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Nonabsolutely convergent series


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NONABSOLUTELY CONVERGENT SERIES

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Summary. Assume that for any $t$ from an interval $[a, b]$ a real number $u(t)$ is given. Summarizing all these numbers $u(t)$ is no problem in case of an absolutely convergent series $\sum_{t \in [a, b]} u(t)$. The paper gives a rule how to summarize a series of this type which is not absolutely convergent, using a theory of generalized Perron (or Kurzweil) integral.

Keywords. Nonabsolutely convergent series, generalized Perron integral.

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Notation. $\mathbb{N}$ is the set of all integers, $\mathbb{R}$ is the set of all real numbers. $[a, b]$, $[a, b)$, $(c, d]$ etc. will be bounded intervals in $\mathbb{R}$. If a point $t \in \mathbb{R}$ and a set $T \in \mathbb{R}$ are given, then $\text{dist}(t; T) = \inf \{|t - s; s \in T|$. If $x \in \mathbb{R}^n$ is an $n$-dimensional vector, then $(x)_j$ denotes the $j$-th component of the vector $x$.

We will make use of the notion of generalized Perron integral, which was defined in [K] in this way:

A finite sequence $A = \{a_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ is a partition of the interval $[a, b]$ if

(1) $a = a_0 < \tau_1 < \ldots < \alpha_{k-1} < \alpha_k = b$ and

(2) $\alpha_{i-1} \leq \tau_i \leq \alpha_i$, $i = 1, 2, \ldots, k$.

An arbitrary positive function $\delta: [a, b] \to (0, \infty)$ is called a gauge on $[a, b]$. Given a gauge $\delta$ on $[a, b]$, a partition $A$ of the interval $[a, b]$ is called $\delta$-fine if

(3) $[\alpha_{i-1}, \alpha_i] \subseteq [\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)]$, $i = 1, 2, \ldots, k$.

The set of all $\delta$-fine partitions of $[a, b]$ will be denoted by $\mathcal{A}(\delta; a, b)$ or briefly $\mathcal{A}(\delta)$.

It is known that for any gauge $\delta$ on $[a, b]$ the set $\mathcal{A}(\delta)$ is nonempty (see [K], Lemma 1,1,1).

Assume that a function $U: [a, b] \times [a, b] \to \mathbb{R}$ and a partition $A = \{a_0, \tau_1, \alpha_1, \ldots, \alpha_{k-1}, \tau_k, \alpha_k\}$ are given. The finite sum

(4) $S(U, A) = \sum_{i=1}^{k} [U(\tau_i, \alpha_i) - U(\tau_i, \alpha_{i-1})]$ is called the integral sum corresponding to the function $U$ and the partition $A$. 

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A function \( U: [a, b] \times [a, b] \to \mathbb{R} \) is called integrable over \([a, b]\) if there exists \( \gamma \in \mathbb{R} \) such that for every \( \varepsilon > 0 \) there exists a gauge \( \delta: [a, b] \to (0, \infty) \) such that for every \( A \in \mathcal{A}(\delta) \) the inequality

\[
|S(U, A) - \gamma| < \varepsilon
\]

holds. The number \( \gamma \in \mathbb{R} \) is called the generalized Perron integral of \( U \) over the interval \([a, b]\) and will be denoted by

\[
\gamma = \int_a^b DU(\tau, t).
\]

In \([K]\) a definition of an integral using the concept of major and minor functions is given, and it is proved that such a definition is equivalent to the definition given above.

The definition using major and minor functions may be formulated in the following way:

A function \( U: [a, b] \times [a, b] \to \mathbb{R} \) is integrable over \([a, b]\) if there exists \( \gamma \in \mathbb{R} \) such that for every \( \varepsilon > 0 \) there exists a gauge \( \delta \) on \([a, b]\) and functions \( M, m: [a, b] \to \mathbb{R} \) such that

\[
(t - \tau)(M(t) - M(\tau)) \geq (t - \tau)(U(\tau, t) - U(\tau, \tau)) \geq
\]

\[
(t - \tau)(m(t) - m(\tau)) \quad \text{whenever} \quad t, \tau \in [a, b] \quad \text{and}
\]

\[
|t - \tau| \leq \delta(\tau) \quad \text{and}
\]

\[
\gamma - \varepsilon < m(b) - m(a) \leq M(b) - M(a) \leq \gamma + \varepsilon.
\]

Then \( \gamma = \int_a^b DU(\tau, t) \).

Let a function \( u: [a, b] \to \mathbb{R} \) be given. The symbol \( \sum_{t \in [a, b]} u(t) \) can be met usually in the following situation: there is an at most countable set of indices \( D \subset [a, b] \) such that \( u(t) = 0 \) for any \( t \in [a, b] \setminus D \); this set \( D \) will be ordered into a sequence in an arbitrary way, say \( D = \{t_1, t_2, \ldots\} \). If the series \( \sum_{k=1}^{\infty} u(t_k) \) is absolutely convergent, i.e. the series \( \sum_{k=1}^{\infty} |u(t_k)| \) is convergent, we have \( \sum_{t \in [a, b]} u(t) = \sum_{k=1}^{\infty} u(t_k) \).

However, if the series is not absolutely convergent, then in order to obtain a reasonable theory we have to give a rule how to order the index set \( D \). In fact, this is the aim of the present paper.

In the following we will deal only with real-valued functions \( u \); if \( u \) is an \( \mathbb{R}^n \)-valued function with \( n > 1 \), then the sum \( \sum_{t \in [a, b]} u(t) \) can be defined componentwise:

\[
(\sum_{t \in [a, b]} u(t))_j = \sum_{t \in [a, b]} (u(t))_j, \quad j = 1, 2, \ldots, n.
\]

**Definition 1.** Assume that a gauge \( \delta: [a, b] \to (0, \infty) \) is given. By \( I(\delta; a, b) \) or briefly \( I(\delta) \) we denote the set of all finite nonempty sets \( B \subset [a, b] \) such that the following holds:
If \( t, t' \in B, \ t < t' \) are neighbouring points, i.e. \((t, t') \cap B = \emptyset\), then 
\[ t' - t < \delta(t) + \delta(t'). \]
Denote \( I = \min B, \ i = \max B; \) then \( i - a < \delta(i) \), 
\[
b - i < \delta(i).\]

**Lemma 1.** (i) For every gauge \( \delta \) on \([a, b]\) the set \( I(\delta) \) is nonempty. (ii) If a gauge \( \delta: [a, b] \to (0, \infty) \) is given and \( a < c < b \), then for any two sets \( B_1 \in I(\delta; a, c) \) and \( B_2 \in I(\delta; c, b) \) the set \( B_1 \cup B_2 \) belongs to \( I(\delta; a, b) \).

**Proof.** (i) For every \( t \in (a, b] \) such that \( t < a + \delta(a) \) the set \( \{a\} \) obviously belongs to \( I(\delta; a, t) \). Denote \[
c = \sup \{ t \in (a, b], \ I(\delta; a, t) \neq \emptyset \}. \]
We have just shown that \( c > a \). There is \( t_0 \in (a, b] \) such that \( I(\delta; a, t_0) \neq \emptyset \) and \( c - \delta(c) < t_0 \). If \( B \in I(\delta; a, t_0) \) then \( B \cup \{c\} \in I(\delta; a, c) \) because denoting \( i = \max B \) we have the estimate \( c - i = (c - t_0) + (t_0 - i) < \delta(c) + \delta(i) \).

Let us assume that \( c < b \); then for every \( c' \in (c, b] \) such that \( c' < c + \delta(c) \) we have \( B \cup \{c\} \in I(\delta; a, c') \) and consequently the set \( I(\delta; a, c') \) is nonempty, but this is impossible because of (8). It means that \( c = b \) and \( I(\delta; a, b) \neq \emptyset \).

