

Bohdan Zelinka

Total edge-domatic number of a graph

Mathematica Bohemica, Vol. 116 (1991), No. 1, 96–100

Persistent URL: <http://dml.cz/dmlcz/126193>

Terms of use:

© Institute of Mathematics AS CR, 1991

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://dml.cz>

TOTAL EDGE-DOMATIC NUMBER OF A GRAPH

BOHDAN ZELINKA, Liberec

(Received March 6, 1989)

Summary. The total edge-domatic number of a graph is introduced as an edge analogue of the total domatic number. Its values are studied for some special classes of graphs. The concept of totally edge-domatically full graph is introduced and investigated.

Keywords: total dominating edge set, total edge-domatic number, totally edge-domatically full graph.

AMS classification: 05C35, 05C70.

We consider finite undirected graphs without loops and multiple edges. The domatic number of a graph was introduced by E. J. Cockayne and S. T. Hedetniemi in [1], the total domatic number by the same authors together with R. M. Dawes in [2].

A subset D of the vertex set $V(G)$ of a graph G is called dominating (or total dominating), if for each vertex $x \in V(G) - D$ (or each $x \in V(G)$, respectively) there exists a vertex $y \in D$ adjacent to x . A partition of $V(G)$, all of whose classes are dominating (or total dominating) sets in G , is called a domatic (or total domatic, respectively) partition of G . The maximum number of classes of a domatic (or total domatic) partition of G is called the domatic (or total domatic, respectively) number of G . The domatic number of a graph G is denoted by $d(G)$, the total domatic number by $d_t(G)$.

Analogous concepts can be defined for edges instead of vertices. Two edges will be called adjacent, if they have one end vertex in common. The number of edges which are adjacent to an edge e in a graph G will be called the degree of e in G . The graph in which the degree of each edge is equal to the same number r will be called an edge-regular graph of degree r .

A subset D of the edge set $E(G)$ of a graph G is called a dominating (or total dominating) edge set, if for each edge $e \in E(G) - D$ (or each $e \in E(G)$, respectively) there exists an edge $f \in D$ adjacent to e . A partition of $E(G)$, all of whose classes are dominating (or total dominating) edge sets, is called an edge-domatic (or total edge-domatic, respectively) partition of G . The maximum number of classes of an edge-domatic (or total edge-domatic) partition of G is called the edge-domatic (or total edge-domatic, respectively) number of G . The edge-domatic number of G is denoted by $ed(G)$, the total edge-domatic number by $ed_t(G)$. The edge-domatic number of a graph was investigated in [3].

Note that the edge-domotic number is well-defined for any graph with at least one edge, while the total edge-domotic number only for graphs having no connected component with exactly one edge. If G contains at least one edge, then the whole set $E(G)$ is a dominating edge set in G and there exists at least one edge-domotic partition of G , namely $|E(G)|$. But if a graph G contains a connected component with exactly one edge e , then no subset of $E(G)$ is total dominating, because there exists no edge adjacent to e .

The following inequality is analogous to an inequality for the domotic number of a graph proved by E. J. Cockayne and S. T. Hedetniemi.

Proposition. *Let G be a finite graph, let $\delta_e(G)$ be the minimum degree of an edge in G . Then*

$$ed_t(G) \leq \delta_e(G).$$

Proof. Consider a total edge-domotic partition \mathcal{D} of G with $ed_t(G)$ classes. Any edge of G must be adjacent to edges from all classes of \mathcal{D} , i.e. to at least $ed_t(G)$ edges, which implies the assertion. \square

If the equality $ed_t(G) = \delta_e(G)$ holds, the graph G is called totally edge-domotically full.

We shall now look for the values of $ed_t(G)$ for certain special graphs.

