Alexandr Vondra
Geometry of second-order connections and ordinary differential equations


Persistent URL: [http://dml.cz/dmlcz/126226](http://dml.cz/dmlcz/126226)

**Terms of use:**

© Institute of Mathematics AS CR, 1995

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*
GEOMETRY OF SECOND-ORDER CONNECTIONS AND
ORDINARY DIFFERENTIAL EQUATIONS

ALEXANDR VONDRA, Brno

(Received December 22, 1993)

Summary. The geometry of second-order systems of ordinary differential equations represented by 2-connections on the trivial bundle \( \pi_1 : \mathbb{R} \times M \rightarrow \mathbb{R} \) is studied. The formalism used, being completely utilizable within the framework of more general situations (partial equations), turns out to be of interest in confrontation with a traditional approach (semisprays), moreover, it amounts to certain new ideas and results. The paper is aimed at discussion on the interrelations between all types of connections having to do with integral sections (geodesics), integrals and symmetries of the equations studied.

Keywords: connection, semispray, differential equation, integral, symmetry

AMS classification: 34A26, 53C05, 70H35

1. INTRODUCTION

The goal of the paper is to direct the attention to some less traditional aspects and points of view applicable to the geometry of first and second-order systems of ordinary differential equations represented by connections on the tangent bundle \( T_M : TM \rightarrow M \) in the autonomous situation, and mainly on the trivial bundle \( \pi_1 : \mathbb{R} \times M \rightarrow \mathbb{R} \) in the time-dependent case. The equations for integral sections of connections (i.e. the equations solved with respect to the highest derivatives or the Pfaffian systems of 'horizontal form') do not represent the most general situation, nevertheless, their role e.g. in both the autonomous and the nonautonomous mechanics is well-known. Consequently, the investigation of generators of such equations, called semisprays (or second-order differential equation fields) is very extensive (cf. [6] and references therein). The methods and ideas used are closely related to particular properties of tangent bundles and the underlying canonical structures and morphisms. It appears

Supported by the GA CR grant No. 201/93/2245.
that the absence of such tools in the general situation on fibered manifolds calls for
the application of a more general approach, the most characteristic feature of which
is the change

\[ \text{semispray} \mapsto \text{2-connection} = \text{semispray distribution} \]

[25], [27]. This change causes the assistance of deep theories acting on fibered mani-
folds - the theory of differential equations (e.g. [22], [26] and references therein) and
the theory of connections (e.g. [14], [19], [20], [25] and references therein). Both the-
ories integrated additionally by the theory of natural operations (e.g. [14], [15] and
references therein) admit a transparent description of the geometry of the systems
studied, even for the case of partial differential equations or in higher-order theories;
we refer to [8], [9], [17], [28], [30], [31] for details.

The paper represents an attempt at the illustration of this approach in the partic-
ular situation of the trivial bundle \( pr_1 : \mathbb{R} \times M \to \mathbb{R} \), where the developed formalism
meets canonical tangent and related structures, hence it can stand for certain dif-
ferent points of view to the traditional results. Moreover, some new results and
motivations appear, as well, whose possible areas of application could be for exam-
ple the theory of variational equations (e.g. [11], [16], [18]) and the formalism of
differential forms along a map [24]. In addition, the role of connections on \( pr_2 \circ \pi_{1,0} : \mathbb{R} \times TM \to M \), studied e.g. in [1], appears to be of interest and a discussion of this
topic will be presented in a forthcoming paper.

All notions and results of the paper are presented in the form with the ambition for
the work to be more or less self-contained; by \( \mathcal{F}(X) \) we mean the module of smooth
functions on \( X \), all manifolds and maps are smooth and the standard summation
convention is used.

2. Connections on \( \tau_M : TM \to M \)

Let \( M \) be an \( n \)-dimensional manifold, \( \tau_M : TM \to M \) its tangent bundle. Denote
by \( (x^i) \) or \( (x^i, \dot{x}^i) \) local coordinates on \( M \) or the induced fibered coordinates on \( TM \),
respectively. The corresponding fibered coordinates on the first jet prolongation
\( J^1\tau_M \) of \( \tau_M \) which is the set of 1-jets of local vector fields on \( M \) will be \( (x^i, \dot{x}^i, \ddot{x}^i) \),
i.e. for \( v = \psi \partial \psi + \dot{v} \), \( \ddot{v} = \ddot{x} \psi + \dot{v} \), we have \( \ddot{x}^i \psi + \dot{v} \). Recall that \( \tau_M \)
\( J^1\tau_M \to M \) is again a vector bundle while according to the general theory of fibered
manifolds, \( (\tau_M)_{1,0} : J^1\tau_M \to TM \) is an affine bundle modelled on the vector bundle

\[ V_{\tau_M}TM \otimes T^*M \to TM. \]
where \( V_{\omega}TM = \text{span}\{\partial/\partial x^i\} \) is a subbundle of \( \tau_M \)-vertical vectors from \( TTM \) (note that \( T^*M \) should be considered pulled-back by \( \tau_M \)). The sections of (2.1) are called soldering forms on \( \tau_M \); the local expression of any such vertical tangent valued 1-form (or equivalently a \((1,1)\)-tensor field on \( M \) or an endomorphism on \( TTM \)) is

\[
\varphi = \varphi^i \frac{\partial}{\partial x^i} \otimes dx^j.
\]

There is a canonical basic soldering form

\[
J = \delta^i_j \frac{\partial}{\partial x^i} \otimes dx^j
\]

on \( \tau_M \), called an almost tangent structure on \( TM \), which finds wide application in the tangent bundle geometry realizing e.g. the vertical lift of vectors from \( TM \) to \( TTM \).