(ii) Denote \( t_1 = \max B_1 \) and \( t_2 = \min B_2 \), then \( c - t_1 < \delta(t_1) \) and \( t_2 - c < \delta(t_2) \) by (7). Then \( t_2 - t_1 < \delta(t_1) + \delta(t_2) \) and consequently the assumption (7) holds for \( B_1 \cup B_2 \) on the interval \([a, b]\).

**Definition 2.** Assume that a function \( u: [a, b] \to \mathbb{R} \) is given. We say that the series 
\[
\sum_{t \in [a, b]} u(t) \]
is convergent and that its sum is equal to \( u \in \mathbb{R} \), if for every \( \varepsilon > 0 \) there is a gauge \( \delta \) on \([a, b]\) such that for every finite set of indices \( \{t_1, t_2, \ldots, t_m\} \) belonging to \( I(\delta) \) the inequality
\[
| \sum_{n=1}^{m} u(t_n) - u | < \varepsilon
\]
holds. The series \( \sum_{t \in [a, b]} u(t) \) is defined as the series \( \sum_{t \in [a, b]} u(t) \) with \( u(b) = 0 \), similarly \( \sum_{t \in [a, b]} u(t) \), \( \sum_{t \in [a, b]} u(t) \).

**Remark.** For a given series \( \sum_{t \in [a, b]} u(t) \) and for any set \( B = \{t_1, t_2, \ldots, t_m\} \subset [a, b] \) let us denote
\[
s(B) = \sum_{n=1}^{m} u(t_n). \]
Then (10) can be written in the form
\[
|s(B) - u| < \varepsilon
\]
for every \( B \in I(\delta) \).
Lemma 2. Let a finite set \( B_0 \subset [a, b] \) and a gauge \( \delta \) on \( [a, b] \) be given. Assume that
\[
(11) \quad \delta(t) \leq \text{dist}(t; B_0 \setminus \{\tau\}) \quad \text{for every} \quad \tau \in [a, b].
\]
Then every set \( B \in I(\delta) \) includes \( B_0 \).

Proof. The condition (11) can be written also in the form
\[
|\tau - \sigma| \geq \delta(t) \quad \text{holds for any} \quad \sigma \in B_0 \quad \text{and} \quad \tau \in [a, b] \quad \text{such that} \quad \tau \neq \sigma.
\]
Assume that there are \( B \in I(\delta) \) and \( \sigma \in B_0 \) such that \( \sigma \notin B \). Let us find neighbouring points \( t', t'' \in B \) such that \( t' < \sigma < t'' \). Then
\[
\delta(t'') + \delta(t') > t'' - t' = (t'' - \sigma) + (\sigma - t') \geq \delta(t'') + \delta(t'),
\]
which is a contradiction.

Proposition 1. Let real functions \( u, v: [a, b] \to \mathbb{R} \) be given. Assume that there are points \( s_1, s_2, \ldots, s_k \in [a, b] \) such that
\[
(12) \quad u(t) = v(t) \quad \text{for every} \quad t \in [a, b] \setminus \{s_1, s_2, \ldots, s_k\}.
\]
If at least one of the series \( \sum_{t \in [a, b]} u(t) \), \( \sum_{t \in [a, b]} v(t) \) is convergent, then the other is also convergent and the equality
\[
\sum_{t \in [a, b]} u(t) - \sum_{j=1}^{k} u(s_j) = \sum_{t \in [a, b]} v(t) - \sum_{j=1}^{k} v(s_j)
\]
holds.

Proof. Assume for instance that the series \( \sum_{t \in [a, b]} u(t) = u \) is convergent. Then for every \( \varepsilon > 0 \) there is a gauge \( \delta \) such that (10)' holds for every \( B \in I(\delta) \). Let us define
\[
\delta'(t) = \min \{\delta(t), \text{dist}(t; C \setminus \{\tau\})\} \quad \text{where} \quad C = \{s_1, s_2, \ldots, s_k\}.
\]
Lemma 2 implies that an arbitrary set \( B = \{t_1, t_2, \ldots, t_m\} \in I(\delta') \) includes all the points \( s_1, s_2, \ldots, s_k \).

From (12) it follows that \( u(t_n) = v(t_n) \) for every \( t_n \in B \) which does not belong to \( C \).

We have an estimate
\[
\left| \sum_{n=1}^{m} v(t_n) - \left[ \sum_{j=1}^{k} v(s_j) - \sum_{j=1}^{k} u(s_j) + u \right] \right| \leq \n \leq \left| \sum_{n=1}^{m} v(t_n) - \sum_{j=1}^{k} u(s_j) + \sum_{n=1}^{m} u(t_n) \right| + \left| \sum_{n=1}^{m} u(t_n) - u \right| = \n = \left| \sum_{n=1}^{m} v(t_n) - \sum_{n=1}^{m} u(t_n) \right| + \left| \sum_{n=1}^{m} u(t_n) - u \right| = \left| \sum_{n=1}^{m} u(t_n) - u \right| < \varepsilon.
\]
Since the set $B \in I(\delta')$ was arbitrary, we get the equality
\[ \sum_{t \in [a, b]} v(t) = \sum_{j=1}^k v(s_j) - \sum_{j=1}^k u(s_j) + u. \]
The proof of the other implication is analogous.

**Corollary.** Let a function $u: [a, b] \to \mathbb{R}$ be given. Then
\[ \sum_{t \in [a, b]} u(t) = \sum_{t \in [a, b]} u(t) + u(b) = u(a) + \sum_{t \in [a, b]} u(t) \]
provided at least one of the three series is convergent.

**Proof.** By Definition 2 the series $\sum_{t \in [a, b]} u(t)$ is identical with a series $\sum_{t \in [a, b]} v(t)$ where $v(t) = u(t)$ for $t \in [a, b]$, $v(b) = 0$, and the series $\sum_{t \in [a, b]} u(t)$ is defined as a series $\sum_{t \in [a, b]} w(t)$ where $w(t) = u(t)$ for $t \in (a, b]$, $w(a) = 0$.

Proposition 1 implies that
\[ \sum_{t \in [a, b]} u(t) - u(a) - u(b) = \sum_{t \in [a, b]} v(t) - v(a) - v(b) = \]
\[ = \sum_{t \in [a, b]} w(t) - w(a) - w(b), \quad \text{i.e.} \]
\[ \sum_{t \in [a, b]} u(t) - u(a) - u(b) = \sum_{t \in [a, b]} u(t) - u(a) = \sum_{t \in [a, b]} u(t) - u(b) \]
provided at least one of the series $\sum_{t \in [a, b]} u(t)$, $\sum_{t \in [a, b]} v(t)$, $\sum_{t \in [a, b]} w(t)$ is convergent.

**Proposition 2.** The series $\sum_{t \in [a, b]} u(t)$ is convergent if and only if for every $\varepsilon > 0$ there is a gauge $\delta: [a, b] \to (0, \infty)$ such that for every two sets $B_1, B_2 \in I(\delta)$ the inequality
\[ |s(B_1) - s(B_2)| < \varepsilon \]
holds.

**Proof.** 1. If the series $\sum_{t \in [a, b]} u(t)$ is convergent and has the sum $u$, then for every $\varepsilon > 0$ there is a gauge $\delta$ such that for every $B \in I(\delta)$ the inequality $|s(B) - u| < \varepsilon/2$ holds. Then
\[ |s(B_1) - s(B_2)| \leq |s(B_1) - u| + |s(B_2) - u| < \varepsilon \]
for every $B_1, B_2 \in I(\delta)$.

2. Assume that for every $n \in \mathbb{N}$ there is a gauge $\delta_n$ on $[a, b]$, such that the inequality
\[ |s(B_1) - s(B_2)| < \frac{1}{n} \]
holds for every $B_1, B_2 \in I(\delta_n)$. We may assume that

$$\delta_1(\tau) \geq \delta_2(\tau) \geq \delta_3(\tau) \geq \ldots, \quad \tau \in [a, b].$$

For every $n \in \mathbb{N}$ let us choose a set $B_n \in I(\delta_n)$; then also $B_k \in I(\delta_k)$ for every $k \leq n$.

