

A groupoid $\Gamma$ is called conservative, if

$$xy = x \vee xy = y$$
for any $x, y$ of $\Gamma$.

Obviously every conservative groupoid is idempotent.

Let $\Gamma$ be an anticommutative conservative groupoid, let $x, y$ be two elements of $\Gamma$. Then either $xy = x$ and $yx = y$, or $xy = y$ and $yx = x$. Therefore we may introduce a one-to-one correspondence between undirected graphs and anticommutative conservative groupoids.

Let $G$ be an undirected graph. Define the groupoid $\Gamma(G)$ on the vertex set $V(G)$ of $G$ in such a way that $xx = x$ for each $x \in V(G)$, $xy = x$ for any two adjacent vertices $x, y$ of $G$ and $xy = y$ for any two distinct non-adjacent vertices $x, y$ of $G$. On the other hand, to every anticommutative conservative groupoid we may assign an undirected graph in such a way that the vertices of the graph are the elements of the groupoid and two vertices $x, y$ are adjacent if and only if $x \neq y$ and $xy = x$.

**Theorem 1.** Let $G$ be an undirected graph. The groupoid $\Gamma(G)$ is a semigroup if and only if $G$ is either a complete graph, or a totally disconnected graph.

**Remark.** A graph is called totally disconnected, if it has no edges.

**Proof.** If $G$ is a complete graph, then for any three elements $x, y, z$ of $\Gamma(G)$ we have

$$(xy)z = xz = x = xy = x(yz)$$

and the multiplication is associative. If $G$ is a totally disconnected graph, then

$$(xy)z = yz = z = xz = x(yz)$$

and the multiplication is again associative.

Now suppose that $G$ is neither complete, nor totally disconnected. Then there exist three distinct vertices $x, y, z$ of $G$ such that $x, y$ are adjacent, while $x, z$ are not. If $y, z$ are adjacent, then

$$(xy)z = xz = z \neq x = xy = x(yz).$$

If $y, z$ are not adjacent, then

$$(xz)y = zy = y \neq x = xy = x(zy).$$

We shall study tolerances and congruences on anticommutative conservative groupoids. A tolerance on a groupoid $\Gamma$ is a reflexive and symmetric binary relation $T$ on $\Gamma$ with the property that $(x_1, y_1) \in T, (x_2, y_2) \in T$ imply $(x_1 x_2, y_1 y_2) \in T$.
for any four elements $x_1, x_2, y_1, y_2$ of $\Gamma$. If moreover $T$ is transitive, it is called a congruence on $\Gamma$.

Let a groupoid $\Gamma$ and a tolerance $T$ on it be given. A subset $B$ of $\Gamma$ is called a block of $T$, if $(x, y) \in T$ for any two elements of $B$ and $B$ is a maximal set with this property (it is not a proper subset of another set with this property). If $T$ is a congruence, then its blocks are called congruence classes.

We shall prove a lemma.

**Lemma.** Let $G$ be a graph, let $T$ be a tolerance on $\Gamma(G)$. Let $M$ be a subset of a block of $T$. Let $u \in \Gamma(G) - M$, let $u$ be adjacent to at least one vertex of $M$ and non-adjacent to at least one vertex of $M$ in $G$. Then $(u, x) \in T$ for each $x \in M$.

**Proof.** Let $X$ (or $Y$) be the set of all vertices of $M$ which are adjacent (or non-adjacent respectively) to $u$. According to the assumption $X \neq \emptyset$, $Y \neq \emptyset$. Let $x \in X$, $y \in Y$. As both $x, y$ are in $M$, we have $(x, y) \in T$. By reflexivity $(u, u) \in T$. Then $(ux, uy) = (u, y) \in T$, $(xu, yu) = (x, u) \in T$ and by symmetry $(u, x) \in T$. The vertex $x$ was chosen arbitrarily in $X$, the vertex $y$ was chosen arbitrarily in $Y$ and $X \cup Y = M$, which proves the assertion. $\square$

Now we prove a theorem.

**Theorem 2.** Let $G$ be a graph, let $B$ be a non-empty subset of $\Gamma(G)$. Then the following two assertions are equivalent:

(i) Each vertex $x \in \Gamma(G) - B$ is either adjacent to all vertices of $B$, or non-adjacent to all vertices of $B$.

(ii) There exists a tolerance $T$ on $\Gamma(B)$ such that $B$ is a block of $T$.