A (generally nonlinear) connection on \( \tau_M \) is a section \( \Lambda: TM \to J^1\tau_M \) of \((\tau_M)_{1,0}\). Local equations of \( \Lambda \) are \( \bar{\xi}^i \circ \Lambda = \Lambda^i_j(x^k, \bar{x}^h) \), where \( \Lambda^i_j \) are the components of \( \Lambda \) transformed like the coordinates \( \bar{x}^i \):

\[
\bar{\xi}^j = \Lambda^j_i \frac{\partial}{\partial x^i} \otimes dx^j - \frac{\partial x^l}{\partial \bar{x}^i} \frac{\partial}{\partial \bar{x}^j} \frac{\partial x^k}{\partial \bar{x}^h} \delta^k_l.
\]

It is easy to see that \( \Lambda \) can be identified in particular with the tangent valued 1-form \( h_\Lambda: TM \to TTM \otimes T^*M \), called a horizontal form of \( \Lambda \). Locally, \( h_\Lambda = D_\Lambda \otimes dx^i \), where

\[
D_\Lambda = \frac{\partial}{\partial x^i} + \Lambda^i_j \frac{\partial}{\partial \bar{x}^j},
\]

for \( i = 1, \ldots, n \), is the \( i \)-th absolute derivative with respect to \( \Lambda \). As \( h_\Lambda \) creates a splitting of the canonical exact sequence

\[
0 \to V_{\omega}TM \to TTM \to TM \times_M TM \to 0
\]

and thus it realizes a horizontal lift of vectors, a connection \( \Lambda \) on \( \tau_M \) is identified with the decomposition \( TTM = V_{\omega}TM \oplus H_\Lambda \) with the horizontal subbundle \( H_\Lambda = \text{span}\{D_\Lambda\} \). The complementary projection \( v_\Lambda = I - h_\Lambda \) on \( TTM \) is called the vertical form of \( \Lambda \).

The structure of \((\tau_M)_{1,0}\) implies the meaning of soldering forms on \( \tau_M \) as deformations of connections on \( \tau_M \); namely, for each pair \( \Lambda_1, \Lambda_2 \) of connections, \( \varphi = h_{\Lambda_1} - h_{\Lambda_2} \) is a soldering form, and conversely, for \( \Lambda \) being a connection and \( \varphi \) a soldering form, \( h_\Lambda + \varphi \) is a horizontal form of a connection.
A connection $\Lambda$ on $\tau_M$ represents a (generally nonlinear) first-order system of partial differential equations. A global (or coordinate-free) expression is given by the submanifold $\text{Im}(\Lambda) \subset J^1\tau_M$, while the coordinate expression is

$$\frac{\partial v^i}{\partial x^j} = \Lambda^i_j(x^k, v^k(x^l))$$

for the unknown family of functions $v^k$. The integral sections of $\Lambda$ are thus (local) vector fields $v$ on $M$ satisfying $j^1v = \Lambda \circ v$ on its domain. Evidently, an equivalent condition for $v$ to be an integral section of $\Lambda$ is to be an integral mapping of the corresponding horizontal distribution $H_\Lambda$, which can be expressed by the condition $Tv \in H_\Lambda$. By

$$M \xrightarrow{j^1} J^1\tau_M \to J^1\tau_M \times_M TM \to J^1\tau_M \times_M J^1\tau_M \to V_\tau TM \otimes TM,$$

the covariant derivative $\nabla_\Lambda v$ of a vector field $v$ with respect to $\Lambda$ is defined. In view of

$$\nabla_\Lambda v = \left( \frac{\partial v^i}{\partial x^j} - \Lambda^i_j \circ v \right) \frac{\partial}{\partial x^i} \otimes dx^j,$$

the integral sections of $\Lambda$ are vector fields parallel with respect to $\Lambda$, which means $\nabla_\Lambda v = 0$. In this arrangement, the integrability of $\Lambda$ means that there is a uniquely determined maximal vector field $v$ on $M$ passing through each point of $TM$ being an integral section of $\Lambda$. Equivalent conditions are e.g. both the integrability of $H_\Lambda$ and the vanishing of the tangent valued 2-form $R_\Lambda = \frac{1}{2} [h_\Lambda, h_\Lambda]: TM \to V_\tau TM \otimes \Lambda^2 TM$ called the curvature of $\Lambda$, which locally means

$$\frac{\partial \Lambda^k_j}{\partial x^i} + \frac{\partial \Lambda^k_j}{\partial x^j} \Lambda^i_l = \frac{\partial \Lambda^k_i}{\partial x^j} + \frac{\partial \Lambda^k_j}{\partial x^i} \Lambda^i_l \quad \text{for any } i, j, k = 1, \ldots, n.$$

