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NOTE ON k-CHROMATIC GRAPHS

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Summary. In this paper we characterize k-chromatic graphs without isolated vertices and connected k-chromatic graphs having a minimal number of edges.

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Graphs, considered here, are finite and simple (without loops and multiple edges), and [1, 2] are followed for terminology and notation. Let \( G = (V, E) \) be an undirected graph, with \( V \) the set of vertices and \( E \) the set of edges, such that \(|V| = n\) and \(|E| = m\). By colouring a graph we mean painting the vertices of the graph with one or more distinct colours. By properly colouring a graph, we mean painting the vertices of the graph in such a way that no two adjacent vertices are painted with the same colour. The chromatic number \( \gamma(G) \) of a graph \( G \) is the least number of distinct colours that can be used to colour the graph properly. A graph is said to be complete, if every two vertices of it are joined by an edge. We shall denote by \( K_n \) the complete graph on \( n \) vertices. If \( v \) is an arbitrary vertex of \( G \), we shall denote by \( G - v \) the subgraph obtained from \( G \) by deleting \( v \) together with its incident edges.

A set of vertices in a graph is said to be an independent set if no two vertices in it are adjacent.

For any real number \( x \), we use \( \lfloor x \rfloor \) to denote the smallest integer greater than or equal to \( x \), and \( \lceil x \rceil \) to denote the greatest integer less than or equal to \( x \).

**Theorem 1.** If \( G = (V, E) \) is a graph without isolated vertices and \( \gamma(G) = k \), then

\[
m \geq \binom{k}{2} + \left\lfloor \frac{n-k}{2} \right\rfloor.
\]
Proof. First, suppose that for each \( v \in V \) the subgraph \( G - v \) contains isolated vertices. Let \( w \) be an isolated vertex of \( G - v \), that is, \( w \) is adjacent only to \( v \) in \( G \). However, the subgraph \( G - w \) also contains isolated vertices. Thus, \((v, w) \in E\) and vertices \( v, w \) are not adjacent to other vertices in \( G \).

Repeating this reasoning, we obtain that if for each \( v \in V \) the subgraph \( G - v \) contains isolated vertices and \( G \) does not contain isolated vertices, then \( n \) is even, \( \gamma(G) = 2 \) and

\[
m = \frac{n}{2} = \binom{2}{2} + \frac{n-2}{2}.
\]

However, this number is the minimal number of edges of \( G \), since \( G \) does not contain isolated vertices and, hence, the degree \( d(v) \) of each vertex of \( G \) is at least equal to 1. Therefore, we have

\[
2m = \sum_{v \in V} d(v) \geq n,
\]

that is,

\[
m \geq \left\lfloor \frac{n}{2} \right\rfloor.
\]

Thus, in this case, the theorem is proved.

In the sequel, we shall prove the theorem by induction on \( n \). So, suppose that the theorem is true for all graphs \( G \) having \( n - 1 \) vertices and the chromatic number equal to \( k \) (\( k \leq n - 1 \)). Let \( G \) be a graph with \( n \) vertices. If \( \gamma(G) = n \), then \( G \) is isomorphic to \( K_n \), and the theorem is proved. Suppose that \( \gamma(G) = k \leq n - 1 \). Let \( v \in V \) be such that \( G - v \) does not contain isolated vertices. If such a vertex does not exist, we have seen above that the theorem is true. We have two cases.

(a) \( \gamma(G - v) = k \). Thus, by the induction hypothesis, the minimal number of edges of the subgraph \( G - v \) is equal to

\[
\binom{k}{2} + \left\lfloor \frac{n-k-1}{2} \right\rfloor.
\]

But \( v \) is not an isolated vertex. Thus, \( d(v) \geq 1 \) and, therefore, the number of edges of \( G \) is greater than or equal to

\[
\binom{k}{2} + \left\lfloor \frac{n-k-1}{2} \right\rfloor + 1 \geq \binom{k}{2} + \left\lfloor \frac{n-k}{2} \right\rfloor.
\]

