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Edge-Domatic Numbers of Cacti

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Summary. The edge-domatic number of a graph is the maximum number of classes of a partition of its edge set into dominating sets. This number is studied for cacti, i.e. graphs in which each edge belongs to at most one circuit.

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In [1] E. J. Cockayne and S. T. Hedetniemi have introduced the domatic number of a graph. One of its variants is the edge-domatic number of a graph; introduced in [2].

We shall consider finite undirected graphs without loops and multiple edges. Two distinct edges are called adjacent, if they have a common end vertex.

A subset $D$ of the edge set $E(G)$ of a graph $G$ is called dominating, if for each $e \in E(G) - D$ there exists an edge $f \in D$ adjacent to $e$. An edge-domatic partition of $G$ is a partition of $E(G)$, all of whose classes are dominating edge sets of $G$. The maximum number of classes of an edge-domatic partition of $G$ is called the edge-domatic number of $G$ and denoted by $ed(G)$.

It is sometimes convenient to consider edge-domatic colourings instead of edge-domatic partitions. A colouring $\mathcal{C}$ of edges of $G$ is called edge-domatic, if each edge of $G$ is adjacent to edges of all colours of $\mathcal{C}$ different from its own. The maximum number of colours of an edge-domatic colouring of $G$ is the edge-domatic number of $G$. This definition is evidently equivalent to the previous one.

In this paper we shall investigate cacti. A cactus is a connected graph which has the property that each of its edges is contained in at most one circuit.

Thus each block of a cactus is either a circuit, or a complete graph $K_2$ with two vertices. If a cactus contains only one block, it will be called trivial; otherwise it will be called non-trivial. A cactus in which all blocks are circuits will be called round.

The edge-domatic number of $G$ is evidently equal to the domatic number [1] of the line graph of $G$. Therefore it easily follows from the results in [1] that $ed(G) \geq 2$ for each graph $G$, none of whose connected components is $K_2$, and $ed(G) \leq \delta_e(G) + 1$, where $\delta_e(G)$ is the minimum degree of an edge of $G$. (The degree of an edge is
the number of edges adjacent to it.) As any circuit is isomorphic to its line graph, the
edge-domatic number of a circuit is equal to its domatic number. Thus we have the
following propositions.

Proposition 1. The edge-domatic number of $K_2$ is 1.

Proposition 2. The edge-domatic number of a circuit is 3 if and only if its length
is divisible by 3; otherwise it is 2.

Thus, in the sequel we shall study only non-trivial cacti. We shall prove a theorem
concerning round cacti. Before formulating it, we prove some lemmas.

Lemma 1. Let $G$ be a round cactus. Then $ed(G) \leq 3$.

Proof. For trivial cacti this follows from Proposition 1 and Proposition 2. Let $G$
be a non-trivial cactus. Let $C$ be a terminal block of $G$, i.e. a block containing only
one articulation of $G$. (Such a block must always exist.) The block $C$ is a circuit and
thus it contains two adjacent vertices $u, v$ which are not articulations of $G$. The
vertices $u, v$ have degree 2 and thus also the degree of the edge $uv$ is 2. Thus $\delta_e(G) \leq 2$
and, according to the above mentioned inequality, $ed(G) \leq \delta_e(G) + 1 \leq 3$. □

Now we shall define a certain property of a graph.
A graph $G$ is said to have the property $P$, if $ed(G) = 3$ and there exists an edge-
domatic colouring of $G$ with colours such that each vertex of $G$ is incident with edges
of at least two colours.

Lemma 2. Let $G$ be a non-trivial round cactus, let $C$ be its terminal block. Let $G_0$
be the union of all blocks of $G$ except $C$. Let $ed(G_0) = 3$ and let $G_0$ have the property $P$.
Then $ed(G) = 3$ and $G$ has the property $P$.