Let $\varphi$ be a soldering form (2.2) on $\tau_M$. The $\varphi$-torsion $T_\varphi$ is another important notion closely related to a connection $\Lambda$ on $\tau_M$, defined again to be a tangent valued 2-form

$$T_\varphi = [h_\Lambda, \varphi]: TM \to V_\tau TM \otimes \Lambda^2 TM.$$

Using a natural choice $\varphi = J$, the corresponding $J$-torsion will be briefly called a torsion $T$ of $\Lambda$. Locally,

$$T = \frac{\partial \Lambda^k_j}{\partial x^i} \frac{\partial}{\partial x^i} \otimes dx^j \wedge dx^k.$$
There is a bijective correspondence between the connections on $\tau_M$ and the endomorphisms on $T\!M$ satisfying $J\!G = J$, $G\!J = -J$, called Grifone connections. The identification is expressed by $G\!\Lambda = 2\!h_\Lambda = I$ and a distinguished soldering form $H$, called the tension of $\Lambda$, is defined by

$$H = \frac{1}{2} \mathcal{L}_G\!A,$$

where $C = \dot{x}^i \partial / \partial x^i$ is the Liouville vector field on $T\!M$ and $\mathcal{L}$ is the Lie derivative. Locally,

$$H = \left( \frac{\partial \Lambda^i_k}{\partial x^k} \dot{x}^i - \Lambda^i_j \right) \frac{\partial}{\partial x^i} \otimes dx^j.
$$

(2.3) It appears that the above situation represents a natural generalization of the classical one concerning linear connections on $M$. An additional requirement for $\Lambda$ to realize a vector bundle morphism between $T\!M$ and $(\!T\!M)_I$ over $M$ implies the linearity of $\Lambda^i_j$ in $\dot{x}^k$, i.e. $\Lambda^i_j(x^k, \dot{x}^k) = -\Lambda^i_j(x^k)\dot{x}^k$, where the functions $\Lambda^i_j$ are the classical Christoffels. By (2.3), this condition is equivalent to $H = 0$. More generally, if $H$ is basic, then evidently $\Lambda$ is affine.

The geodesics of a connection $\Lambda$ on $\tau_M$ are the curves $c: \mathbb{R} \ni J \to M$ whose tangent vector field $\dot{c}: J \to T\!M$ along $c$ is parallel with respect to $\Lambda$, which means $\dot{c} \in H_\Lambda$. Locally, this condition is represented by a (generally again nonlinear) system of second-order differential equations (ODE) for $c^i = x^i \circ c$:

$$\begin{align*}
\frac{d^2 c^i}{dt^2} &= \Lambda^i_j \left( c^k, \frac{dc^k}{dt} \right) \frac{dc^j}{dt},
\end{align*}
$$

(2.4) Clearly, if $v$ is an integral section of $\Lambda$ then each integral curve $c$ of $v$ is a geodesic of $\Lambda$. Accordingly, for $\Lambda$ being integrable (recall that under the condition $R_\Lambda = 0$, $\Lambda$ is more traditionally called flat), the second-order system (2.4) is reduced to the first-order system $dc^j / dt = v^j(c^k)$ on a certain neighbourhood of each point from $T\!M$.

For a more detailed discussion on the above concepts and for the relations to the classical theory of linear connections we refer to [14], [20], [25] and [6].
When speaking of the second-order tangent bundle, we bear in mind an embedded submanifold \( T^2M \subset TTM \) of those tangent vectors from \( TTM \) for which the bundle projections \( T^2M \) and \( \tau_M \) coincide. Local fibered coordinates on \( T^2M \) are \((x^i, \dot{x}^i, \ddot{x}^i)\), where briefly \( \ddot{x}^i = \frac{d\dot{x}^i}{dt} = \frac{d^2x^i}{dt^2} \). A semispray on \( TM \) is a (global) section \( w \) of \( \tau^2_M = \tau_M|_{T^2M} : T^2M \to TM \), i.e. a vector field on \( TM \) of the particular type

\[
(3.1) \quad w = \dot{x}^i \frac{\partial}{\partial x^i} + \dot{\omega}^i(x^j, \dot{x}^j) \frac{\partial}{\partial \dot{x}^i}.
\]

An essential property of a semispray is that of its integral curves \( c: \mathbb{R} \supset J \to TM \),

\[
(3.2) \quad c = T\tau_M \circ \dot{c}.
\]

Consistently, a curve \( c: \mathbb{R} \supset J \to M \) is called a geodesic of a semispray \( w \) if \( \dot{c} \): \( J \to TM \) is an integral curve of \( w \). Relative to (3.2), the geodesics satisfy \( T\tau_M \circ \dot{c} = \dot{c} \) and the equations for them are

\[
\frac{d^2c^i}{dt^2} = w^i \left( x^j \frac{d\dot{x}^j}{dt} \right),
\]

which corresponds to an alternative name for semisprays: second-order differential equation fields (SODE).

A semispray \( w \) on \( TM \) is called a spray if \( w = [C, w] \), which means that the components \( w^i \) of \( w \) are functions homogeneous of order two in \( \dot{x}^i \), i.e.

\[
\frac{\partial w^i}{\partial \dot{x}^j} \dot{x}^j = 2w^i.
\]

There is a uniquely determined (global) semispray \( w_A \) on \( TM \) associated to a connection \( \Lambda \) on \( \tau_M \), which is a spray in the case of \( \Lambda \) linear. Globally it can be defined as a generator of the one-dimensional distribution \( H_w = H_\Lambda \cap T^2M \), which locally gives

\[
(3.3) \quad w_A = \dot{x}^i \frac{\partial}{\partial x^i} + \Lambda^j_i \dot{x}^j \frac{\partial}{\partial \dot{x}^i},
\]

and consequently \( \Lambda \) and \( w_A \) have the same geodesics.