Theorem 1. *Let C_n be the circuit of length n . If n is divisible by 4, then $ed_t(C_n) = 2$; otherwise $ed_t(C_n) = 1$.*

Proof. Evidently the total edge-domotic number of a graph is equal to the total domotic number of its line graph. Every circuit is isomorphic to its own line graph and therefore its total edge-domotic number is equal to its total domotic number. The assertion follows from the corresponding assertion on the total domotic number in [2]. \square

Theorem 2. *Let $K_{m,n}$ be the complete bipartite graph with the bipartition classes of cardinalities m, n . Then*

$$\begin{aligned} ed_t(K_{m,n}) &= \lceil n/2 \rceil \quad \text{for } m = 1, \quad n \geq 2, \\ ed_t(K_{m,n}) &= \max(m, n) \quad \text{for } m \geq 2, \quad n \geq 2. \end{aligned}$$

Proof. Let A, B be the bipartition classes of $K_{m,n}$, let $|A| = m, |B| = n$. If $m = 1, n \geq 2$, then $K_{m,n}$ is a star with n edges. Any subset of the edge set of a star which contains at least two edges is total dominating. Hence an edge-domotic partition of $K_{m,n}$ with the maximum number of classes consists of classes of cardinality 2, except at most one class of cardinality 3; the number of classes is $\lceil n/2 \rceil$. Now suppose $m \geq 2, n \geq 2$ and let D be a subset of $E(K_{m,n})$. If each vertex of A is incident with an edge of D , then each edge of $K_{m,n}$, being incident with a vertex of A , is

adjacent to an edge of D ; the set D is dominating. Analogously, if each vertex of B is incident with an edge of D , then the set D is dominating. If there exist $a \in A$ and $b \in B$ such that none of these vertices is an end vertex of an edge of D , then the edge joining a and b is adjacent to no edge of D and D is not dominating. Thus if D is dominating edge set, we must have $|D| \geq \min(m, n)$. As $K_{m,n}$ has mn edges, we have $ed_r(G) \leq ed(G) \leq mn/\min(m, n) = \max(m, n)$. Without loss of generality suppose that $m \leq n$. For each $x \in B$ let $D(x)$ be the set of all edges joining x with vertices of A . Evidently the sets $D(x)$ for $x \in B$ form a total edge-domestic partition of $K_{m,n}$ with $|B| = n = \max(m, n)$ classes and $ed_r(K_{m,n}) = \max(m, n)$. \square

The following theorem will concern totally edge-domestically full graphs.

Theorem 3. *Let G be a finite connected edge-regular graph of degree r . Then G is totally edge-domestically full if and only if it is obtained from a regular graph G' of degree r having a decomposition into r linear factors by inserting one vertex onto each edge.*

Proof. If $r = 2$, then the assertion follows from Theorem 1. Thus suppose $r \geq 3$. Suppose that G is totally edge-domestically full. Then there exists a total edge-domestic partition $\mathcal{D} = \{D_1, \dots, D_r\}$ of G . Each edge e of G is adjacent to exactly one edge from each class of \mathcal{D} . Let the end vertices of e be u and v . Let i be such a number that $e \in D_i$. The edge e is adjacent to exactly one edge $f \in D_i$; without loss of generality let u be the common end vertex of e and f . Suppose that there exists an edge e' incident with u and different from e and f . Then e' is adjacent to at least two edges of D_i and thus to at most $r - 2$ edges of other classes of \mathcal{D} ; this implies that there exists a class of \mathcal{D} such that e' is adjacent to no edge from it, which is a contradiction. Hence u has degree 2 in G . All edges which are adjacent to e and different from f are incident with v and thus v has degree r . As e was chosen arbitrarily, we have proved that each edge of G joins a vertex of degree 2 with a vertex of degree r . Let V' be the set of all vertices of degree r in G . Let G' be the graph whose vertex set is V' and in which two vertices x, y are adjacent if and only if there exists a vertex z of degree 2 in G adjacent to both x and y . The graph G' is evidently regular of degree r . In G any two edges adjacent to the same vertex of degree 2 belong to the same class of \mathcal{D} . Let $\mathcal{F} = \{F_1, \dots, F_r\}$ be the partition of $E(G')$ such that an edge of G' belongs to F_i if and only if the two edges of the path of length 2 connecting its end vertices in G belong to D_i (for all $i \in \{1, \dots, r\}$). As each vertex of V' is incident in G with exactly one edge from each class of \mathcal{D} , it is incident in G' with exactly one edge from each class of \mathcal{F} ; the edges of any class of F form a linear factor of G' . The graph G is obtained from G' by inserting one vertex onto each edge.