For a given $\eta > 0$ let us find $n_0 \in \mathbb{N}$ such that $1/n_0 \leq \eta$. For every $m, n \in \mathbb{N}$ such that $m > n \geq n_0$ we have an estimate

$$|s(B_n) - s(B_m)| < \frac{1}{n} \leq \eta.$$ 

This means that $\{s(B_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, which has a limit $u \in \mathbb{R}$. Passing to the limit with $m \to \infty$ in (15) we get

$$|s(B_n) - u| \leq \frac{1}{n}.$$ 

Let $\varepsilon > 0$ be given. Let us find $n' \in \mathbb{N}$ such that $1/n' \leq \varepsilon/2$; then for every $B \in I(\delta_n)$ we have the inequality

$$|s(B) - u| \leq |s(B) - s(B_{n'})| + |s(B_{n'}) - u| < \frac{2}{n'} \leq \varepsilon.$$ 

Consequently $\sum_{t \in [a,b]} u(t) = u$.

**Lemma 3.** Assume that a convergent series $\sum_{t \in [a,b]} u(t)$ is given; for $\varepsilon > 0$ let a gauge $\delta$ on $[a, b]$ be given such that the inequality (13) holds for every $B_1, B_2 \in I(\delta; a, b)$. Then $|s(C_1) - s(C_2)| < \varepsilon$ for every interval $[c, d] \subset [a, b]$ and every $C_1, C_2 \in I(\delta; c, d)$.

**Proof.** Assume that $C_1 = \{s_1, s_2, \ldots, s_k\}$, $C_2 = \{t_1, t_2, \ldots, t_m\}$. Let us choose sets $B = \{\tau_1, \ldots, \tau_p\} \in I(\delta; a, c)$ and $D = \{\sigma_1, \ldots, \sigma_q\} \in I(\delta; d, b)$ (if $a = c$ or $d = b$ then $B = \emptyset$ or $D = \emptyset$, respectively). According to Lemma 1 (ii) the sets $B \cup C_1 \cup D$ and $B \cup C_2 \cup D$ belong to $I(\delta; a, b)$. By (13) we get the inequality

$$|s(C_1) - s(C_2)| = |\sum_{i=1}^{k} u(s_i) - \sum_{i=1}^{m} u(t_i)| =$$

$$= \left| \left\{ \sum_{i=1}^{k} u(s_i) + \sum_{i=1}^{p} u(\tau_i) + \sum_{i=1}^{q} u(\sigma_i) \right\} - \left\{ \sum_{i=1}^{m} u(t_i) + \sum_{i=1}^{p} u(\tau_i) + \sum_{i=1}^{q} u(\sigma_i) \right\} \right| =$$

$$= |s(B \cup C_1 \cup D) - s(B \cup C_2 \cup D)| < \varepsilon.$$
Proposition 3. (i) If the series \( \sum_{t \in [a, b]} u(t) \) is convergent, then \( \sum_{t \in [c, d]} u(t) \) is convergent for every interval \([c, d] \subseteq [a, b]\).

(ii) For \( \varepsilon > 0 \) let a gauge \( \delta \) be given such that \( |s(B) - \sum_{t \in [a, b]} u(t)| < \varepsilon \) holds for every \( B \in I(\delta; a, b) \). Then \( |s(C) - \sum_{t \in [c, d]} u(t)| \leq \varepsilon \) holds for every \( C \in I(\delta; c, d) \), where \([c, d] \subseteq [a, b]\).

Proof. This is a consequence of Proposition 2 and Lemma 3.

Theorem 1. Assume that a convergent series \( \sum_{t \in [a, b]} u(t) \) is given. Let us define

\[
(16) \quad f(a) = u(a), \quad f(\tau) = \sum_{t \in [a, \tau]} u(t) \quad \text{for} \quad \tau \in (a, b].
\]

Then the function \( f \) is regulated (i.e. has one-sided limits) and

\[
(17) \quad \lim_{s \to \tau-} f(s) = f(\tau) - u(\tau), \quad \tau \in (a, b],
\]

\[
\lim_{s \to \tau+} f(s) = f(\tau), \quad \tau \in [a, b).
\]

Proof. Let \( \varepsilon > 0 \) be given. Let us find a gauge \( \delta \) on \([a, b]\) such that \( |s(B) - \sum_{t \in [a, b]} u(t)| < \varepsilon \) holds for every \( B \in I(\delta; a, b) \).

a) Assume that \( \tau \in (a, b] \). Let \( s \in [a, \tau) \) be such that \( \tau - \delta(\tau) < s \). Take any set \( B \in I(\delta; a, s) \) such that \( s \in B \). Since \( \{\tau\} \in I(\delta; s, \tau) \), by Lemma 1 the set \( B \cup \{\tau\} \) belongs to \( I(\delta; a, \tau) \). According to Proposition 3 (ii) the following estimate holds:

\[
(18) \quad |f(\tau) - u(\tau) - f(s)| \leq |f(\tau) - [u(\tau) + s(B)]| + |f(s) - s(B)| =
\]

\[
= |f(\tau) - s(B \cup \{\tau\})| + |f(s) - s(B)| \leq 2\varepsilon.
\]

b) Assume that \( a \leq \tau < b \), let \( C \in I(\delta; a, \tau) \) be such a set that \( \tau \in C \) (if \( \tau = a \) then \( C = \{\tau\} \)). For every \( s \in (\tau, b] \) such that \( s < \tau + \delta(\tau) \) the set \( \{\tau\} \) belongs to \( I(\delta; s, \tau) \) and consequently \( C \in I(\delta; a, s) \). Then

\[
(19) \quad |f(s) - f(\tau)| \leq |f(s) - s(C)| + |f(\tau) - s(C)| \leq 2\varepsilon.
\]

The relations (18), (19) imply (17).

Corollary 1. If the series \( \sum_{t \in [a, b]} u(t) \) is convergent, then the set \( \{t \in [a, b]; u(t) \neq 0\} \) is at most countable.

Proof. Since the function \( f \) defined by (16) is regulated, it can be discontinuous only in an at most countable set; according to (17)

\[
f(\tau-) \neq f(\tau) \quad \text{if and only if} \quad u(\tau) \neq 0.
\]
Corollary 2. If the series $\sum_{t\in[a,b]} u(t)$ is convergent then
$$\lim_{s\to\tau} u(s) = 0 \text{ for every } \tau \in [a, b].$$

Proof. Let $\tau \in (a, b]$ and $\varepsilon > 0$ be given. There is $\lambda > 0$ such that the following holds: If $\tau - \lambda < s < \tau$, then $|f(\tau) - f(s)| \leq \varepsilon$. Then also $|f(\tau -) - f(s -)| \leq \varepsilon$ for every $s \in (\tau - \lambda, \tau)$. Hence
$$|u(s)| = |f(s) - f(s -)| \leq |f(s) - f(\tau -)| + |f(\tau -) - f(s -)| \leq 2\varepsilon,$$
if $s \in (\tau - \lambda, \tau)$. This means that $\lim_{s\to\tau-} u(s) = 0$. Similarly $\lim_{s\to\tau+} u(s) = 0$ for every $\tau \in [a, b)$.