Moreover, the integral sections of \( \Lambda \) can be thus equivalently defined as vector fields on \( TM \) such that \( w_A \circ v = T\nu \circ v \).
Conversely, if \( w \) is a semispray on \( T\mathcal{M} \), one might ask for connections on \( \tau\mathcal{M} \) associated to \( w \). It turns out (e.g. [2], [3], [5], [10]) that

\[
h_{\Lambda} = \frac{1}{2} (I_{\tau\mathcal{M}} - \mathcal{L}_w J)
\]

is a horizontal form of a canonically determined connection \( \Lambda \) with the components

\[
\Lambda^i_j = \frac{1}{2} \frac{\partial w^i}{\partial x^j}.
\]

Obviously, \( \Lambda \) is torsion free but generally it is not associated to \( w \) except for \( w \) being a spray. If this is the case, \( \Lambda \) is linear with a zero strong torsion \( T \), which is a soldering form

\[
\mathcal{T} = i_w T - \mathcal{H},
\]

locally expressed by

\[
\mathcal{T} = \left( \Lambda^i_j - \frac{\partial \Lambda^i_j}{\partial x^k} \epsilon^k \right) \frac{\partial}{\partial x^i} \otimes dx^j,
\]

where \( \mathcal{H} \) and \( \mathcal{T} \) are the tension and the torsion of \( \Lambda \), and \( w \) is an arbitrary semispray on \( T\mathcal{M} \). It is worth mentioning here that all first-order natural operators transforming semisprays on \( T\mathcal{M} \) into connections on \( \tau\mathcal{M} \) form a one-parameter family expressed by \( h_{\Lambda} + kJ \), where \( k \in \mathbb{R} \) and \( h_{\Lambda} \) is given by (3.4) (see [4] or [14]), which corresponds to the fact that \( J \) is the only natural soldering form on \( \tau\mathcal{M} \).

Both the necessary and the sufficient form of deformations of (3.1) for the obtained connection to be associated to the given \( w \), and the role of the strong torsion in the previous considerations are described in the so-called Decomposition Theorem [6]:

for any semispray \( w \) on \( T\mathcal{M} \) and a soldering form \( \varphi \) on \( \tau\mathcal{M} \) such that

\[
2i_w \varphi = w - [C, w],
\]

there exists a unique connection \( \Lambda \) on \( \tau\mathcal{M} \) whose associated semispray is \( w \) and its strong torsion \( \mathcal{T} = 2\varphi \); this connection is given by

\[
h_{\Lambda} = \frac{1}{2} (I_{\tau\mathcal{M}} - \mathcal{L}_w J) + \varphi.
\]

For a related discussion of the material we refer to Sec. 6; it should be noticed here that the distinguished role of sprays in the above considerations can be seen e.g. from the local expression of (3.5):

\[
\varphi^i_j \epsilon^j = w^i - \frac{1}{2} \frac{\partial w^i}{\partial x^j} \epsilon^j.
\]
Notice the role played by the relationships between semisprays on $TM$ and connections on $\tau_M$ in the theory of derivations of forms (even vector valued) along $\tau_M$, (see [21] and references therein). In this situation, a semispray $w$ is combined with the prolonged objects studied and the connection (3.4) coming from $w$ is used essentially, as well.

4. First and Second-Order Connections on $pr_1 : R \times M \to R$

Let now $M$ be an $m$-dimensional manifold with local coordinates redenoted (for some technical reasons) to $(q^a)$ on $M$ and $(q^a, q^{\alpha}_1)$ on $TM$. Fibered coordinates on the total space $Y = R \times M$ of the trivial bundle $\pi := pr_1 : R \times M \to R$ are consequently $(t, q^a)$ with $(t)$ being a global canonical coordinate on $R$. Local sections of $\pi$ are evidently of the form $\gamma = (id_R, c)$, where $c$ is a curve in $M$. By $j_t^0 \gamma \to \dot{c}(0)$, where $\gamma$ is an arbitrary section of $\pi$ on a neighborhood of zero, a canonical isomorphism $J^0\pi \cong TM$ is realized. Analogously, by $j_t^1 \gamma \to (t, \dot{c}(t))$ one gets the well-known isomorphism $J^1\pi \cong R \times TM$. This implies that in contradistinction to the general situation where $\pi_{1,0} : J^1\pi \to Y$ is an affine bundle, $\pi_{1,0} \cong id_{R \times TM} : R \times TM \to R \times M$

is now a vector bundle with fibers $T_x M$ over $(t, z)$. Another useful identification

\[ J^1\pi \cong V_y Y \]

(4.1)

of $R \times TM$ with the submanifold $V_y Y \subset T(R \times M)$ of $\pi$-vertical vector fields on $R \times M$ is given by $j_t^1 \gamma \to \dot{\gamma}(t)$, and locally by $t = 1$.

In this arrangement, a (first-order) connection on $\pi$, i.e. a section $\Gamma : Y \to J^1\pi$, can be identified with a $\pi$-vertical vector field

\[ v = \Gamma^a(t, q^\alpha) \frac{\partial}{\partial q^a} \]

(4.2)

on $R \times M$, which is equivalently a vector field along $pr_2 : R \times M \to M$ called a time-dependent vector field on $M$. The one-dimensional $\pi$-horizontal distribution $H_\Gamma \subset TY$ defining the decomposition $TY = H_\Gamma \oplus V_y Y$, is thus generated by the absolute derivative $D_v$ with respect to $\Gamma$, $D_v = \partial/\partial t + v$, where $D_v = D \circ \Gamma$ with $D = \partial/\partial t + q^{\alpha}_1 \partial/\partial q^\alpha$ is the formal derivative (a vector field along $\pi_{1,0}$). The horizontal form of $\Gamma$ is a tangent valued 1-form $h_\Gamma = D_v \otimes dt$ while $v_\Gamma = I_{TY} - h_\Gamma$ is the vertical form of $\Gamma$.

