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On some conditions which imply the continuity of almost all sections $x \rightarrow f(t, x)$


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ON SOME CONDITIONS WHICH IMPLY THE CONTINUITY
OF ALMOST ALL SECTIONS x → f(t, x)

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Summary. Let I be an open interval, X a topological space and Y a metric space. Some local conditions implying continuity and quasicontinuity of almost all sections x → f(t, x) of a function f : I × X → Y are shown.

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Let \( \mathbb{R} \) be the set of reals and let \( \mu \) (resp. \( \mu^* \)) be the Lebesgue measure (resp. the outer Lebesgue measure) in \( \mathbb{R} \). The upper outer density \( d_{u,e}(A, x) \) of a set \( A \subset \mathbb{R} \) at a point \( x \in \mathbb{R} \) is defined as \( \limsup_{h \to 0} \mu^*(A \cap [x - h, x + h])/2h \). If the set \( A \) is measurable (in the Lebesgue sense) then upper outer density of \( A \) at \( x \) is called the upper density of \( A \) at \( x \) and it is denoted as \( d_u(A, x) \). The corresponding lower limits are called lower outer density and lower density of \( A \) at \( x \) and denoted by \( d_{l,e}(A, x) \) and \( d_l(A, x) \) respectively. The family of all measurable sets \( A \subset \mathbb{R} \) such that if \( x \in A \) then \( d_l(A, x) = 1 \) is a topology called the density topology \( \mathcal{T}_d [1, 5] \). Moreover, the family \( \mathcal{I}_{ae} \) of all sets \( A \in \mathcal{T}_d \) such that \( \mu(A - \text{int} A) = 0 \) is a topology [5] (\( \text{int} A \) denotes the Euclidean interior of \( A \)). Let \( I \subset \mathbb{R} \) be an open interval, let \( (X, \mathcal{T}) \) be a topological space, and let \( (Y, g) \) be a metric space. In [2] the following condition \( (a_0) \) is introduced for a function \( f : I \times X \to Y \):

\[
(a_0) \quad f \text{ satisfies } (a_0) \text{ if for every point } (t, x) \in I \times X \text{ there is a measurable set } A(t, x) \subset I \text{ such that } d_l(A(t, x), t) = 1 \text{ and the sections } f_s(x) = f(s, x), s \in A(t, x), \text{ are } \mathcal{T}-\text{equicontinuous at } x, \text{ i.e. for every } \varepsilon > 0 \text{ there is a set } U \in \mathcal{T} \text{ such that } x \in U \text{ and } f_s(U) \subset K(f_s(x), \varepsilon) = \{u \in Y; g(f(s, x), u) < \varepsilon\} \text{ for every } s \in A(t, x).
\]

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In [2] this condition is used to investigate Carathéodory's superposition $h(t) = f(t, g(t))$ and it is proved that if $X = Y$ is a separable Banach space and if $f$ satisfies the condition $(a_0)$ then almost all sections $f_t$ are $\mathcal{T}$-continuous. Moreover, if $f$ is a bounded function and all its sections $f^*(t) = f(t, x)$ are derivatives then all sections $f_t$ are continuous. In this article I examine some analogous conditions as $(a_0)$.

A function $f: I \times X \to Y$ satisfies the condition:

\begin{enumerate}
    \item[(a_1)] if for every point $(t, x) \in I \times X$ there is a measurable set $A(t, x) \subset I$ such that $d_u(A(t, x), t) > 0$ and the sections $f_s, s \in A(t, x)$, are $\mathcal{T}$-equicontinuous at $x$;
    \item[(a_2)] if for every point $(t, x)$ there is a measurable set $A(t, x) \subset I$ such that $d_u(A(t, x), t) > 0$ and the sections $f_s, s \in A(t, x)$, are $\mathcal{T}$-quasi-equicontinuous at $x$;
    \item[(a_3)] if for every point $(t, x)$ there is a measurable set $A(t, x) \subset I$ such that $d_u(A(t, x), t) > 0$ and the sections $f_s, s \in A(t, x)$, are $\mathcal{T}$-equicontinuous at $x$, i.e. for every $\varepsilon > 0$ and for every $\mathcal{T}$-open set $U \ni x$ there is a nonempty $\mathcal{T}$-open set $V \subset U$ such that $f_s(V) \subset K(f(s, x), \varepsilon)$ for every $s \in A(t, x)$;
    \item[(b_1)] if for every point $(t, x)$ there is a set $A(t, x) \subset I$ having the Baire property and of the second category at $t$ such that the sections $f_s, s \in A(t, x)$, are $\mathcal{T}$-equicontinuous at $x$;
    \item[(b_2)] if for every point $(t, x)$ there is a set $A(t, x) \subset I$ having the Baire property and of the second category at $t$ such that the sections $f_s, s \in A(t, x)$, are $\mathcal{T}$-continuous at $x$;
    \item[(b_3)] if for every point $(t, x)$ there is a set $A(t, x) \subset I$ having the Baire property and of the second category at $x$ such that the sections $f_s, s \in A(t, x)$, are $\mathcal{T}$-quasi-equicontinuous at $x$.
\end{enumerate}