**Proof.** (i)$\Rightarrow$(ii). Let (i) be satisfied. Let us define a tolerance $T$ such that $(x, y) \in T$ if and only if either $x = y$, or $x \in B$ and $y \in B$. Evidently $T$ is reflexive and symmetric (and moreover transitive). Let $x_1, y_1, x_2, y_2$ be four elements of $\Gamma(B)$ such that $(x_1, y_1) \in T$, $(x_2, y_2) \in T$. If $x_1 = y_1$, $x_2 = y_2$, then $(x_1x_2, y_1y_2) = (x_1x_2, x_1x_2) \in T$. Suppose $x_1 \in B$, $y_1 \in B$, $x_2 = y_2 \not\in B$. Then by (i) either $x_2 = y_2$ is adjacent to all vertices of $B$, or non-adjacent to all of them. In the first case $(x_1x_2, y_1y_2) = (x_1, y_1) \in T$, in the second case $(x_1x_2, y_1y_2) = (x_2, x_2) \in T$. Analogously in the case where $x_1 = y_1 \not\in B$, $x_2 \in B$, $y_2 \in B$. If all the elements $x_1, x_2, y_1, y_2$ are in $B$, then so are the products $x_1x_2, y_1y_2$, because $\Gamma(G)$ is conservative; again $(x_1x_2, y_1y_2) \in T$ and $T$ is a tolerance on $\Gamma(G)$.

(ii)$\Rightarrow$(i). Suppose that there exists $x \in \Gamma(G) - B$ adjacent to at least one vertex of $B$ and non-adjacent to at least one vertex of $B$. Then, by Lemma, the set $B \cup \{x\}$ has the property that any two of its elements are in $T$ and thus $B$ is not maximal with this property, i.e. it is not a block of $T$. $\square$
The family of all non-empty subsets of $\Gamma(G)$ satisfying the condition (i) will be denoted by $B(G)$.

We shall prove a theorem concerning $B(G)$.

**Theorem 3.** Let $G$ be an undirected graph. Then $B(G) \cup \{\emptyset\}$ is a complete lattice with respect to set inclusion.

**Proof.** Let $C$ be a non-empty subset of $B(G)$ and consider the intersection $D = \bigcap_{C \in C} C$. If $D = \emptyset$, then $D \in B(G) \cup \{\emptyset\}$. If $D \neq \emptyset$, then let $x \in \Gamma(G) - D$. Then there exists $C_0 \in C$ such that $x \in \Gamma(G) - C_0$. As $C_0 \in B(G)$, the vertex $x$ is either adjacent to all vertices of $C_0$ and thus also to all vertices of $D \subseteq C_0$, or non-adjacent to all of them; we have proved that $D \in B(G)$. Therefore there exists the meet $\bigwedge_{C \in C} C = \bigcap_{C \in C} C$. Now consider the set $D$ of all elements of $B(G)$ which contain $\bigcup_{C \in C} C$ as a subset; this set is non-empty, because $\Gamma(G) \in D$. There exists the meet $\bigwedge_{D \in D} D = \bigcap_{D \in D} D$ and this is $\bigvee_{C \in C} C$. $\square$

**Theorem 4.** Let $G$ be an undirected graph, let $B \in B(G)$, $C \in B(G)$, $B \cap C \neq \emptyset$. Then $B \lor C = B \cup C$.

**Proof.** Let $x \in \Gamma(G) - (B \cup C)$. Then $x \in \Gamma(G) - B$ and $x \in \Gamma(G) - C$. As $x \in \Gamma(G) - B$, it is either adjacent to all vertices of $B$, or non-adjacent to all vertices of $B$. In the first case it is adjacent to all vertices of $B \cap C \subseteq B$. As $B \cap C \neq \emptyset$, it is adjacent to at least one vertex of $C$ and, as $C \in B(G)$, to all vertices of $C$ and hence also to all vertices of $B \cup C$. In the second case it is non-adjacent to all vertices of $B \cup C$. Therefore $B \cup C \in B(G)$ and $B \lor C = B \cup C$. $\square$

**Proposition 1.** The lattice $B(G) \cup \{\emptyset\}$ is not distributive in general, but each of its complete sublattices not containing $\emptyset$ as an element is distributive.