The integral sections of $\Gamma$ are the sections $\gamma$ of $\pi$ satisfying $j^1 \gamma = \Gamma \circ \gamma$ on its domain. In terms of the above identifications, integral sections of $\Gamma$ are the 'graphs'
of integral curves of $v$ (4.2), i.e. $\gamma = (id_R, c)$ such that $v \circ \gamma = \dot{c}$. The corresponding first-order system of ODE reads

$$\frac{dc'}{dt} = \Gamma^v(t, c').$$

In general, the equations (4.3) are globally represented by a submanifold $\Gamma(R \times M) \subset R \times TM$; in particular, in case that $v$ does not depend on $t$, this submanifold is $R \times Im v \subset R \times TM$.

If we are interested in the structure of some other prolongations of fibrations under consideration we find that for $\pi_1: R \times TM \rightarrow R$ we have

$$J^1\pi_1 \cong R \times TTM \quad \text{and} \quad J^2\pi \cong R \times T^2M,$$

where the additional induced coordinates on the first prolongation $J^1\pi_1$ of $\pi_1$ or on the second prolongation $J^2\pi$ of $\pi$ are denoted by $q_1^\alpha, q_2^\alpha$ or $q_2^\alpha$, respectively. Moreover, evidently $T^2M \cong \mathcal{J}_{\pi}^2\pi$ and also for $\pi_1$-vertical vectors on $R \times TM$ we have $V_{\pi_1}J^1\pi \cong R \times TTM \subset T(R \times TM)$; it is important to note that $R \times TTM$ is a total space of $(id_R \times \tau_{TM})$ in the above identifications. On the other hand, by (4.1) we have $J^1(\pi \circ \gamma_{|V_{\pi_1}}) \cong J^1\pi_1 \cong R \times TTM$, however, now the presented fibration over $R \times TM$ is $(id_R \times \tau_{TM})$ and accordingly the realization of the isomorphism

$$J^1(\pi \circ \gamma_{|V_{\pi_1}}) \cong V_{\pi_1}J^1\pi$$

calls for the canonical involution $\kappa_M: TTM \rightarrow TTM$ (see [6], [14] etc.) by $(id_R, \kappa_M)$.

A second-order connection (briefly a 2-connection) on $\pi$ is in accordance with [19], [25] a section $\Gamma^{(2)}$ of $\pi_{2,1}: J^2\pi \rightarrow J^1\pi$, which means

$$\Gamma^{(2)}: R \times TM \rightarrow R \times T^2M$$

in the situation studied. Any 2-connection $\Gamma^{(2)}$ on $\pi$ is equivalently characterized by its horizontal form $h_{\Gamma^{(2)}} = D_{\Gamma^{(2)}} \otimes dt$, where the absolute derivative $D_{\Gamma^{(2)}}$ with respect to $\Gamma^{(2)}$

$$D_{\Gamma^{(2)}} = \frac{\partial}{\partial t} + q_2^\alpha \frac{\partial}{\partial q_2^\alpha} + \Gamma^{(2)}_\alpha \frac{\partial}{\partial q_2^\alpha}$$

is a semispray on $R \times TM$ where its alternative name semispray connection comes from. Again, $D_{\pi_{2,1}} = D \circ \Gamma^{(1)}$, where the formal derivative $D = \partial/\partial t + q_1^\alpha \partial/\partial q_1^\alpha + q_2^\alpha \partial/\partial q_1^\alpha$ is now a vector field along $\pi_{2,1}$. Thus, $\Gamma^{(2)}$ can be identified with a one-dimensional $\pi_1$-horizontal distribution $H_{\Gamma^{(2)}} = Im h_{\Gamma^{(2)}}$, realizing a decomposition $TJ^2\pi = H_{\Gamma^{(2)}} \oplus V_{\pi_1}J^2\pi$; recall that evidently $H_{\Gamma^{(2)}} \subset C_{\pi_{2,1}}$, where $C_{\pi_{2,1}}$ is a
canonical Cartan distribution on \( J^1 \pi \), and that the soldering forms on \( \pi_1 \) realizing the deformations of 2-connections on \( \pi \) are the sections of the vector bundle \( V_{\pi_1} J^1 \pi \otimes \pi_1^* (T^* \mathbb{R}) \to J^1 \pi \), associated to \( \pi_{2,1} \), locally expressed by

\[
\varphi^{(2)} = \frac{\partial}{\partial q^{(1)}_1} \otimes dt.
\]

Due to the previous considerations, \( D_{\pi_2} \) and thus \( \Gamma^{(2)} \) itself can be represented by the (generally time-dependent) semispray

\[
w = q^{(1)}_1 \frac{\partial}{\partial q^{(1)}_1} + \Gamma^{(2)}_1 (t, q, q^{(1)}_1) \frac{\partial}{\partial q^{(1)}_1}
\]

on \( TM \), which can be viewed as a vector field along \( pr_2 : \mathbb{R} \times TM \to TM \). Clearly, the autonomous situation (denoted here by \( w = w(q, q^{(1)}_1) \)) means that (4.5) is an ordinary semispray on \( TM \) in the sense of Sec. 3.

The integral sections of \( \Gamma^{(2)} \) are the sections \( \gamma \) of \( \pi \) such that \( j^2 \gamma = \Gamma^{(2)} \circ j^1 \gamma \), hence they are the ‘graphs’ of geodesics of \( w \) from (4.5) which means \( \gamma = (id_R, c) \) such that \( w \circ j^1 \gamma = \tilde{c} \). Consequently, the corresponding second-order system of ODE, represented by \( \Gamma^{(2)}(R \times TM) \subset \mathbb{R} \times T^2 M \), is

\[
\frac{d^2 c^*}{dt^2} = \Gamma^{(2)}_2 \left(t, c^*, \frac{dc^*}{dt}\right).
\]

Recall that both for \( \Gamma \) and for \( \Gamma^{(2)} \), the integral sections coincide with the maximal integral mappings of the corresponding horizontal distributions.