**Theorem 1.** Suppose that $(X; \mathcal{T})$ is a topological space having a countable basis of open sets. If the function $f: I \times X \to Y$ satisfies the condition $(a_1)$ then there is a set $Z \subset I$ of measure zero such that all sections $f_t, t \in I - Z$, are $\mathcal{T}$-continuous.

**Proof.** Assume that the set $B = \{t \in I; f_t$ is not continuous at some point $x(t) \in X\}$ is of positive outer measure. Then there are a set $C \subset B$ of positive outer measure and a positive number $s$ such that for every $t \in C$ the oscillation $\text{osc} f_t(x(t)) = \inf \{\sup \{g(f(t, u), f(t, v)) ; u, v \in U\} ; U \in \mathcal{T}, x(t) \in U\} > s$. Let $U_1, \ldots, U_n, \ldots$ be an enumeration of all open sets of a basis of the topology $\mathcal{T}$ and let $C_n = \{t \in C; x(t) \in U_n\}$ and $D_n = \{t \in C_n; d_{t, e}(C_n, t) < 1\}$, $n = 1, 2, \ldots$. Evidently, $\mu(D_n) = 0$ for every $n = 1, 2, \ldots$. Let $D = C - (D_1 \cup D_2 \cup \ldots)$. Then $\mu(C - D) = 0$ and $D \subset C$ is a set of positive outer measure. Let $t \in D$ be a point such that $d_{t, e}(D, t) = 1$. Since $f$ satisfies the condition $(a_1)$, there is
a measurable set $A(t, x(t)) \subset I$ such that $d_u(A(t, x(t)), t) > 0$ and the sections $f_r, r \in A(t, x(t))$, are equicontinuous at $x(t)$. Consequently, there is an integer $n$ such that $x(t) \in U_n$ and $\text{osc}_{f_r} < \frac{1}{2}s$ on $U_n$ for every $r \in A(t, x(t))$. Since $t \in D = C - (D_1 \cup D_2 \cup \ldots) = (C - D_1) \cap (C - D_2) \cap \ldots$, we have $d_{i, \varepsilon}(\{r \in C; x(r) \in U_n\}, t) = 1$. Observe that the set $E = A(t, x(t)) \cap \{r \in C; x(r) \in U_n\} \neq \emptyset$. If $p \in E$ then $x(p) \in U_n$ and $\text{osc}_{f_p}(x_p) > s$, in a contradiction with the fact that $\text{osc}_{f_p} < \frac{1}{2}s$ on $U_n$. This completes the proof. □

**Theorem 2.** Suppose that a topological space $(X, \mathcal{T})$ has a countable basis of open sets. If the function $f : I \times X \to Y$ satisfies the condition $(a_3)$ then there is a set $Z \subset I$ of measure zero such that all sections $f_t, t \in I - Z$, are $\mathcal{T}$-quasicontinuous, i.e. for every $\varepsilon > 0$, for every $x \in X$ and for every set $U \in \mathcal{T}$ with $x \in U$ there is a nonempty set $V \subset U$ such that $V \in \mathcal{T}$ and $f_t(V) \subset K(f(t, x), \varepsilon)$ [6].