We obtain equality, that is,

\[
m = \binom{k}{2} + \left\lfloor \frac{n-k}{2} \right\rfloor,
\]

only if \( n - k \) is odd, \( d(v) = 1 \) and the subgraph \( G - v \) has a minimal number of edges.
(b) $\gamma(G - v) = k - 1$. In this case there exists a partition of $V$ consisting of independent sets in the form \{v\}, $C_1$, $C_2$, ..., $C_{k-1}$, and $v$ is joined by an edge to at least one vertex from each class $C_1$, $C_2$, ..., $C_{k-1}$. Thus $d(v) \geq k - 1$, as otherwise $\gamma(G) \leq k - 1$, which contradicts the hypothesis, that is, the fact that $\gamma(G) = k$. Hence, the number of edges of $G - v$ plus $k - 1$ is a lower bound for $m$ and, by the induction hypothesis, we have
\[
m \geq \binom{k-1}{2} + \left\lfloor \frac{n-1-(k-1)}{2} \right\rfloor + k - 1 = \binom{k}{2} + \left\lfloor \frac{n-k}{2} \right\rfloor.
\]
The equality holds only if $d(v) = k - 1$ and the subgraph $G - v$ has a minimal number of edges.

Following the above proof and the cases when inequalities become equalities, we obtain, by induction, the characterization of graphs $G$ without isolated vertices, with $n$ vertices and $\gamma(G) = k$, which have a minimal number of edges, as follows.

If $n - k$ is even, the graph $G$ with a minimal number of edges is unique (up to an isomorphism) and consists of a subgraph $K_k$ and $n - k$ vertices which are pairwise joined by $\frac{1}{2}(n - k)$ edges.

If $n - k$ is odd, then there are two types of non-isomorphic graphs which have a minimal number of edges: a graph consisting of subgraph $K_k$, $n - k - 1$ vertices which are pairwise joined by $\frac{1}{2}(n - k - 1)$ edges, and another vertex which is joined by an edge to an arbitrary vertex of $K_k$. The other type consists of a subgraph $K_k$, $n - k - 1$ vertices which are pairwise joined by $\frac{1}{2}(n - k - 1)$ edges, and another vertex which is joined by an edge to a vertex which does not belong to $K_k$. Obviously, for $k = 2$, these two types of graphs coincide.

Indeed, in case (a), in order to obtain the minimal value of $m$, the number $n - k - 1$ must be even. Thus, the subgraph $G - v$ having a minimal number of edges is unique, and for $v$ we have two possibilities of joining it by an edge such that $d(v) = 1$.

In case (b), the vertex $v$ is joined to all vertices of the subgraph $K_{k-1}$ of $G - v$ which has a minimal number of edges, as otherwise we obtain $\gamma(G) < k$, contradicting the hypothesis ($\gamma(G) = k$). Hence, the minimal graph must have necessarily the above indicated structure. If $G$ has $n$ vertices, $\gamma(G) = k$ and no restriction is imposed on $G$, then the minimal number of edges is equal to $\binom{k}{2}$, since between two arbitrary classes of a partition of $V$ consisting of $k$ independent sets there exists at least one edge, as otherwise $\gamma(G) < k$, contradicting the hypothesis ($\gamma(G) = k$). It is easy to show similarly, by induction on $n$, that the single graph having this minimal number of edges consists of a subgraph $K_k$ and $n - k$ isolated vertices. Thus, we have obtained
\[
m \geq \frac{k^2 - k}{2} + \frac{n - k}{2}.
\]
or
\[ k^2 - 2k + n - 2m \leq 0, \]
wherefrom
\[ k \leq 1 + \sqrt{2m - n + 1}. \]

**Corollary.** If \( G = (V, E) \) is a graph without isolated vertices, then
\[ \gamma(G) \leq 1 + \sqrt{2m - n + 1}. \]

It is easy to see that this inequality becomes equality, for example, if \( G \) is isomorphic to \( K_n \).

According to [3], if \( G \) is connected, then
\[ \gamma(G) \leq \left\lfloor \frac{3 + \sqrt{9 + 8(m - n)}}{2} \right\rfloor. \]

Thus, if \( G \) is connected and \( \gamma(G) = k \), we have
\[ m \geq \binom{k}{2} + n - k. \]

The connected graph having this minimal number of edges is not unique. For example, it consists of a subgraph \( K_k \) and \( n - k \) vertices, each of them being joined by an edge to a vertex of \( K_k \), or it consists of a subgraph \( K_k \) and a path with \( n - k \) vertices which is joined by an edge to a vertex of \( K_k \).