The following ideas will appear to be profitable in Sec. 7.

Let \( \Gamma : Y \to J^1 \pi \) be a connection on \( \pi \). Using the vertical functor \( V \) one gets the mapping \( VT : V_\pi Y \to V_\pi J^1 \pi \), i.e. \( VT : \mathbb{R} \times TM \to \mathbb{R} \times TT M \). With regard to (4.4)

\[
VT = (id_R, \kappa_M) \circ V \Gamma
\]

is the so-called vertical prolongation of \( \Gamma \) realizing the only connection on \( q := \pi \circ \gamma : V_\pi Y \to \mathbb{R} \) naturally generated by \( \Gamma \) [13], [14]. In fact, in view of (4.1), \( VT \) is a connection on \( \pi_1 : \mathbb{R} \times TM \to \mathbb{R} \), locally

\[
(t, q^*, q^{(1)}_1) \xrightarrow{VT} \left(t, q^*, q^{(1)}_1, \Gamma^*, \frac{\partial \Gamma^*}{\partial q^{(1)}_1} q^{(1)}_1\right).
\]

In particular, for time-independent \( v \) in (4.1) one gets \( VT = (id_R, Tv) \) and

\[
VT = (id_R, v^*),
\]

154
where \( v^c \) is the complete lift of \( v \) (see e.g. [6]).

Applying analogous ideas and isomorphisms \( J^2\pi_1 \cong R \times T^2 TM \) together with the canonical involution \( \sigma_M^{(2)} : T^2 TM \to TT^2 M \) we obtain the vertical prolongation \( \nu \Gamma^{(2)} : J^1 \pi_1 \to J^2 \pi_1 \) as a naturally determined 2-connection on \( \pi_1 \); in coordinates

\[
(t, q^a, q^\alpha_1, \tilde{q}^a, \tilde{q}^\alpha_1 \Gamma^\alpha_2, \frac{\partial \Gamma^\alpha_2}{\partial q^a} \tilde{q}^a + \frac{\partial \Gamma^\alpha_2}{\partial q^\alpha_1} q^\alpha_1)
\]

5. CONNECTIONS ON \( \pi_{1,0} : R \times TM \to R \times M \)

The first jet prolongation \( J^1 \pi_{1,0} \) of \( \pi_{1,0} \) is the manifold of 1-jets of (local) connections on \( \pi \) with fibered coordinates \( (t, q^a, q^\alpha_1, z^\alpha, z_\alpha^n) \), where

\[
z^\alpha(j_1^t \Gamma) = \frac{\partial \Gamma^\alpha}{\partial t}(y), \quad z_\alpha^n(j_1^t \Gamma) = \frac{\partial \Gamma^\alpha}{\partial q^n}(y).
\]

Since our particular concern in this paper is with the relations between autonomous and time-dependent situations, the following fact appears to be of importance. An immediate verification shows that there is a canonical inclusion

\[
R \times J^1 \tau_M \hookrightarrow J^1 \pi_{1,0}
\]

over \( J^1 \pi = R \times TM \). For any pair \( (t, j^1_t v) \in R \times J^1 \tau_M \) we have \( \sigma(t, j^1_t v) = j^1_t j_\Gamma^t \Gamma \), where \( \Gamma \) and \( v \) are identified by (4.2). Local equations for \( \sigma(R \times J^1 \tau_M) \subset J^1 \pi_{1,0} \) are thus

\[
z^\alpha = 0.
\]

A connection on \( \pi_{1,0} \) is a section \( \Xi : J^1 \pi \to J^1 \pi_{1,0} \). The horizontal form of \( \Xi \) is \( h_\Xi = D_{\Xi_0} \otimes dt + D_{\Xi_1} \otimes dq^a \), where the absolute derivatives

\[
D_{\Xi_0} = \frac{\partial}{\partial t} + \Xi^\alpha \frac{\partial}{\partial q^\alpha_1}, \quad D_{\Xi_1} = \frac{\partial}{\partial q^a} + \Xi^\alpha_1 \frac{\partial}{\partial q^\alpha_1}
\]

are the generators of the \( \pi_{1,0} \)-horizontal \((m+1)\)-dimensional distribution \( H_\Xi \) realizing a decomposition \( T J^1 \pi = H_\Xi \oplus V_{1,0} J^1 \pi \). Notice that evidently

\[
V_{1,0} J^1 \pi \cong R \times V_{1,0} TM.
\]
The integral sections of a connection $\Xi$ on $\pi_{1,0}$ are (local) connections on $\pi$ satisfying $j^1\Gamma = \Xi \circ \Gamma$, which locally means a first-order system of PDE of the form

$$\frac{\partial \Gamma^\nu}{\partial t} = \Xi^\nu(t, q^\beta, \Gamma^\beta), \quad \frac{\partial \Gamma^\nu}{\partial q^\lambda} = \Xi^\nu(t, q^\beta, \Gamma^\beta),$$

where $\Xi^\nu, \Xi^\nu_\lambda$ are the components of $\Xi$.