**Proof.** Let $U_1, \ldots, U_n, \ldots$ be an enumeration of all open sets of a basis in $X$. Assume that the set $B = \{t \in I; f_t$ is not $\mathcal{T}$-quasicontinuous at some point $x(t) \in X\}$ is of positive outer measure. Consequently, there are a positive number $s$ and a set $U_k$ such that the set $C = \{t \in B; x(t) \in U_k$ and $\text{osc}_{f_t} > s$ on $V \cup \{x(t)\}\}$ for every nonempty set $V \in \mathcal{T}$ such that $V \subset U\}$ is of positive outer measure. For $n = 1, 2, \ldots$, let $C_n = \{t \in C; x(t) \in U_n\}, D_n = \{t \in C; d_{i, \varepsilon}(C_n, t) < 1\}$, and $D = C - (D_1 \cup D_2 \cup \ldots)$. Evidently, $D \subset C$ is of positive outer measure. Let $t \in D$ be such that $d_{i, \varepsilon}(D, t) = 1$. Since $f$ satisfies the condition $(a_3)$ there are a measurable set $A(t, x(t))$ and a set $U_n \subset U_k$ such that $d_u(A(t, x(t)), t) = 1$ and $\text{osc}_{f_r} < \frac{1}{2}s$ on $U_n \cup \{x(t)\}$ for every $r \in A(t, x(t))$. Observe that $d_{i, \varepsilon}(C_n, t) = 1$. So, $A(t, x(t)) \cap C_n \neq \emptyset$. If $p \in A(t, x(t)) \cap C_n$ then $x(p) \in U_n \subset U_k$ and $\text{osc}_{f_p} < \frac{1}{2}s$ on $U_n$, in a contradiction with the fact that $\text{osc}_{f_p} > s$ on $V \cup \{x(p)\}$ for every nonempty set $V \in \mathcal{T}$ such that $V \subset U_k$. This contradiction completes the proof. □

**Theorem 3.** Suppose that $(X, \mathcal{T})$ is a topological space having a countable basis of open sets. If $f : I \times X \to Y$ satisfies the condition $(b_1)$ then there is a set $Z \subset I$ of the first category such that all sections $f_t, t \in I - Z$, are $\mathcal{T}$-continuous.

**Proof.** Assume that the set $B = \{t \in I; f_t$ is not continuous at some point $x(t) \in X\}$ is of the second category. Then there are a set $C \subset B$ of the second category and a positive number $s$ such that $\text{osc}_{f_t}(x(t)) > s$ for each $t \in C$. Let $U_1, \ldots, U_n, \ldots$ be an enumeration of all open sets of a basis in $(X, \mathcal{T})$ and let $C_n = \{t \in C; x(t) \in U_n\}$, and $D_n = \{t \in C_n; C_n$ is of the first category at $t\}, n = 1, 2, \ldots$. Every set $D_n, n = 1, 2, \ldots, n$ is of the first category. Put $D = C - (D_1 \cup D_2 \cup \ldots)$. Let $t \in D$ be a point. There is an open interval $J \subset I$ such that $t \in J$ and every set $K \subset J - D$ having the Baire property is of the first
category. Since \( f \) satisfies the condition \((b_1)\), there is a set \( A(t, x(t)) \subset J \) having the Baire property and of the second category at \( t \) and such that all sections \( f_r, r \in A(t, x(t)) \), are \( \mathcal{T} \)-equicontinuous at \( x(t) \). Consequently, there is an integer \( n \) such that \( x(t) \in U_n \) and for every \( r \in A(t, x(t)) \) we have \( \text{osc} f_r < \frac{1}{2}s \) on \( U_n \). Since \( t \in D = C - (D_1 \cup D_2 \cup \ldots) \), there is an open interval \( L \subset J \) such that \( t \in L \) and every set \( K \subset L - \{ r \in C; x(r) \in U_n \} \) with the Baire property is of the first category. So the set \( E = A(t, x(t)) \cap \{ r \in C \cap L; x(r) \in U_n \} \) is nonempty. If \( p \in E \) then \( x(p) \in U_n \) and \( \text{osc} f_p(x(p)) > s \), in a contradiction with the fact \( \text{osc} f_p < \frac{1}{2}s \) on \( U_n \). This contradiction finishes the proof.

**Remark 1.** The Continuum Hypothesis \( CH \) implies that there is a function \( f: \mathbb{R}^2 \to \mathbb{R} \) satisfying the conditions \((a_2), (b_2)\) (with respect to the Euclidean metric in \( \mathbb{R} = X = Y \)) and such that all its sections \( f_t \) are not quasicontinuous. Really, there is a nonmeasurable set \( D \subset \mathbb{R}^2 \) which has not the Baire property and which is such that all its sections \( D_t = \{ z \in \mathbb{R}; (t, z) \in D \} \) are singletons or contain two points. The construction of such set \( D \) is analogous to the construction of Sierpinski's set.
in [7]. Then the function \( f(t, x) = 1 \) for \((t, x) \in D\) and \( f(t, x) = 0 \) otherwise satisfies the conditions \((a_2), (b_2)\), but all its sections \( f_t \) are not quasicontinuous.