**Proof.** Let the vertex set of $G$ be $V(G) = \{v, x, y, z\}$, let $G$ have exactly one edge $vx$. Evidently each one-element subset of $V(G)$ is in $B(G)$ and thus the sets $\{x\}$, $\{y\}$, $\{z\}$ are in $B(G)$. Evidently

$$\{x\} \lor (\{y\} \land \{z\}) = \{x\} \lor \emptyset = \{x\}.$$ 

The set $\{x\} \lor \{y\}$ is the least set which contains $x$ and $y$ and is in $B(G)$. The vertex $v$ is adjacent to $x$ and not to $y$, therefore $v \in \{x\} \lor \{y\}$. The set $\{v, x, y\} \in B(G)$ and therefore $\{x\} \lor \{y\} = \{v, x, y\}$. Analogously $\{x\} \lor \{z\} = \{v, x, z\}$. We have

$$\left((\{x\} \lor \{y\}) \land (\{x\} \lor \{z\})\right) = \{v, x, y\} \cap \{v, x, z\} = \{v, x\} \neq \{x\}.$$
and the lattice $B(G) \cup \{0\}$ is not distributive.

Now let $G$ be an arbitrary undirected graph. Let $B_0$ be a sublattice of $B(G) \cup \{0\}$ which does not contain $0$. Let $B_0$ be the meet of all elements of $B_0$; as $B_0$ is complete, $B_0$ is the least element of $B_0$ and $B_0 \neq \emptyset$. Any two elements of $B_0$ have a non-empty intersection, because they both contain $B_0$. Therefore the join in $B_0$ is equal to the set union and $B_0$ is a sublattice of the lattice of all subsets of $\Gamma(G)$, hence it is distributive. 

Note that $B(G)$ contains always the set $\Gamma(G)$ and all of its one-element subsets.

**Proposition 2.** Let $G$ be an undirected graph with at least two vertices. Then the lattice $B(G) \cup \{0\}$ is generated by its atoms.

**Proof.** As it was mentioned above, every one-element subset of $\Gamma(G)$ is in $B(G)$ and therefore the set of all atoms of $B(G) \cup \{0\}$ is equal to the set of all one-element subsets of $\Gamma(G)$. If $B \in B(G)$, then evidently $B = \bigvee_{x \in B} x$. If $x, y$ are two different elements of $\Gamma(G)$, then $\{x\} \land \{y\} = \emptyset$. This implies the assertion. 

Now we shall study the lattice $\text{Tol}(\Gamma(G))$ of all tolerances on $\Gamma(G)$.

**Theorem 5.** Let $G$ be an undirected graph. The lattice $\text{Tol}(\Gamma(G))$ is a sublattice of the lattice of all reflexive and symmetric binary relations on $\Gamma(G)$.

**Proof.** Let $T_1, T_2$ be two tolerances on $\Gamma(G)$. It is well-known that the meet of two tolerances on an algebra is equal to their intersection, $T_1 \land T_2 = T_1 \cap T_2$.

Consider the relation $T_1 \cup T_2$. Let $(x_1, y_1) \in T_1 \cup T_2$ and $(x_2, y_2) \in T_1 \cup T_2$. If they both belong to $T_1$ or they both belong to $T_2$, it is evident that $(x_1x_2, y_1y_2) \in T_1 \cup T_2$. If $x_1$ is adjacent to $x_2$ or $x_1 = x_2$ and $y_1$ is adjacent to $y_2$ or $y_1 = y_2$, then $(x_1x_2, y_1y_2) = (x_1, y_1) \in T_1 \subseteq T_1 \cup T_2$. If $x_1$ is non-adjacent to $x_2$ or $x_1 = x_2$ and $y_1$ is non-adjacent to $y_2$ or $y_1 = y_2$, then $(x_1x_2, y_1y_2) = (x_2, y_2) \in T_2 \subseteq T_1 \cup T_2$. Now suppose that $x_1$ is adjacent to $x_2$ and $y_1$ is non-adjacent to $y_2$. Then $(x_1x_2, y_1y_2) = (x_1, y_2)$. If $x_1$ is adjacent to $y_2$, then $(x_1, y_2) = (x_1y_2, y_1y_2) \in T_1 \subseteq T_1 \cup T_2$. If $x_1$ is non-adjacent to $y_2$, then $(x_1, y_2) = (x_1y_1, x_1y_2) \in T_2 \subseteq T_1 \cup T_2$. Hence $T_1 \cup T_2 \in \text{Tol}(\Gamma(G))$ and $T_1 \lor T_2 = T_1 \cup T_2$. We have proved that $\text{Tol}(\Gamma(G))$ is a sublattice of the lattice of all reflexive and symmetric relations on $\Gamma(G)$. 