Following (5.2), any connection $\Lambda$ on $\mathcal{T}M$ can be considered as a connection on $\pi_{1,0}$ of the particular type

$$\Xi = (\text{id}_\mathcal{R}, \Lambda): \mathcal{R} \times \mathcal{T}M \to \mathcal{X}(\mathcal{R} \times J^1\mathcal{T}M),$$

whose components are $\Xi^\nu = 0, \Xi^\nu_\lambda = \Lambda^\nu_q (q^\beta, q^\beta_1)$. The corresponding horizontal form is now

$$h_\Xi = \frac{\partial}{\partial t} \otimes dt + \left( \frac{\partial}{\partial q^\lambda} + \Lambda^\nu_q \frac{\partial}{\partial q^\nu_1} \right) \otimes dq^\lambda = I_{\mathcal{T}Q} + h_\Lambda,$$

and the integral sections can be identified with vector fields on $M$ (cf. Sec. 2).

The deformations of connections on $\pi_{1,0}$ are soldering forms on $\pi_{1,0}$, i.e. the sections

$$\varphi: J^1\pi \to V_{\pi_{1,0}} J^1\pi \otimes \pi^*_{1,0}(T^*Y)$$

of the vector bundle associated to $(\pi_{1,0})_{\text{loc}}: J^1\pi_{1,0} \to J^1\pi$; locally

$$\varphi = \frac{\partial}{\partial q^\nu_1} \otimes (\varphi^\nu dt + \varphi^\nu_\lambda dq^\lambda).$$

By [7], all natural soldering forms on $\pi_{1,0}$ are of the form

$$k_1J + k_2C \otimes dt, \quad k_1, k_2 \in \mathcal{F}(\mathcal{R})$$

and the key-role played by

$$S = J - C \otimes dt$$

will become apparent below.

According to [20], for any soldering form $\varphi$ (5.5) the $\varphi$-torsion of the connection $\Xi$ on $\pi_{1,0}$ is defined by

$$T_\varphi = [h_\Xi, \varphi].$$

For $\varphi = S$ the corresponding $S$-torsion will be called a torsion, locally

$$T = \frac{\partial \Xi^\nu}{\partial q^\nu_1} \frac{\partial}{\partial q^\nu_1} \otimes dq^\nu \wedge dq^\lambda + \left( \Xi^\nu - \frac{\partial \Xi^\nu}{\partial q^\nu_1} q^\nu_1 - \frac{\partial \Xi^\nu_\lambda}{\partial q^\nu_1} \right) \frac{\partial}{\partial q^\nu_1} \otimes dt \wedge dq^\lambda.$$
Evidently, for \( \Xi \) given by (5.4) we can see that the torsions of \( \Xi \) and \( \Lambda \) coincide if and only if \( \Lambda \) is linear.

Let \( \zeta \) be an arbitrary semispray on \( R \times TM \). Then \( i_\zeta \tau \) is a soldering form on \( \pi_{1,0} \), locally expressed by

\[
(5.8) \quad i_\zeta \tau = \left( \frac{\partial \Xi^\sigma}{\partial q_{(1)}^i} q_{(1)}^i \right) \frac{\partial}{\partial q_{(1)}^i} \otimes dt + \left( \Xi^\sigma - \frac{\partial \Xi^\sigma}{\partial q_{(1)}^i} q_{(1)}^i \right) \frac{\partial}{\partial q_{(1)}^i} \otimes dq^\sigma.
\]

The tension of a connection \( \Sigma \) on \( \pi_{1,0} \) can be defined as a soldering form

\[
H = \mathcal{L}_{\tau} \Sigma
\]

locally expressed by

\[
(5.9) \quad H = \left( \frac{\partial \Xi^\sigma}{\partial q_{(1)}^i} q_{(1)}^i - \Xi^\sigma \right) \frac{\partial}{\partial q_{(1)}^i} \otimes dt + \left( \Xi^\sigma - \frac{\partial \Xi^\sigma}{\partial q_{(1)}^i} q_{(1)}^i \right) \frac{\partial}{\partial q_{(1)}^i} \otimes dq^\sigma.
\]

By means of \( H \) the family of linear connections on the vector bundle \( \pi_{1,0} \) can be characterized; namely, \( \Sigma \) is linear if and only if its tension vanishes, which by (5.9) means \( \Xi^\sigma = \Psi^\sigma(t,q^\alpha)q_{(1)}^\alpha \), \( \Xi^\sigma = \Psi^\sigma(t,q^\alpha)q_{(1)}^\alpha \) with \( \Psi^\sigma = \frac{\partial \Xi^\sigma}{\partial q_{(1)}^i} \) and \( \Psi^\sigma = \frac{\partial \Xi^\sigma}{\partial q_{(1)}^i} \). Globally, \( \Xi: J^1 \pi \to J^1 \pi_{1,0} \) is a linear fibered morphism between \( \pi_{1,0} \) and \( (\pi_{1,0})_1 \) over \( Y \). In particular, any linear connection \( \Lambda \) on \( TM \) defines a linear connection \( \Sigma \) on \( \pi_{1,0} \) by (5.4), i.e. \( \Psi^\sigma = \Lambda^\sigma_{1,0} \).