Remark 2. Observe that all sections \( f_t \) of the function \( f \) from Remark 1 are almost everywhere (with respect to the Lebesgue measure) continuous. CH implies that there exists a function \( g : \mathbb{R}^2 \to \mathbb{R} \) satisfying the conditions \((a_2), (b_2)\), but all its sections \( g_t \) are not quasicontinuous.

For every \( \alpha < \Omega \) there is a nowhere dense closed set \( A_\alpha \) of positive measure such that \( a_\alpha \) is not in \( A_\alpha \) for \( \alpha < \beta < \Omega \) and \( \Omega \) denotes the first uncountable ordinal number. For every \( \alpha < \Omega \) there is a nowhere dense closed set \( A_\alpha \) of positive measure such that \( a_\alpha \) is not in \( A_\alpha \) for \( \beta < \alpha \). Let \( g(t, x) = 1 \) for \( t = a_\alpha \) and \( x \in A_\alpha \), \( \alpha < \Omega \), and \( g(t, x) = 0 \) otherwise. Then \( g \) satisfies the conditions \((a_2), (b_2)\) and any section \( g_t \) is not quasicontinuous at a point \( x \in A_\alpha \), where \( \alpha \) is such that \( t = a_\alpha \).

Remark 3. Suppose that \( X = Y = \mathbb{R} \) and consider \( X \) with the topology \( \mathcal{T}_{ae} \) and \( Y \) with the Euclidean metric. There is a function \( f : \mathbb{R}^2 \to \mathbb{R} \) satisfying the conditions \((a_1), (b_1)\) (with respect to the topology \( \mathcal{T}_{ae} \) in \( X \)) and such that any section \( f_t \), \( t \in \mathbb{R} \), is not \( \mathcal{T}_d \)-continuous. Really, let \( C \subseteq \mathbb{R} \) be a Cantor set of measure zero and let \( g : \mathbb{R} \to C \) be an one-to-one function. Put \( f(t, x) = 1 \) if \( t \in \mathbb{R} \) and \( x = g(t) \) and \( f(t, x) = 0 \) otherwise. Since \( f / (\mathbb{R}^2 - (\mathbb{R} \times C)) = 0 \), for every \((t, x) \in \mathbb{R}^2 \) we can take the set \( \mathbb{R} - \{t\} \) as \( A(t, x) \). So, \( f \) satisfies the conditions \((a_1), (b_1)\), but any section \( f_t \), \( t \in \mathbb{R} \), is not \( \mathcal{T}_d \)-continuous at the point \( g(t) \).

In connection with Remarks 1, 2, 3 we will prove the following:

**Theorem 5.** Let \( J \subseteq \mathbb{R} \) be an open interval and let \( \mathcal{T} \) be a topology in \( J \) such that every set \( Z \in \mathcal{T} \) is measurable and if \( x \in Z \) then \( d_\mu(Z, x) > 0 \). Then for every function \( f : I \times J \to Y \) satisfying the condition \((a_1)\) there is a set \( U \subseteq I \) of measure zero such that for every \( t \in I - U \) the section \( f_t \) is almost everywhere (with respect to the Lebesgue measure) \( \mathcal{T} \)-continuous.