Now suppose that G has the described structure. As G' has a decomposition into r linear factors, there exists a partition $\mathcal{F} = \{F_1, \dots, F_r\}$ of $E(G')$ such that each class of \mathcal{F} is the edge set of one of these factors. Now we may define the partition $\mathcal{D} = \{D_1, \dots, D_r\}$ of G in such a way that if an edge of G' belongs to F_i , then both

edges of G obtained from it by inserting one vertex belong to D_i (for all $i \in \{1, \dots, r\}$). The partition D is evidently a total edge-domatic partition of G and G is totally edge-domatically full. \square

A wheel W_n , where n is an integer, $n \geq 3$, is the graph obtained from a circuit C_n of length n by adding a new vertex and joining it by edges with all vertices of C_n .

Theorem 4. *The total edge-domatic number of every wheel is 3.*

Proof. Let the vertices of C_n be v_1, \dots, v_n and let its edges be $v_i v_{i+1}$, where the sum $i + 1$ is taken modulo n , for all $i \in \{1, \dots, n\}$. Let the added vertex of W_n be w . We shall construct a total edge-domatic partition $\{D_1, D_2, D_3\}$ of W_n . If $n \equiv 0 \pmod{3}$, then each D_j for $j \in \{1, 2, 3\}$ consists of all edges $v_i v_{i+1}$ and $v_i w$ for $i \equiv j \pmod{3}$. If $n \equiv 1 \pmod{3}$, then each D_j for $j \in \{1, 2, 3\}$ contains all edges $v_i v_{i+1}$ and $v_i w$ for $i \equiv j \pmod{3}$ and $i \leq n - 3$; further, D_1 contains $v_{n-1} w$, D_2 contains $v_{n-2} v_{n-1}$ and $v_{n-1} v_n$ and D_3 contains $v_{n-2} w$, $v_n w$ and $v_n v_1$. If $n \equiv 2 \pmod{3}$, then D_j for $j \in \{1, 2, 3\}$ contains $v_i v_{i+1}$ and $v_i w$ for $i \equiv j \pmod{3}$ and $i \leq n - 2$; further, D_1 contains $v_{n-1} w$, D_2 contains $v_n w$ and D_3 contains $v_{n-1} v_n$ and $v_n v_1$. The reader may verify himself that in all these cases $\{D_1, D_2, D_3\}$ is a total edge-domatic partition of W_n . We have $\delta_e(W_n) = 4$ and according to Theorem 3 the wheel W_n is not totally edge-domatically full; hence $ed_i(W_n) = 3$. \square

A graph G for which $d(G) = \delta(G) + 1$, where $d(G)$ is the domatic number and $\delta(G)$ is the minimum degree of a vertex of G , is called domatically full.