Proof. Let $\varphi_0$ be the colouring of $G_0$ satisfying the condition of the property $P$.
Let $a$ be the articulation of $G$ contained in $C$. By the assumption the vertex $a$ is
incident in $G_0$ with edges of at least two colours of $\varphi_0$; without loss of generality we
may assume that these colours are 2 and 3. Let $c$ be the length of $C$ and let the vertices
of $C$ be $u_1, \ldots, u_c$ and its edges $u_iu_{i+1}$ for $i = 1, \ldots, c - 1$ and $u_cu_1$. Let $a = u_1$.
We shall colour the edges of $C$ in such a way that each edge $u_iu_{i+1}$ for $i = 1, \ldots, c - 1$
obtains the colour congruent with $i$ modulo 3 and the edge $u_cu_1$ obtains the
colour congruent with $c$ modulo 3. This colouring together with $\varphi_0$ gives
a colouring $\varphi$ of $G$ with the property that each vertex of $G$ is incident with edges of
at least two colours of $\varphi$. It remains to prove that $\varphi$ is edge-domatic. As $\varphi_0$ is an
edge-domatic colouring of $G_0$, any edge of $G_0$ is adjacent to edges of all colours
different from its own. The edge $u_1u_2$ has this property, too, because its colour is 1
and it is adjacent to edges of $G_0$ of the colours 2 and 3 which are incident to $a = u_1$.
The edge $u_cu_1$ is adjacent also to these two edges of $G_0$ and moreover to $u_1u_2$ of the
colour 1. If $2 \leq i \leq c - 2$, then the edge $u_iu_{i+1}$ has the colour congruent with $i$.
modulo 3 and is adjacent to the edge $u_{i-1}u_{i}$ of the colour congruent with $i - 1$
and to the edge $u_{i+1}u_{i+2}$ of the colour congruent with $i + 1$ modulo 3. This proves
the assertion. \hfill \Box

**Lemma 3.** Let $G$ be a cactus consisting of two circuits $C_1, C_2$ of lengths $c_1, c_2$, respectively, let $c_1 \not\equiv 1 \pmod{3}$. Then $ed(G) = 3$ and $G$ has the property $P$.

**Proof.** Denote the vertices of $C_1$ by $u_1, \ldots, u_{c_1}$ and the vertices of $C_2$ by $v_1, \ldots, v_{c_2}$ in an analogous way as in the proof of Lemma 2. Let the articulation of $G$ be $a = u_1 = v_1$. We colour the edges of $C_1$ in such a way that $u_iu_{i+1}$ is coloured with the colour congruent with $i$ modulo 3 for each $i = 1, \ldots, c_1 - 1$ and $u_{c_1}u_1$ with the colour congruent with $c_1$ modulo 3. As $c_1 \not\equiv 1 \pmod{3}$, the edges incident with $a$ have different colours. Now let $\varphi$ be a cyclic permutation of $\{1, 2, 3\}$ such that $\varphi(1)$ is the colour different from the colours of the edges of $C_1$ incident with $a$. We colour the edges of $C_2$ in such a way that $v_iv_{i+1}$ for $i = 1, \ldots, c_2 - 1$ is coloured with the colour $\varphi(j)$, where $j \in \{1, 2, 3\}, j \equiv i \pmod{3}$, and $v_{c_2}v_1$ with the colour $\varphi(j)$, where $j \in \{1, 2, 3\}, j \equiv c_2 \pmod{3}$. Analogously as in the proof of Lemma 2 we prove that this colouring is edge-domatic and satisfies the condition of the property $P$. \hfill \Box

**Lemma 4.** Let $G$ be a cactus consisting of two circuits of lengths congruent with 1 modulo 3. Then $ed(G) = 2$.

**Proof.** Suppose $ed(G) = 3$. Denote the circuits and their vertices in the same way as in the proof of Lemma 3. Without loss of generality let $u_1u_2$ be coloured with 1. Then $u_2u_3$, having the degree 2, must have a colour other than 1; without loss of generality let it be 2. Then the colouring of all edges $u_iu_{i+1}$ for $i = 1, \ldots, c_1 - 1$ is uniquely determined; each edge $u_iu_{i+1}$ must have the colour congruent with $i$ modulo 3. The edge $u_{c_1}u_1$ must have the colour 1. Thus both the edges of $C_1$ incident with $a$ have the colour 1. Analogously the edges of $C_2$ must be coloured in such a way that both edges incident with $a$ have the same colour. If this colour is 2 (or 3), then $u_1u_2$ (or $u_1u_{c_1}$) is not adjacent to an edge of the colour 3 (or 2, respectively). If this colour is 1, then $u_1u_2$ is not adjacent to an edge of the colour 3, either, and $u_1u_{c_1}$ is not adjacent to an edge of the colour 2. This is a contradiction and therefore $ed(G) = 2$. \hfill \Box