For \( k = 2 \), these graphs are trees with \( n \) vertices. For \( k = 3 \), such a minimal connected graph is composed by an odd cycle with \( p \) vertices \((3 \leq p \leq n)\), such that the other \( n - p \) vertices either are joined to a vertex of the cycle or form paths joined by an edge to a vertex of the cycle. More generally, we have

**Theorem 2.** The minimal number of edges of a connected graph \( G \) with \( n \) vertices and \( \gamma(G) = k \) \((2 \leq k \leq n)\) is equal to
\[ \binom{k}{2} + n - k. \]

The graphs having this minimal number of edges are of the following kind:

1. **for** \( k = 2 \), **they** are trees with \( n \) vertices;
2. **for** \( k = 3 \), **they** consist of an odd cycle with \( p \) vertices \((3 \leq p \leq n)\) and \( n - p \) vertices such that if the vertices of the cycle are identified to a single vertex, then the resulting graph is a tree;
(3) for $k \geq 4$, they consist of a subgraph $K_k$ and $n - k$ vertices such that if the vertices of $K_k$ are identified to a single vertex, the resulting graph is a tree.

Proof. Obviously, for $k = 2$, the theorem is true. The connected graph $G$ with $n$ vertices and $\gamma(G) = 2$ which has a minimal number of edges is a tree with $n - 1$ edges, since the existence of a cycle is in contradiction with the hypothesis of minimality for the number of edges. For $k \geq 3$, we proceed by induction on $n$. Obviously, for $n = 2, 3$, the theorem is true. So, suppose that the theorem is true for all graphs with $n - 1$ vertices and let $G$ be a connected graph with $n$ vertices and $\gamma(G) = k$. For $n \geq 3$, there exists a vertex $v$ such that the subgraph $G - v$ is connected as well. We have two cases.

(a) If $\gamma(G - v) = k$, then, by the induction hypothesis, the minimal number of edges of $G - v$ is equal to

\[
\binom{k}{2} + n - k - 1,
\]

and $G - v$ is of one of the above kinds. Thus, $\binom{k}{2} + n - k$ is a lower bound for the number of edges of $G$ since, $G$ being connected, we must have $d(v) \geq 1$.

The connected graph $G$ has a minimal number of edges only if $G - v$ has a minimal number of edges and $d(v) = 1$. Hence, $G$ is of a kind specified in the theorem.

(b) If $\gamma(G - v) = k - 1$, then $G$ has a colouring consisting of classes $\{v\}, C_1, C_2, \ldots, C_{k-1}$, and $v$ is joined by an edge to at least one vertex of each independent set $C_1, C_2, \ldots, C_{k-1}$. Thus, $d(v) \geq k - 1$. Then

\[
\binom{k - 1}{2} + n - k + k - 1 = \binom{k}{2} + n - k
\]

is a lower bound for the number of edges of $G$, and $G$ has a minimal number of edges only if $d(v) = k - 1$ and the connected graph $G - v$ has a minimal number of edges.

If $k \geq 5$, then by the induction hypothesis, the minimal connected subgraph $G - v$ is of kind 3. Thus, the vertex $v$ is joined to each vertex of the subgraph $K_{k-1}$ of $G - v$, as otherwise we obtain $\gamma(G) = k - 1$, contradicting the hypothesis ($\gamma(G) = k$). Hence, in this case, $G$ is also of kind 3.

If $k = 4$, the minimal subgraph $G - v$ consists of a triangle and $n - 4$ vertices which form trees which are joined by an edge to a variable vertex of the triangle, and the vertex $v$ is joined to all vertices of the triangle due to the fact that $d(v) = 3$, since, otherwise, $\gamma(G) = 3$. In this case, the minimal graph $G$ is of kind 3.

If $k = 3$, the subgraph $G - v$ is a tree with $n - 1$ vertices and $d(v) = 2$. Thus, the graph $G$ contains a single odd cycle since $\gamma(G) = 3$, the other vertices being vertices of some trees which are joined by an edge to a variable vertex of the odd cycle. In this case, the minimal connected graph $G$ is of kind 2. \qed
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References


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