Corollary 3. Assume that the series $\sum_{t\in[a,b]} u(t)$ is convergent. Let us define
\begin{equation}
 g(a) = 0, \quad g(\tau) = \sum_{t\in[a,\tau]} u(t) \text{ for } \tau \in (a, b].
\end{equation}
Then the function $g$ is regulated and
\begin{equation}
\lim_{s\to\tau-} g(s) = g(\tau), \quad \tau \in (a, b],
\end{equation}
\begin{equation}
\lim_{s\to\tau+} g(s) = g(\tau) + u(\tau), \quad \tau \in [a, b).
\end{equation}

Proof. By Proposition 1 we have $g(\tau) = f(\tau) - u(\tau)$ for every $\tau \in [a, b]$. If $\tau \in (a, b]$ then
$$\lim_{s\to\tau-} g(s) = \lim_{s\to\tau-} f(s) - \lim_{s\to\tau-} u(s) = f(\tau -) - u(\tau) = g(\tau);$$
if $\tau \in [a, b)$ then
$$\lim_{s\to\tau+} g(s) = \lim_{s\to\tau+} f(s) + \lim_{s\to\tau+} u(s) = f(\tau) = g(\tau) + u(\tau).$$

Theorem 2. Assume that a function $u: [a, b] \to \mathbb{R}$ is given. Let us define a function $U: [a, b] \times [a, b] \to \mathbb{R}$ by
$$U(\tau, t) = u(t) \quad \text{for } \tau < t,$$
$$U(\tau, t) = 0 \quad \text{for } \tau = t,$$
$$U(\tau, t) = -u(\tau) \quad \text{for } \tau > t.$$
Then the series $\sum_{t\in[a,b]} u(t)$ is convergent if and only if $U(\tau, t)$ is integrable over $[a, b]$. We have the equality
$$\int_a^b DU(\tau, t) = \sum_{t\in[a,b]} u(t).$$
Proof. (i) Assume that the function \( U \) is integrable and denote
\[ \gamma = \int_a^b DU(\tau, t). \]
For a given \( \varepsilon > 0 \) there is a gauge \( \delta \) on \([a, b]\) such that
\[ |S(U, A) - \gamma| < \varepsilon \]
holds for every \( A \in \mathcal{A}(\delta; a, b) \). Let us define
\[ \delta'(\tau) = \min \{ \delta(\tau), b - \tau, \tau - a \} \quad \text{for} \quad \tau \in (a, b), \]
\[ \delta'(\tau) = \min \{ \delta(\tau), b - a \} \quad \text{for} \quad \tau = a, b. \]
Let an arbitrary finite set \( B = \{t_1, t_2, \ldots, t_m\} \in I(\delta') \) be given. By Lemma 2 the set \( B \) contains the points \( a, b \). Assume that
\[ a = t_1 < t_2 < \ldots < t_m = b. \]
For any \( i = 1, 2, \ldots, m - 1 \) we have by (7)
\[ i_{i+1} - t_i < \delta'(t_i) + \delta'(t_{i+1}), \quad \text{i.e.} \quad t_{i+1} - \delta'(t_{i+1}) < t_i + \delta'(t_i). \]
Hence the open interval \((t_i, t_{i+1}) \cap (t_{i+1} - \delta'(t_{i+1}), t_i + \delta'(t_i))\) is nonempty. Corollary 1 of Theorem 1 implies that there is \( \alpha_i \in (t_i, t_{i+1}) \cap (t_{i+1} - \delta'(t_{i+1}), t_i + \delta'(t_i)) \) such that \( u(\alpha_i) = 0 \). Denote \( \alpha_0 = a, \alpha_m = b. \)
The set \( A = \{\alpha_0, t_1, \alpha_1, \ldots, t_m, \alpha_m\} \) obviously belongs to \( \mathcal{A}(\delta'; a, b) \). Consequently
\[ |\sum_{n=1}^{m} u(t_n) - [u(a) + \gamma]| = |\sum_{n=2}^{m} u(t_n) - \gamma| = \]
\[ = |[\sum_{n=2}^{m} u(t_n) + \sum_{n=1}^{m-1} u(\alpha_n)] - \gamma| = \]
\[ = |[\sum_{t_n < \alpha_n} u(t_n) + \sum_{t_n < \alpha_n} u(\alpha_n)] - \gamma| = |S(U, A) - \gamma| < \varepsilon. \]
According to Definition 2 the series \( \sum_{t \in [a, b]} u(t) \) is convergent and \( \sum_{t \in [a, b]} u(t) = u(a) + \gamma. \)
Hence \( \gamma = \sum_{t \in [a, b]} u(t). \)
(ii) Assume that the series \( \sum_{t \in [a, b]} u(t) = u \) is convergent. For every gauge \( \delta \) and \( t \in (a, b) \) let us denote by \( I_t(\delta) \) the set of all \( B \in I(\delta; a, i) \) such that \( t \in B. \) For \( t = a \) the set \( I_a(\delta) \) will consist of a single element \( \{a\}. \)
Let \( \varepsilon > 0 \) be given. There is a gauge \( \delta \) on \([a, b]\) such that
\[ |s(B) - u| < \varepsilon \quad \text{holds for any} \quad B \in I(\delta; a, b). \]
Let us define \( m(t) = \inf_{B \in I_t(\delta)} s(B), \) \( M(t) = \sup_{B \in I_t(\delta)} s(B), \) \( t \in [a, b]. \) Let us notice that \( m(a) = u(a), M(a) = u(a). \) From (22) it follows that \( u - \varepsilon < s(B) < u + \varepsilon \) for every \( B \in I_t(\delta) \subset I(\delta; a, b) \), and consequently
\[
\begin{align*}
u - \varepsilon & \leq m(b) \leq M(b) \leq u + \varepsilon, \\
(23) \quad u - u(a) - \varepsilon & \leq m(b) - m(a) \leq M(b) - M(a) \leq u - u(a) + \varepsilon.
\end{align*}
\]
Assume that \(a \leq \tau < t \leq b\) and \(t < \tau + \delta(\tau)\). For arbitrary \(\lambda > 0\) there are \(B_1, B_2 \in I_\iota(\delta)\) such that
\[
s(B_1) < m(\tau) + \lambda, \quad s(B_2) > M(\tau) - \lambda.
\]
Since \(\{\tau, t\} \in I(\delta; \tau, t)\), by Lemma 1 (ii) the sets \(B_1 \cup \{\tau, t\} = B_1 \cup \{t\}\) and \(B_2 \cup \{\tau, t\} = B_2 \cup \{t\}\) belong to \(I(\delta; a, t)\); these sets also belong to \(I_\iota(\delta)\) because they contain \(t\). Hence
\[
m(t) \leq s(B_1 \cup \{t\}) = s(B_1) + u(t) < m(\tau) + \lambda + u(t), \\
M(t) \geq s(B_2 \cup \{t\}) = s(B_2) + u(t) > M(\tau) - \lambda + u(t).
\]
Since the number \(\lambda > 0\) was arbitrary, we get inequalities .

\[
(24) \quad m(t) - m(\tau) \leq u(t) = U(\tau, t) - U(\tau, \tau) \leq M(t) - M(\tau).
\]
Similarly, if \(a \leq t < \tau \leq b\) where \(\tau - \delta(\tau) < t\), then for an arbitrary \(\eta > 0\) we can find \(C_1, C_2 \in I_\iota(\delta)\) such that
\[
s(C_1) < m(t) + \eta, \quad s(C_2) > M(t) - \eta.
\]
Since \(\{\tau\} \in I(\delta; \tau, \tau)\), the sets \(C_1 \cup \{\tau\}, C_2 \cup \{\tau\}\) belong to \(I_\iota(\delta)\) and consequently
\[
m(\tau) \leq s(C_1 \cup \{\tau\}) = s(C_1) + u(\tau) < m(t) + \eta + u(\tau), \\
M(\tau) \geq s(C_2 \cup \{\tau\}) = s(C_2) + u(\tau) > M(t) - \eta + u(\tau).
\]
We get the inequality
\[
(25) \quad m(\tau) - m(t) \leq u(t) = U(\tau, \tau) - U(\tau, t) \leq M(\tau) - M(t).
\]
According to the definition of integral using major and minor functions (see (5), (6)) it follows from (23), (24), (25) that the function \(U\) is integrable over \([a, b]\) and
\[
\int_a^b DU(\tau, t) = u - u(a) = \sum_{t \in [a, b]} u(t).
\]