**Proof.** We may assume that \( I \) and \( J \) are of finite measure. Assume that Theorem 5 does not hold. Then there are a set \( B \subseteq I \) of positive outer measure and a positive number \( s \) such that for every \( t \in B \) the set \( C(t) = \{ x \in J ; \text{osc} f_t(x) > s \} \) is of positive outer measure. Observe that the set \( D = \bigcup_{t \in B} \{(t) \times C(t)\} \) is of positive outer measure in \( I \times J \). Let \( \Phi_1 \) be the family of all sets \( K \times L \) such that \( K \subseteq I \) is a measurable set of positive measure and \( L \in \mathcal{T} \) is a nonempty set such that \( \text{osc} f_t < \frac{1}{2}s \) on \( L \) for every \( t \in K \). Since \( f \) satisfies the condition \((a_1)\), the family \( \Phi_1 \) is nonempty. Let \( s_1 = \sup\{\mu_2(K \times L) ; K \times L \in \Phi_1\} \), where \( \mu_2 \) denotes the Lebesgue measure in \( \mathbb{R}^2 \). Evidently, \( 0 < s_1 \leq \mu_2(I \times J) \). Let \( K_1 \times L_1 \in \Phi_1 \) be such that \( \mu_2(K_1 \times L_1) > \frac{1}{2}s_1 \). If \( \mu_2((I \times J) - (K_1 \times L_1)) > 0 \) then we denote by
The family $\Phi_2$ is nonempty. Really, for this let $E \subset (I \times J) - (I_1 \times J_1)$ be an $F_\sigma$ set such that $\mu_2((I \times J) - (K_1 \times L_1)) = 0$ and for every $(t, x) \in E$ we have $d_1(E_t, x) = 1$, $d_2(E^x, t) = 1$ $(E^x = \{ r \in I; (r, x) \in E \})$ [3]. Let $(t, x) \in E$ be a point. Since $f$ satisfies the condition $(a_1)$, there is a measurable set $A(t, x) \subset I$ and a nonempty set $J(t, x) \in \mathcal{T}$ such that $x \in J(t, x)$, osc $\{ f(t, x) \}$ on $L_n$ is a derivative if for every $r \in A(t, x)$ and $d_n(A(t, x), t) > 0$. Observe that $\mu(J(t, x) \cap E) = 0$ and for every $(t, x) \in E$ we have $\mu((I \times J) - (K_1 \times L_1)) > 0$. In general, for $n \geq 2$, if $\mu_2((I \times J) - (K_1 \times L_1)) > 0$ we find a set $K_n \times L_n \in \Phi_1$ such that

$$\mu_2((K_n \times L_n) - \bigcup_{i < n} (K_i \times L_i)) > \frac{1}{2} s_n,$$

where $s_n = \sup \{ \mu_2((K \times L) - \bigcup_{i < n} (K_i \times L_i)); K \times L \in \Phi_1 \}$. Since $\mu_2(I \times J) < \infty$, $\lim_{n \to \infty} s_n = 0$. From this and from (i) it follows that $\mu_2((I \times J) - \bigcup_{i < n} (K_n \times L_n)) = 0$. Since $D$ is of positive outer measure, there are an integer $n$ and a point $(t, x) \in D \cap (K_n \times L_n)$. Consequently, osc $f_t < \frac{1}{2} s$ on $L_n$, in a contradiction with the fact that $x \in C(t)$ and osc $f_t(x) > s$. This contradiction finishes the proof.

Evidently, the Euclidean topology $\mathcal{T}$ in $\mathbb{R}$ and the topology $\mathcal{T}_d$ and the topology $\mathcal{T}_{ae}$ satisfy the hypothesis of Theorem 5.

**Problem 1.** Let $(J, \mathcal{T})$ be the same as in Theorem 5 and let $f: I \times J \to Y$ satisfies the condition $(b_1)$. If a set $U \subset I$ of the first category and such that for every $t \in I - U$ the section $f_t$ is almost everywhere $\mathcal{T}$-continuous?

**Theorem 6.** If $X = Y = \mathbb{R}$ and $\mathcal{T} = \mathcal{T}_d [\mathcal{T} = \mathcal{T}_{ae}]$ and a function $f: I \times \mathbb{R} \to \mathbb{R}$ satisfies the condition $(a_3) [(a_2)]$ and all its sections $f^x(t) = f(t, x)$ are measurable [have the Baire property] then $f$ is measurable [has the Baire property] as the function of two variables.