Theorem 5. *Let G be a regular graph of degree 3 and let G be domatically full. Then $ed_i(G) = 3$.*

Proof. As G is a regular graph of degree 3, it is edge-regular of degree 4 and $ed_i(G) \leq 4$ according to Proposition. According to Theorem 3 the graph G is not totally edge-domatically full and therefore $ed_i(G) \leq 3$. As G is domatically full, $d(G) = 4$. Let $\mathcal{D} = \{D_1, D_2, D_3, D_4\}$ be a domatic partition of G . Evidently each vertex of G is adjacent to exactly one vertex of any class of \mathcal{D} different from its own and to no vertex of its own class. For each $i \in \{1, 2, 3\}$ let D'_i be the set of all edges of G which join a vertex of D_i with a vertex of $D_{i-1} \cup D_4$, where the subscript $i - 1$ is taken modulo 3. Let e be an edge of G ; then $e \in D'_i$ for some $i \in \{1, 2, 3\}$. If e joins a vertex $v_i \in D_i$ with a vertex $v_{i-1} \in D_i$, then the vertex v_i is its common end vertex with an edge of D'_i (distinct from e) and with an edge of D'_{i+1} , and the vertex v_{i-1} is its common end vertex with two edges of D'_{i-1} . If e joins a vertex $v_i \in D_i$ with a vertex $v_4 \in D_4$, then the vertex v_i is its common end vertex with an edge of D'_i distinct from e and with an edge of D'_{i+1} , and the vertex v_4 is its common end vertex with an edge of D'_{i-1} and an edge of D'_{i+1} . Therefore e is adjacent to edges from all classes D'_1, D'_2, D'_3 and these classes form a total edge-domatic partition of G and $ed_i(G) = 3$.

Domatically full graphs which are regular of degree 3 are studied in [4].

In the end we shall prove an inequality for the total edge-domatic number of a complete graph.

Theorem 6. *Let K_n be a complete graph with n vertices. Then*

$$ed_t(K_n) \leq 3n/4.$$

Proof. Let D be a total dominating edge set of minimum cardinality in K_n . There exists at most one vertex of K_n which is not an end vertex of an edge of D ; otherwise an edge joining two such vertices would not be adjacent to an edge of D . Further, each edge of D must be adjacent to another edge of D . Therefore the subgraph of K_n formed by the edges of D and their end vertices contains all vertices of K_n except at most one, and each of its connected components has at least two edges. If $n \equiv 1 \pmod{3}$, then such a graph consists of $(n-1)/3$ vertex-disjoint paths of length 2 and $|D| = 2(n-1)/3$. In the case $n \equiv 2 \pmod{3}$ we have $|D| = 2(n-2)/3 + 1 = (2n-1)/3 > 2(n-1)/3$; in the case $n \equiv 0 \pmod{3}$ we have $|D| = 2n/3 > 2(n-1)/3$. In any case $|D| \geq 2(n-1)/3$. Thus $ed_t(K_n)$ is less than or equal to the number $n(n-1)/2$ of edges of K_n divided by $2(n-1)/3$, i.e. $ed_t(K_n) \leq 3n/4$.

References

- [1] *E. J. Cockayne, S. T. Hedetniemi: Towards a theory of domination in graphs. Networks 7 (1977), 247–261.*
- [2] *E. J. Cockayne, R. M. Dawes, S. T. Hedetniemi: Total domination in graphs. Networks 10 (1980), 211–219.*
- [3] *B. Zelinka: Edge-domatic number of a graph. Czechoslovak Math. J. 33 (1983), 107–110.*
- [4] *B. Zelinka: Domination in cubic graphs. In: Topics in Combinatorics and Graph Theory Physica-Verlag Heidelberg 1990, 727–735.*

Souhrn

TOTÁLNÍ HRANOVĚ DOMATICKÉ ČÍSLO GRAFU

BOHDAN ŽELINKA

Podmnožina D množiny hran $E(G)$ grafu G se nazývá totální dominantní, jestliže ke každé hraně $e \in E(G)$ existuje hrana $f \in D$ mající společný koncový uzel s hranou e . Totální hranově domatické číslo $ed_t(G)$ grafu G je maximální počet tříd rozkladu množiny $E(G)$ na totální dominantní množiny. Zkoumají se základní vlastnosti totálního hranově domatického čísla grafu.

Author's address: Katedra matematiky VŠST, Sokolská 8, 460 01 Liberec 1.