**Theorem 3.** Assume that real functions \(u, v : [a, b] \to \mathbb{R}\) are given. Let us define a function \(V : [a, b] \times [a, b] \to \mathbb{R}\) by
\[
(26) \quad V(\tau, t) = u(t) + v(\tau) \quad \text{for} \quad \tau < t, \\
V(\tau, t) = 0 \quad \text{for} \quad \tau = t, \\
V(\tau, t) = -u(\tau) - v(t) \quad \text{for} \quad \tau > t.
\]
Then the series \(\sum_{t \in [a, b]} (u(t) + v(t))\) is convergent if and only if the function \(V\) is integrable over \([a, b]\). We have the equality
\[
\int_a^b DV(\tau, t) = v(a) + \sum_{t \in [a, b]} (u(t) + v(t)) + u(b).
\]
Proof. Let us define

\[ R(\tau, t) = u(t) + v(t) \quad \text{for} \quad \tau < t, \]
\[ R(\tau, t) = 0 \quad \text{for} \quad \tau = t, \]
\[ R(\tau, t) = -u(\tau) - v(\tau) \quad \text{for} \quad \tau > t. \]

By Theorem 2 the series \( \sum_{\tau\in[a,b]} (u(t) + v(t)) \) is convergent if and only if \( R \) is integrable over \([a, b] \), and

\[ \int_{a}^{b} DR(\tau, t) = \sum_{\tau\in[a,b]} (u(t) + v(t)) \]

holds. Using the definition of the generalized Perron integral, it can be easily proved that the function \( V(\tau, t) - R(\tau, t) = v(\tau) - v(t) \) is integrable over \([a, b] \), and

\[ \int_{a}^{b} D[V(\tau, t) - R(\tau, t)] = v(a) - v(b). \]

Then the function \( V \) is integrable if and only if \( R \) is integrable. From (27), (28) we obtain

\[ \int_{a}^{b} DV(\tau, t) = \int_{a}^{b} DR(\tau, t) + \int_{a}^{b} D[V(\tau, t) - R(\tau, t)] = \]
\[ = \sum_{\tau\in[a,b]} (u(t) + v(t)) + (u(b) + v(b)) + (v(a) - v(b)) = \]
\[ = v(a) + \sum_{\tau\in[a,b]} (u(t) + v(t)) + u(b). \]

Corollary 4. The series \( \sum_{\tau\in[a,b]} u(t) \) is convergent if and only if the function

\[ U': [a, b] \times [a, b] \to \mathbb{R} \]

is integrable over \([a, b] \); the equality

\[ \int_{a}^{b} DU'(\tau, t) = \sum_{\tau\in[a,b]} u(t) \]

is satisfied.

Theorem 4. Assume that functions \( u, v: [a, b] \to \mathbb{R} \) are given. Let us define a function \( W: [a, b] \times [a, b] \to \mathbb{R} \) by

\[ W(\tau, t) = v(\tau) \quad \text{for} \quad \tau < t, \]
\[ W(\tau, t) = 0 \quad \text{for} \quad \tau = t, \]
\[ W(\tau, t) = -u(\tau) \quad \text{for} \quad \tau > t. \]
If the function $W$ is integrable over $[a, b]$, then the series \( \sum_{t \in (a,b)} (u(t) + v(t)) \) is convergent, and the equality
\[
\int_a^b DW(\tau, t) = v(a) + \sum_{t \in (a,b)} (u(t) + v(t)) + u(b)
\]
holds.

**Proof.** Denote $\int_a^b DW(\tau, t) = \gamma$. Since the values $u(a), v(b)$ have no influence on the values of $W(\tau, t)$, we can assume that
\[
(29) \quad u(a) = v(b) = 0.
\]
For a given $\varepsilon > 0$ there is a gauge $\delta$ such that $|S(W, A) - \gamma| < \varepsilon$ holds for every $A \in \mathcal{A}(\delta; a, b)$. Let us define
\[
\delta'(\tau) = \min \{ \delta(\tau), b - \tau, \tau - a \} \quad \text{for} \quad \tau \in (a, b),
\]
\[
\delta'(\tau) = \min \{ \delta(\tau), b - a \} \quad \text{for} \quad \tau = a, b.
\]
Let an arbitrary set $\{t_1, t_2, \ldots, t_m\} \in I(\delta'; a, b)$ be given. Lemma 2 implies that this set includes the points $a, b$. We can assume that
\[
a = t_1 < t_2 < \ldots < t_m = b.
\]
Define $\alpha_0 = a$, $\alpha_m = b$; for every $i = 2, 3, \ldots, m - 1$ it follows from (7) that there exists a point $\alpha_i \in (t_i, t_{i+1}) \cap (t_{i+1} - \delta(t_{i+1}), t_i + \delta(t_i))$ similarly as in the proof of Theorem 2. Then $A = \{\alpha_0, t_1, \alpha_1, \ldots, \alpha_{m-1}, t_m, \alpha_m\} \in \mathcal{A}(\delta; a, b)$. Let us note that $\alpha_0 = t_1 < \alpha_1; \alpha_{m-1} < t_m = \alpha_m; \alpha_{i-1} < t_i < \alpha_i$ for $i = 2, \ldots, m - 1$. We have the estimate
\[
\varepsilon > |S(W, A) - \gamma| = |[W(t_1, \alpha_1) - W(t_1, t_1) +
\sum_{i=2}^{m-1} (W(t_i, \alpha_i) - W(t_i, \alpha_{i-1})) + W(t_m, t_m) - W(t_m, \alpha_{m-1})] - \gamma| =
= |[v(t_1) + \sum_{i=2}^{m-1} (v(t_i) + u(t_i)) + u(t_m)] - \gamma| =
= |\sum_{i=1}^{m} (u(t_i) + v(t_i)) - \gamma|.
\]
Consequently,
\[
\gamma = \sum_{t \in [a,b]} (u(t) + v(t)) = (u(a) + v(a)) + \sum_{t \in (a,b)} (u(t) + v(t)) +
+ (u(b) + v(b)) = v(a) + \sum_{t \in (a,b)} (u(t) + v(t)) + u(b)
\]
(we take (29) into consideration).

If we use the known properties of the integrals of functions $U$ or $U'$ as defined in Theorem 2 or Corollary 4, we can obtain several properties of the series $\sum_{t \in [a,b]} u(t)$:
Proposition 4. Let \( \alpha \in \mathbb{R} \) be given. If the series \( \sum_{t \in [a, b]} u(t) \) is convergent then the series \( \sum_{t \in [a, b]} (\alpha u(t)) \) is convergent and
\[
\sum_{t \in [a, b]} (\alpha u(t)) = \alpha \sum_{t \in [a, b]} u(t).
\]
(See [S], Th. 1.5.)

Proposition 5. If the series \( \sum_{t \in [a, b]} u(t), \sum_{t \in [a, b]} v(t) \) are convergent, then
\[
\sum_{t \in [a, b]} (u(t) + v(t)) = \sum_{t \in [a, b]} u(t) + \sum_{t \in [a, b]} v(t).
\]
(See [S], Th. 1.6.)

Proposition 6. If \( c \in (a, b) \) and the series \( \sum_{t \in [a, c]} u(t) \) and \( \sum_{t \in [c, b]} u(t) \) are convergent then
\[
\sum_{t \in [a, b]} u(t) = \sum_{t \in [a, c]} u(t) + \sum_{t \in [c, b]} u(t).
\]
(See [S], Th. 1.10.)

Proposition 7. Assume that for every \( c \in (a, b) \) the series \( \sum_{t \in [a, c]} u(t) \) is convergent and that there exists a finite limit \( \lim_{c \to b-} \sum_{t \in [a, c]} u(t) = \alpha \). Then the series \( \sum_{t \in [a, b]} u(t) \) is convergent and \( \alpha = \sum_{t \in [c, b]} u(t) \).
(See [S], Th. 1.13.)