**Lemma 5.** Let $G$ be a round cactus with three blocks. Then $ed(G) = 3$ and $G$ has the property $P$.

**Proof.** Let $C_1, C_2, C_3$ be the blocks of $G$; they are circuits. If some of them has the length not congruent to 1 modulo 3, then the assertion follows from Lemma 3 and Lemma 2. Thus suppose that the lengths $c_1, c_2, c_3$ of $C_1, C_2, C_3$ are all congruent with 1 modulo 3. The graph $G$ can have either one or two articulations. Consider
the first case. Let \( \varphi_1 \) be the identity permutation of \( \{1, 2, 3\} \), let \( \varphi_2, \varphi_3 \) be the cyclic permutations of \( \{1, 2, 3\} \) such that \( \varphi_2(1) = 2, \varphi_3(1) = 3 \). Let the vertices of \( C_j \) for \( j = 1, 2, 3 \) be \( u^{(j)}_i, \ldots, u^{(j)}_{c_j} \), and let the edges be \( u^{(j)}_iu^{(j)}_{i+1} \) for \( i = 1, \ldots, c_j - 1 \) and \( u^{(j)}_{c_j}u^{(j)}_1 \). Let the articulation of \( G \) be \( a = u^{(1)}_1 = u^{(2)}_1 = u^{(3)}_1 \). We colour any edge \( u^{(j)}_iu^{(j)}_{i+1} \) with the colour congruent with \( \varphi_j(i) \) modulo 3 and any edge \( u^{(j)}_{c_j}u^{(j)}_1 \) by \( j \). The reader may verify that \( G \) has the property \( P \). Now consider the second case. The vertices and edges of \( C_1 \) and \( C_3 \) will be the same as in the preceding case. The articulations will be \( a_1 = u^{(1)}_1 \) and \( a_2 = u^{(3)}_1 \). Both \( a_1, a_2 \) will be contained in \( C_2 \). Then \( C_2 \) is the union of two edge-disjoint paths \( P_1, P_2 \) connecting \( a_1 \) with \( a_2 \). Let \( p_1, p_2 \) be their lengths. We have \( p_1 + p_2 \equiv c_2 \equiv 1 \mod 3 \); therefore (without loss of generality) either \( p_1 \equiv 1 \mod 3 \) and \( p_2 \equiv 0 \mod 3 \), or \( p_1 \equiv p_2 \equiv 2 \mod 3 \). Let the vertices of \( P_1 \) (or \( P_2 \)) be \( v_0, \ldots, v_{p_1} \) (or \( w_0, \ldots, w_{p_2} \)) and let the edges be \( v_iv_{i+1} \) (or \( w_iw_{i+1} \)) for \( i = 0, \ldots, p_1 - 1 \) (or \( i = 0, \ldots, p_2 - 1 \), respectively). The notation will be chosen so that \( v_i = w_i = a_1 \), \( v_{p_1} = w_{p_2} = a_2 \). If \( p_1 \equiv 1 \mod 3 \) and \( p_2 \equiv 0 \mod 3 \), we colour each edge \( v_iv_{i+1} \) with the colour congruent with \( \varphi_3(i) \) modulo 3 and each edge \( w_iw_{i+1} \) with the colour congruent with \( i \) modulo 3. Then the edges of \( C_2 \) incident with \( a_1 \) have the colours 2 and 3 and the edges of \( C_2 \) incident with \( a_2 \) have the colours 1 and 2. Now we colour the edges of \( C_1 \) and \( C_3 \) in the same way as in the preceding case. The graph \( G \) has the property \( P \), as the reader may verify himself. If \( p_1 \equiv p_2 \equiv 2 \mod 3 \), then we colour the edges of \( C_2 \) in the same way. The edges of \( C_2 \) incident with \( a_1 \) have again the colours 2 and 3, and the edges of \( C_2 \) incident with \( a_2 \) have the colours 1 and 3. The edges of \( C_1 \) will be coloured as in the preceding case and the edges of \( C_3 \) in such a way as the edges of \( C_2 \) in the case of the articulation. Again \( G \) has the property \( P \). \( \square \)

Now we can prove a theorem.

**Theorem.** Let \( G \) be a non-trivial round cactus. Then \( ed(G) = 2 \) if and only if \( G \) consists of two circuits of lengths congruent with 1 modulo 3; otherwise \( ed(G) = 3 \).

**Proof.** According to Lemma 1 we have \( ed(G) \leq 3 \). If \( G \) consists of two circuits of lengths congruent with 1 modulo 3, then \( ed(G) = 2 \), according to Lemma 4. Otherwise \( G \) contains a subcactus \( G_0 \) consisting either of two circuits, at least one of which has a length non-congruent with 1 modulo 3, or of three circuits. Then from Lemma 3 or Lemma 5 by using iteratively Lemma 2 we obtain the assertion. \( \square \)

For cacti which are not round the theorem does not hold. For trees (which are a particular case of cacti) it was proved in \([2]\) that \( ed(G) = \delta_1(G) + 1 \).

**References**


Souhrn

HRANOVĚ DOMATICKÁ ČÍSLA KAKTUSŮ

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Hranově domatické číslo grafu je maximální počet tříd rozkladu množiny jeho hran na dominantní množiny. Toto číslo je v článku studováno pro kaktusy, tj. grafy, v nichž každá hrana patří do nejvýše jednoho cyklu.

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