**Proof.** For the proof of this theorem see the proofs of Theorems 2 and 4 from [4].

**Remark 5.** In [2] it is proven that if $Y$ is a separable Banach space and a bounded function $f: I \times Y \to Y$ satisfies the condition $(a_0)$ and all its sections $f^x$ are derivatives then all sections $f_t$ are continuous. ($f^x$ is a derivative if for every $t \in I$, $\lim_{h \to 0} (1/h) f_{t+h}^x f(s) ds = f(t, x)$). Obviously, it is also true for locally bounded...
We shall show that there is a function \( f: \mathbb{R}^2 \to \mathbb{R} \) satisfying the condition \((a_0)\) and such that all its sections \( f^x \) are derivatives and the section \( x \mapsto f(0, x) \) is not continuous. For this, let \( a_n = 1/n, b_n = a_n - 4^{-n}, c_n = a_n + 4^{-n}, d_n = 1/n - 1/(n + 1) \) and let \( g_n (n = 1, 2, \ldots) \) be defined as follows: \( g_n(t) = 4^k \) for \( t = a_k, k > n, g_n(t) = 0 \) for \( t \geq c_n \) or \( t \in [c_{k+1}, b_k], k \geq n, \) \( g_n(0) = 1, \) \( g_n(t) = g_n(-t) \) for \( t < 0. \) Then the function \( f(t, x) = g_n(x)g_n(t) \min(|x - b_n|, |x - c_n|) \) for \( x \in [b_n, c_n], n = 1, 2, \ldots, \) and \( f(t, x) = 0 \) otherwise, satisfies required conditions.

In connection with Remark 5 we have also:

Remark 6. Let \( X = Y = \mathbb{R} \) and \( \mathcal{T} = \mathcal{T}_c. \) There is a bounded function \( f: \mathbb{R}^2 \to \mathbb{R} \) satisfying the condition \((a_1)\), having derivatives as its sections \( f^x, x \in \mathbb{R}, \) and such that its section \( x \mapsto f(0, x) \) is discontinuous. For this, let \( a_n = 1/n, b_n = \frac{1}{2}(a_{n+1} + a_n), c_n = b_n + 10^{-n}, d_n = a_n - 10^{-n} \) and let \( g_n, n = 1, 2, \ldots, \) be defined as follows: \( g_n(t) = 1 \) for \( t \in [a_{k+1}, b_k], k \geq n, \) \( g_n(t) = 0 \) for \( t \in [c_k, d_k], k \geq n, \) or \( t \geq a_1, \) \( g_n \) is linear in the intervals \([b_k, c_k]\) and \([d_k, a_k]\), \( g_n(0) = \frac{1}{2}, \) \( g_n(t) = g_n(-t) \) for \( t < 0. \) Then the function \( f(t, x) = g_n(x)g_n(t) \min(|x + 4^{-n} - a_n|, |a_n + 4^{-n} - x|) \) for \( x \in [a_n - 4^{-n}, a_n + 4^{-n}], n = 1, 2, \ldots, \) and \( f(t, x) = 0 \) otherwise, satisfies all required conditions.

Theorem 7. Let \( J \subset \mathbb{R} \) be an open interval, \( \mathcal{T} = \mathcal{T}_c \) and let \((Y, \rho)\) be a metric space. If a function \( f: I \times J \to Y \) satisfies the condition \((a_1)\) and all its sections \( f^x \) are \( \mathcal{T}_c \)-continuous then all sections \( f_t, t \in \mathbb{R}, \) are \( \mathcal{T}_c \)-continuous.

Proof. If Theorem 7 does not hold then there are \( t \in I, x \in J \) such that \( \rho(f_t(x), x) > 5s. \) Consequently, there is a sequence of points \( x_n \in J \) such that \( \lim_{n \to \infty} x_n = x \) and \( \rho(f(t, x_n), f(t, x)) > 2s \) for \( n = 1, 2, \ldots. \) Since \( f \) satisfies the condition \((a_1)\) there are a measurable set \( A(t, x) \subset I \) and an open set \( K \subset J \) such that \( d_u(A(t, x), t) > 0, \) \( x \in K \) and \( \rho(f_t, x) < \frac{1}{2}s \) on \( K \) for each \( t \in A(t, x). \) Let \( x_n \in K. \) Since the sections \( t \mapsto f(t, x_n) \) and \( t \mapsto f(t, x) \) are \( \mathcal{T}_c \)-continuous, there is a measurable set \( B \subset I \) such that \( d_1(B, t) = 1, \) \( \rho(f(r, x_n), f(t, x)) < \frac{1}{2}s, \) and \( \rho(f(r, x), f(t, x)) < \frac{1}{2}s \) for each \( r \in B. \) Evidently, \( B \cap A(t, x) \neq \emptyset. \) Let \( p \in B \cap A(t, x). \) Then \( 2s < \rho(f(t, x_n), f(t, x)) \leq \rho(f(t, x_n), f(p, x_n)) + \rho(f(p, x_n), f(p, x)) + \rho(f(p, x), f(t, x)) < \frac{1}{2}s + \frac{1}{2}s + \frac{1}{2}s = \frac{3}{2}s. \) This contradiction completes the proof. \( \square \)
References


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