Proposition 8. Assume that for every \( c \in (a, b) \) the series \( \sum_{t \in [c, b]} u(t) \) is convergent and that there exists a finite limit \( \lim_{c \to a+} \sum_{t \in [c, b]} u(t) = \beta \). Then the series \( \sum_{t \in [a, b]} u(t) \) is convergent and \( \beta = \sum_{t \in [a, b]} u(t) \).
(See [S], Remark 1.14.)

Proposition 9. Assume that \( \varphi : [a, b] \to [c, d] \) is a continuous strictly monotone function such that \( \varphi(a) = c, \varphi(b) = d \), or \( \varphi(a) = d, \varphi(b) = c \). If one of the series \( \sum_{t \in [c, d]} u(t), \sum_{t \in [a, b]} u(\varphi(t)) \) is convergent, then also the other is convergent and
\[
\sum_{t \in [c, d]} u(t) = \sum_{t \in [a, b]} u(\varphi(t)).
\]
(See [S], Th. 1.24.)

Theorem 5. Assume that a convergent series \( \sum_{t \in [a, b]} u(t) = u \) is given. Then there is a sequence \( \{t_n\}_{n=1}^\infty \) of pairwise different points from \([a, b]\), such that
\begin{equation}
\sum_{t \in [a, b]} u(t) = \sum_{n=1}^{\infty} u(t_n)
\end{equation}
and \( \{t \in [a, b]; u(t) \neq 0\} \subset \{t_1, t_2, t_3, \ldots\}. \)

**Proof.** Let us denote \( M = \{t \in [a, b]; u(t) \neq 0\}. \) Since the set \( M \) is at most countable, there is a sequence \( \{\sigma_n\}_{n=1}^{\infty} \subset [a, b] \) such that \( M \subset \{\sigma_1, \sigma_2, \sigma_3, \ldots\}. \) Let us denote \( C_k = \{\sigma_1, \sigma_2, \ldots, \sigma_k\} \) for every \( k \in \mathbb{N}. \) For any \( k = 1, 2, 3, \ldots \) there is a gauge \( \delta_k \) on \([a, b]\) such that
\begin{equation}
|s(B) - u| < \frac{1}{k}
\end{equation}
holds for any finite set \( B \in I(\delta_k). \)

Let us choose a set \( B_1 \in I(\delta_1). \) There is an integer \( p_1 \) such that \( B_1 \cap M \subset C_{p_1}. \) Let us define
\[ A_2(\tau) = \min \{\delta_2(\tau), \text{dist} (\tau; B_1 \cup C_{p_1} \setminus \{\tau\})\} \quad \text{for any} \quad \tau \in [a, b]. \]

Let us choose a set \( B_2 \in I(A_2); \) then \( B_2 \subset B_1 \cup C_{p_1} \) holds according to Lemma 2. Further, if the set \( B_k \) has been defined for an integer \( k, \) we can find an integer \( p_k \) such that \( B_k \cap M \subset C_{p_k}, \) and we will denote
\[ A_{k+1}(\tau) = \min \{\delta_{k+1}(\tau), A_k(\tau), \text{dist} (\tau; B_k \cup C_{p_k} \setminus \{\tau\})\} \quad \text{for any} \quad \tau \in [a, b]. \]

Then let us choose a set \( B_{k+1} \in I(A_{k+1}). \)

In this way we can obtain a sequence \( \{p_k\} \) of integers, a sequence \( \{A_k\} \) of gauges and a sequence of finite sets \( B_1 \subset B_2 \subset \ldots \subset B_k \subset B_{k+1} \subset \ldots \subset [a, b] \) such that
\begin{equation}
B_k \cap M \subset C_{p_k} \subset B_{k+1}
\end{equation}
hold for any integer \( k. \)

Let us denote the elements of \( B_1 \) by \( t_1 < t_2 < \ldots < t_m. \) If \( t_1, t_2, \ldots, t_m \) have been defined for an integer \( k, \) let us denote the elements of \( B_{k+1} \setminus B_k \) by \( t_{m_k+1} < t_{m_k+2} < \ldots < t_{m_k+1}. \) We obtain a sequence of pairwise different points \( \{t_n\}_{n=1}^{\infty} \) such that \( B_k = \{t_1, t_2, \ldots, t_{m_k}\}. \) (31) implies that
\[ \{t_1, t_2, t_3, \ldots\} = \bigcup_{k=1}^{\infty} B_k \subset \bigcup_{k=1}^{\infty} C_{p_k} = M. \]

Let us prove that \( \sum_{n=1}^{\infty} u(t_n) = u. \) For a given \( \varepsilon > 0 \) let us find an integer \( k_0 \) such that \( 1/k_0 \leq \varepsilon. \) If an arbitrary integer \( N \geq m_{k_0} \) is given, we will find such \( k \geq k_0 \) that \( m_k < N \leq m_{k+1}. \) In case that \( N = m_{k+1}, \) the set \( \{t_1, t_2, \ldots, t_N\} \) coincides with \( B_{k+1} \) which belongs to \( I(A_{k+1}); \) hence
\[ |\sum_{n=1}^{\infty} u(t_n) - u| = |s(B_{k+1}) - u| < \frac{1}{k+1} < \frac{1}{k_0} \leq \varepsilon. \]
Now assume that \( N < m_{k+1} \). Let \( t_r \) be the neighbour of \( t_N \) inside \( B_{k+1} \cap [t_N, b] \), i.e. a point from \( B_{k+1} \) satisfying \((t_N, t_r) \cap B_{k+1} = \emptyset\). Then \( t_r - t_N < \Delta_{k+1}(t_r) + \Delta_{k+1}(t_N) \) according to Definition 1. There is \( c \in (t_N, t_r) \) such that \( t_r - \Delta_{k+1}(t_r) < c < t_N + \Delta_{k+1}(t_N) \).

It is quite evident that \( \{t_1, t_2, \ldots, t_N\} \cap [a, c] \in \mathcal{I}(A_{k+1}; a, c) \), while \( \{t_1, t_2, \ldots, t_N\} \cap [c, b] = \{t_1, t_2, \ldots, t_m\} \cap [c, b] \in \mathcal{I}(A_k; c, b) \). According to Lemma 1 (ii) we can conclude that \( \{t_1, t_2, \ldots, t_N\} \in \mathcal{I}(A_k; a, b) \); consequently

\[
|\sum_{n=1}^{N} u(t_n) - u| < \frac{1}{k} \leq \frac{1}{k_0} \leq \varepsilon
\]

holds by (30).

**Proposition 10.** Assume that a convergent series of real numbers \( \sum_{n=1}^{\infty} a_n \) is given. If \( \{t_n\}_{n=1}^{\infty} \subset [a, b] \) is any increasing sequence and we define

\[
u(t) = a_n \quad \text{for} \quad t = t_n ,
\]
\[
u(t) = 0 \quad \text{for} \quad t \in [a, b] \setminus \{t_1, t_2, \ldots\} ,
\]

then the series \( \sum_{t \in [a, b]} \nu(t) \) is convergent and \( \sum_{t \in [a, b]} \nu(t) = \sum_{n=1}^{\infty} a_n \).

**Proof.** Denote \( \sum_{n=1}^{\infty} a_n = \alpha \). Since the sequence \( \{t_n\} \) is increasing in the compact interval \([a, b]\), it has a limit \( c \in (a, b) \). For any \( \varepsilon > 0 \) there is an integer \( N \) such that

\[
|\sum_{n=1}^{m} a_n - \alpha| < \varepsilon \quad \text{holds for any} \quad m \geq N .
\]

Let us define

\[
\delta(\tau) = t_1 - \tau \quad \text{for} \quad \tau \in [a, t_1] ;
\]
\[
\delta(t_1) = t_2 - t_1 ;
\]
\[
\delta(\tau) = \min \{ \tau - t_n, t_{n+1} - \tau \} \quad \text{for} \quad \tau \in (t_n, t_{n+1}) , \quad n \in N ;
\]
\[
\delta(t_n) = \min \{ t_{n+1} - t_n, t_n - t_{n-1} \} \quad \text{for} \quad n \geq 2 ;
\]
\[
\delta(c) = c - t_N ;
\]
\[
\delta(\tau) = \tau - c \quad \text{for} \quad \tau \in (c, b] .
\]

Let an arbitrary set \( B \in \mathcal{I}(\delta; a, b) \) be given. Since \( \delta(\tau) \leq |\tau - c| \) holds for any \( \tau \in [a, b] \setminus \{c\} \) and \( \delta(\tau) \leq |\tau - t_n| \) holds for any \( \tau \in [a, b] \setminus \{t_n\} \), the points \( t_n \) and \( c \) belong to \( B \).