Let $x, y$ be two distinct elements of $\Gamma(G)$. By $T(x, y)$ we shall denote the least tolerance on $\Gamma(G)$ containing the pair $(x, y)$, i.e. the intersection of all tolerances on $\Gamma(G)$ containing that pair.
Theorem 6. Let $G$ be an undirected graph, let $x, y$ be two distinct vertices of $G$. Then $T(x, y)$ is a congruence on $\Gamma(G)$ which has exactly one class with more than one element.

Proof. Let $B(x, y)$ be the set of all elements of $B(G)$ which contain the vertices $x, y$. This set is non-empty, because $\Gamma(G) \in B(x, y)$. Let $B_0(x, y)$ be the intersection of all elements of $B(x, y)$, i.e. their meet in the lattice $B(G) \cup \{\emptyset\}$. Obviously $\{x, y\} \subseteq B_0(x, y)$. In any tolerance $T$ every pair of elements being in $T$ belongs to at least one block of $T$. Therefore there exists a block $B$ of $T(x, y)$ such that $\{x, y\} \subseteq B$. By Theorem 2 we have $B \in B(G)$ and hence $B_0(x, y) \subseteq B$. We have then $(u, v) \in T(x, y)$ whenever $u \in B_0(x, y)$ and $v \in B_0(x, y)$. On the other hand, let the relation $T_0$ be defined so that $(u, v) \in T_0$ if and only if either $u \in B_0(x, y)$ and $v \in B_0(x, y)$, or $u = v$. Then by Theorem 2 the relation $T_0$ is a tolerance on $\Gamma(G)$; hence $T_0 \subseteq T(x, y)$ and by the minimality of $T(x, y)$ we have $T_0 = T(x, y)$. From the definition of $T_0$ it is clear that it has the required properties. \hfill $\square$

At the end we shall prove a theorem concerning the relationship between different blocks of a tolerance.

Theorem 7. Let $G$ be an undirected graph, let $T \in \text{Tol}(\Gamma(G))$, let $B_1, B_2$ be two distinct blocks of $T$. Then $B_1 - B_2 \neq \emptyset$, $B_2 - B_1 \neq \emptyset$ and either all vertices of $B_1 - B_2$ are adjacent to all vertices of $B_2$ and all vertices of $B_2 - B_1$ are adjacent to all vertices of $B_1$, or all vertices of $B_1 - B_2$ are non-adjacent to all vertices of $B_2$ and all vertices of $B_2 - B_1$ are non-adjacent to all vertices of $B_1$.

Proof. We have $B_1 - B_2 \neq \emptyset$ and $B_2 - B_1 \neq \emptyset$, because no block of a tolerance is a proper subset of another block. Let $x_1 \in B_1 - B_2$, $x_2 \in B_2 - B_1$. If $x_1$ is adjacent to $x_2$, then it is adjacent to all vertices of $B_2$, because $B_2 \in B(G)$. But then $x_2$ is adjacent to $x_1$ and thus $x_2$ is adjacent to all vertices of $B_1$, because $B_1 \in B(G)$. As $x_1, x_2$ were chosen arbitrarily, the assertion holds. If $x_1$ is non-adjacent to $x_2$, the proof is analogous. \hfill $\square$

We shall add some final remarks.

We may introduce a factor-graph $G/T$ of the graph $G$ by the tolerance $T \in \text{Tol}(\Gamma(G))$ in such a way that the vertex set of $G/T$ is the set of all blocks of $T$ and two such blocks $B_1, B_2$ are adjacent in $G/T$ if and only if all vertices of $B_1 - B_2$ are adjacent to all vertices of $B_2$. The corresponding groupoid $\Gamma(G/T)$ is called the factor-groupoid of $\Gamma(G)$ by $T$ and may be denoted by $\Gamma(G)/T$. If $T$ is a congruence, this is the factor-groupoid of $\Gamma(G)$ by $T$ in the usual sense.

Note that conservative groupoids do not form a variety; the direct product of two conservative groupoids need not be conservative.
References


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