Let us denote \( m = \max \{ n \in N ; \ t_n \in B \} \). Then \( m \geq N \). The gauge \( \delta \) is defined so that

\[
\delta(\tau) \leq \text{dist} (\tau; \{t_1, t_2, \ldots, t_m\} \setminus \{\tau\})
\]
holds for any $t \in [a, t_m]$. By Lemma 2 the set $B$ contains all points $t_1, t_2, \ldots, t_m$, consequently

$$s(B) = \sum_{n=1}^{m} u(t_n) = \sum_{n=1}^{m} \alpha_n.$$  

Since $m \geq N$, (32) yields

$$|s(B) - \alpha| = \left| \sum_{n=1}^{m} \alpha_n - \alpha \right| < \varepsilon.$$

**Theorem 6.** Let an absolutely convergent series $\sum_{n=1}^{\infty} \alpha_n$ of real numbers and a sequence of pairwise different points $\{s_n\}_{n=1}^{\infty} \subset [a, b]$ be given. Let us define $u(t) = \alpha_n$ if $t = s_n, n \in \mathbb{N}, u(t) = 0$ if $t \in [a, b] \setminus \{s_n\}_{n=1}^{\infty}$. Then the series $\sum_{t \in [a, b]} u(t)$ is convergent, the function $W: [a, b] \times [a, b] \to \mathbb{R}$ defined by

$$W(\tau, t) = u(\tau) \quad \text{if} \quad \tau < t, \quad W(\tau, t) = 0 \quad \text{if} \quad \tau \geq t$$

is integrable over $[a, b]$, and

$$\int_{a}^{b} DW(\tau, t) = \sum_{t \in [a, b]} u(t) = \sum_{n=1}^{\infty} \alpha_n.$$

**Proof.** Denote $\alpha = \sum_{n=1}^{\infty} \alpha_n$. Let $\varepsilon > 0$ be given. There is an integer $n_0$ such that

$$\sum_{n=n_0+1}^{\infty} |\alpha_n| < \varepsilon.$$  

Let us define

$$\delta(\tau) = \min \{|\tau - s_n|; n = 1, 2, \ldots, n_0\} \quad \text{for} \quad \tau \in [a, b] \setminus \{s_n\}_{n=1}^{n_0};$$

$$\delta(\tau) = \min \{|\tau - s_n|; n = 1, 2, \ldots, n_0, n \neq k\} \quad \text{for} \quad \tau = s_k,$$

$$k = 1, 2, \ldots, n_0.$$  

Let a partition $A \in \mathcal{A}(\delta; a, b)$ be given, $A = \{\alpha_0, \tau_1, \ldots, \tau_k, \alpha_k\}$. Lemma 2 implies that the set $\{s_1, s_2, \ldots, s_{n_0}\}$ is contained in the set $\{\tau_1, \tau_2, \ldots, \tau_k\}$. Moreover, for every $s_n, n = 1, 2, \ldots, n_0$ there is an integer $i$ such that $s_n = \tau_i < \alpha_i$ (if $s_n = \tau_i = \alpha_i < \tau_{i+1}$ then $s_n \in (\tau_{i+1} - \delta(\tau_{i+1}), \tau_{i+1})$ which contradicts (33)). Denote $J = \{n \in \mathbb{N}; s_n = \tau_i < \alpha_i \text{ for some } i\}$; then $J \subset \{s_1, s_2, \ldots, s_{n_0}\}$. We have the estimate

$$|S(W, A) - \alpha| = \left| \sum_{i=1}^{k} u(\tau_i) - \alpha \right| = \left| \sum_{n \in J} u(s_n) - \sum_{n=1}^{\infty} \alpha_n \right| =$$

$$= \left| \sum_{n=1}^{n_0} \alpha_n \right| \leq \sum_{n=n_0+1}^{\infty} |\alpha_n| < \varepsilon.$$  

Consequently, the function $W$ is integrable over $[a, b]$ and $\int_{a}^{b} DW(\tau, t) = \alpha$. Theorem 4 (with $u(\tau)$ and 0 instead of $v(\tau)$ and $u(t)$) implies that the series $\sum_{t \in [a, b]} u(t)$ is convergent and has the sum $\alpha$.  

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Theorem 7. Assume that functions $u, v: [a, b] \to \mathbb{R}$ satisfy $|u(t)| \leq v(t)$ for $t \in [a, b]$. If the series $\sum_{t \in [a, b]} v(t)$ is convergent, then

(i) the series $\sum_{t \in [a, b]} u(t)$ is convergent and $| \sum_{t \in [a, b]} u(t) | \leq \sum_{t \in [a, b]} v(t)$;

(ii) for every sequence of pairwise different points $\{s_n\}_{n=1}^{\infty} \subset [a, b]$ such that $\{t \in [a, b]; u(t) \neq 0\} \subset \{s_1, s_2, s_3, \ldots\}$ the equality

$$\sum_{n=1}^{\infty} u(s_n) = \sum_{t \in [a, b]} u(t)$$

holds.

Proof. (i) Let $\varepsilon > 0$ be given. By Proposition 2 there is a gauge $\delta$ on $[a, b]$ such that

$$| \sum_{n=1}^{m} \tau(t_n) - \sum_{j=1}^{k} \tau(t_j) | < \varepsilon$$

holds for every two sets $\{t_1, t_2, ..., t_m\}, \{\tau_1, \tau_2, ..., \tau_k\} \in \delta(\delta)$.

Let $B_0 = \{t_1, t_2, ..., t_m\} \in I(\delta)$ be fixed. Let us denote

$$\delta'(\tau) = \min \{\delta(\tau), \text{dist} (\tau; B_0 \setminus \{\tau\})\}$$

for any $\tau \in [a, b]$.

Then by Lemma 2 arbitrary sets $\{s_1, s_2, ..., s_k\}, \{\sigma_1, \sigma_2, ..., \sigma_l\} \in I(\delta')$ contain all points from $B_0$. We have an estimate

$$| \sum_{i=1}^{m} u(s_i) - \sum_{j=1}^{l} u(\sigma_j) | = | \sum_{i=1}^{k} u(s_i) - \sum_{j=1}^{l} u(\sigma_j) | \leq$$

$$\leq | \sum_{i=1}^{k} u(s_i) | + | \sum_{j=1}^{l} u(\sigma_j) | \leq \sum_{i=1}^{k} v(s_i) + \sum_{j=1}^{l} v(\sigma_j) =$$

$$= \sum_{i=1}^{k} v(t_i) - \sum_{n=1}^{m} v(t_n) + \sum_{j=1}^{l} v(\sigma_j) - \sum_{n=1}^{m} v(t_n) < 2\varepsilon.$$

According to Proposition 2 the series $\sum_{t \in [a, b]} u(t)$ is convergent. Since for every finite set $\{t_1, t_2, ..., t_m\} \subset [a, b]$ the inequality

$$| \sum_{n=1}^{m} u(t_n) | \leq \sum_{n=1}^{m} v(t_n)$$

holds, we conclude that

$$| \sum_{t \in [a, b]} u(t) | \leq \sum_{t \in [a, b]} v(t).$$

(ii) By Theorem 5 there is a sequence $\{t_n\}_{n=1}^{\infty} \subset [a, b]$ of pairwise different points such that

$$\{t \in [a, b]; v(t) \neq 0\} \subset \{t_1, t_2, t_3, \ldots\} \quad \text{and} \quad \sum_{t \in [a, b]} v(t) = \sum_{n=1}^{\infty} v(t_n).$$
Let an arbitrary sequence of pairwise different points \( \{s_j\}_{j=1}^\infty \subset [a, b] \) be given such that
\[
\{t \in [a, b]; \ u(t) \neq 0\} = \{s_1, s_2, s_3, \ldots\}.
\]
For a given \( \varepsilon > 0 \) there is such an integer \( N \) that
\[
| \sum_{t \in [a, b]} t(t) - \sum_{n=1}^m v(t_n)| < \varepsilon
\]
holds for any \( m \geq N \). There is such an integer \( K \) that
\[
\{s_1, s_2, \ldots, s_K\} \cap \{t_n\}_{n=1}^\infty = \{t_1, t_2, \ldots, t_N\}.
\]
Let us mention that if \( t \notin \{t_n\}_{n=1}^\infty \) then \( v(t) = 0 \). For any \( k \geq K \) we have
\[
[a, b] \setminus \{s_1, s_2, \ldots, s_k\} \cap \{t_n\}_{n=1}^\infty \subset [a, b] \setminus \{t_1, t_2, \ldots, t_N\}.
\]
Then
\[
| \sum_{t \in [a, b]} v(t) - \sum_{j=1}^k v(s_j)| = | \sum_{t \in [a, b]} v(t)| \leq \sum_{t \in [a, b]} v(t) = \sum_{t \in [a, b]} v(t) - \sum_{n=1}^N v(t_n) < \varepsilon.
\]
Consequently
\[
\sum_{t \in [a, b]} v(s_j) = \sum_{t \in [a, b]} v(t).
\]

**Definition 3.** Assume that for every \( \alpha \) from some index set \( C \) a series \( \sum_{t \in [a, b]} u^\alpha(t) \) is given. We say that the series \( \sum_{t \in [a, b]} u^\alpha(t) = u_\alpha \), \( \alpha \in C \) are equiconvergent, if for every \( \varepsilon > 0 \) there is a gauge \( \delta \) on \( [a, b] \) such that
\[
| \sum_{n=1}^m u^\alpha(t) - u_\alpha| < \varepsilon \quad \text{for every} \quad \{t_1, t_2, \ldots, t_m\} \in I(\delta) \quad \text{and} \quad \alpha \in C.
\]

**Theorem 8.** Let for every \( \alpha \in C \) a series \( \sum_{t \in [a, b]} u^\alpha(t) \) be given. Assume that there are convergent series \( \sum_{t \in [a, b]} v(t) = v \), \( \sum_{t \in [a, b]} w(t) = w \) such that \( v(t) \leq u^\alpha(t) \leq w(t) \) for every \( t \in [a, b] \), \( \alpha \in C \). Then the series \( \sum_{t \in [a, b]} u^\alpha(t) \), \( \alpha \in C \) are equiconvergent and there is a sequence \( \{t_n\}_{n=1}^\infty \) such that
\[
\{t_n\}_{n=1}^\infty \subset \{t \in [a, b]; u^\alpha(t) \neq 0 \text{ for some } \alpha \in C\} ;
\]
\[
t_n \neq t_m \text{ if } n \neq m ; \sum_{t \in [a, b]} u^\alpha(t) = \sum_{n=1}^\infty u^\alpha(t_n) \text{ for every } \alpha \in C.
\]

**Proof.** Let \( \varepsilon > 0 \) be given. Let \( \delta_0 \) be a gauge such that
\[
| \sum_{n=1}^k v(t_n) - v| < \varepsilon \quad \text{and} \quad | \sum_{n=1}^k w(t_n) - w| < \varepsilon \quad \text{for all}
\]
\( \{t_1, t_2, \ldots, t_k \} \in I(\delta_0) \).

Let \( S = \{s_1, s_2, \ldots, s_p \} \in I(\delta_0) \) be a fixed set. Let us define
\[
\delta(\tau) = \min \{\delta_0(\tau), \text{dist}(\tau; \{s_1, \ldots, s_p\} \setminus \{\tau\}) \} \quad \text{for} \quad \tau \in [a, b].
\]

An arbitrary set \( \{t_1, t_2, \ldots, t_m \} \in I(\delta) \) includes all the points \( s_1, s_2, \ldots, s_p \). Then for every \( \alpha \in C \) we have estimates
\[
\sum_{n=1}^{m} u'(t_n) - \sum_{k=1}^{p} u'(s_k) = \sum_{n=1}^{m} u'(t_n) \leq \sum_{n=1}^{m} w(t_n) =
\]
\[
= \sum_{n=1}^{m} w(t_n) - \sum_{k=1}^{p} w(s_k) = (\sum_{n=1}^{m} w(t_n) - w) + (w - \sum_{k=1}^{p} w(s_k)) < 2\varepsilon.
\]

Analogously \( \sum_{n=1}^{m} u'(t_n) - \sum_{k=1}^{p} u'(s_k) \geq (\sum_{n=1}^{m} v(t_n) - v) + (v - \sum_{k=1}^{p} v(s_k)) > -2\varepsilon. \)

Consequently
\[
(34) \left| \sum_{k=1}^{m} u'(t_n) - \sum_{n=1}^{p} u'(s_k) \right| < 2\varepsilon.
\]

Proposition 2 implies that \( \sum_{t \in [a, b]} u(t) \) is a convergent series and has a sum \( u_\alpha \). From

(34) it follows that \( \left| \sum_{n=1}^{m} u'(t_n) - u_\alpha \right| \leq 2\varepsilon \), hence the series \( \sum_{t \in [a, b]} u'(t), \alpha \in C \) are equi-

convergent.

By Theorem 5 and Corollary 1 there is a sequence \( \{t_n\}_{n=1}^{\infty} \) such that \( t_n \neq t_m \) for \( n \neq m \),

(35) \( \sum_{t \in [a, b]} v(t) = \sum_{n=1}^{\infty} v(t_n) \).

(36) \( \{t_n\}_{n=1}^{\infty} \subset \{t \in [a, b]; v(t) \neq 0\} \),

and

(37) \( \{t_n\}_{n=1}^{\infty} \subset \{t \in [a, b]; w(t) \neq 0\} \).

Let \( \alpha \in C \). Then \( u'(t) = v(t) + (u(t) - v(t)) \) where \( u'(t) - v(t) \geq 0. \) By Proposition

5 the series \( \sum_{t \in [a, b]} (u(t) - v(t)) \) is convergent. Since \( u'(t) - v(t) \geq 0 \) for \( t \in [a, b] \)

and \( u'(t) - v(t) = 0 \) for every \( t \in \{t_n\}_{n=1}^{\infty} \) according to (36), (37), Theorem 7 implies that

\[
\sum_{t \in [a, b]} (u'(t) - v(t)) = \sum_{n=1}^{\infty} (u'(t_n) - v(t_n)).
\]

Then
\[
\sum_{t \in [a, b]} u'(t) = \sum_{t \in [a, b]} v(t) + \sum_{t \in [a, b]} (u(t) - v(t)) =
\]
\[
= \sum_{n=1}^{\infty} v(t_n) + \sum_{n=1}^{\infty} (u'(t_n) - v(t_n)) = \sum_{n=1}^{\infty} u'(t_n).
\]
References


Souhrn

NEABSOLUTNĚ KONVERGENTNÍ ŘADY

DANA FRAŇKOVÁ

Nechť pro každé t z intervalu [a, b] je dáno reálné číslo u(t). Než problém sečíst všechna tato čísla u(t) v případě, že řada \( \sum_{t \in [a,b]} u(t) \) je absolutně konvergentní. Článek podává návod, jak sečíst řadu tohoto typu, která však není absolutně konvergentní. Používá se zde teorie zobecněného Perronova (neboli Kurzweilova) integrálu.

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