According to the results of [8], there is a unique natural transformation of \( J^1 \pi_{1,0} \) into \( J^1 \pi_1 \) (in fact, into \( J^2 \pi \)) over \( J^1 \pi \), which is of the form

\[
j^1_\Gamma \xrightarrow{\cong} J^1(\Gamma, id_R) \circ \Gamma(y),
\]

where \( J^1(\Gamma, id_R) \) means the first jet prolongation of the morphism \( \Gamma \) over \( R \). Equivalently,

\[
j^1_\Gamma \xrightarrow{\cong} j^2_\gamma,
\]

where \( \gamma \) is the maximal integral section of \( \Gamma \) passing through \( y = (t,z) \). In coordinates

\[
(5.10) \quad q_{(2)}^\sigma \circ \gamma = z^\sigma + z_{(1)}^\sigma.
\]

In particular, due to (5.1) the only natural transformation \( f_0^M \) of \( J^1 \tau_M \) into \( T^2 M \) over \( TM \) is determined as

\[
j^1_\psi \xrightarrow{\cong} T \circ \psi
\]
and presented here in the notation of Sections 2, 3: $\dot{x}^i \circ T_0^M = \dot{x}^j \circ p^j$. As an immediate consequence, there is a unique naturally determined 2-connection $\Gamma^{(2)} = \xi \circ \Xi$ for any connection $\Xi$ on $\pi_{1,0}$, called characteristic to $\Xi$, whose components are

$$\Gamma^{(2)} = \Xi^\alpha + \Xi^\gamma \xi^1.$$  

(5.11)

Evidently, the generator $D_{\Gamma^{(2)}}$ of the characteristic distribution $H_{\Gamma^{(2)}}$ is just the semispray associated to $\Xi$ (cf. [6]). In particular, (5.11) implies (3.3).

The relationship between $\Xi$ on $\pi_{1,0}$ and its characteristic 2-connection $\Gamma^{(2)}$ on $\pi$ is based on the essential property of the corresponding horizontal distributions which reads $H_{\Gamma^{(2)}} \subseteq H_{\Xi}$. This \textquoteleft horizontal\textquoteright considerations lead to a description of some indirect integration methods for the connections studied (we refer to [17] for full details).

First, the integral sections of $\Xi$ are foliated (and thus can be \textquoteleft glued together\textquoteright) by the integral manifolds of $H_{\Gamma^{(2)}}$, called characteristics, which are just the first jet prolongations of the integral sections of $\Gamma^{(2)}$. Secondly, if $\Gamma$ is a (local) integral section of $\Xi$ then

$$J^1(\Gamma, \text{Id}_\mathbb{R}) \circ \Gamma = \Gamma^{(3)} \circ \Gamma$$

(5.12)

on the domain of $\Gamma$; in other words, $\Gamma$ represents a (local) first-order system whose prolongation is just $\Gamma^{(2)}$. According to [17], any integral section $\Gamma$ of $\Xi$ is a \textit{field of geodesics} of the characteristic $\Gamma^{(2)}$ which means in case of $\Xi$ integrable (see Sec. 6) that each integral section of $\Gamma^{(2)}$ is locally an integral section of a certain integral section $\Gamma$ of $\Xi$. Roughly speaking, a second-order problem for geodesics of $\Gamma^{(2)}$ can be reduced to a first-order problem for integral sections of $\Xi$. In the autonomous situation (5.4) the fields of geodesics are exactly the vector fields parallel to $\Lambda$; in fact, (5.12) reads $T \circ v = \nu_\Lambda \circ v$.

The geometry of connections on $\pi_{1,0}$ is rich in structures related, the above \textquoteleft horizontal\textquoteright ideas can be supplemented by some \textquoteleft vertical\textquoteright ones. While the above considerations had to do with integral sections themselves, the following ones are closely related to the infinitesimal symmetries (cf. Sec. 7).

By [31], any connection $\Xi$ on $\pi_{1,0}$ can be identified with an $f(3,-1)$-structure $F_{\Xi} = 2 h_{\Xi} - h_{\gamma_{1,0}} - I$ of rank $2m$ on $J^1 \pi$, where $\Gamma^{(2)}$ is the characteristic connection and $F_{\Xi} = \nu_{\gamma_{1,0}}$. The eigenspaces of $F_{\Xi}$ are generated by the projections

$$I - F_{\Xi}^2 = h_{\Gamma^{(2)}}, \quad \frac{1}{2} (F_{\Xi}^2 + F_{\Xi}) = h_{\Xi} - h_{\gamma_{1,0}}, \quad \frac{1}{2} (F_{\Xi}^2 - F_{\Xi}) = \nu_{\Xi},$$

hence the decomposition of $T J^1 \pi$ generated by $\Xi$ through $F_{\Xi}$ is

$$T J^1 \pi = V_{\gamma_{1,0}} J^1 \pi \oplus H_{F_{\Xi}} \oplus H_{\Gamma^{(2)}}.$$
The m-dimensional \( H_{Fz} = \text{Im}(h_{z} - h_{Fz}) = \text{span}(Dx) \) is in accordance with [6] called strong horizontal, which means

\[
V_{\pi,0}J^{1}\pi \otimes H_{Fz} = V_{\pi,0}J^{1}\pi
\]

Evidently

\[
H_{Fz} = \text{Im } h_{z}\big|_{\pi_{*,0}(V,Y)},
\]

where \( h_{z} \) is now regarded as a horizontal lift \( \pi_{*,0}(TY) \to T^{J^{1}}\pi \) (rather than the projector on \( T^{J^{1}}\pi \)), which corresponds to the autonomous situation (5.4), where \( H_{Fz} \) coincides with \( H_{z} \).

Notice finally that in terms of the characteristic connection, (5.8) can be rewritten to

\[
i_{T}T = \left(2\Xi^{\sigma} + \frac{\partial T_{(2)}}{\partial q^{(1)}_{(1)}} - 2\pi^{(3)}\right) \frac{\partial}{\partial q^{(1)}_{(1)}} \otimes dt + \left(2\Xi^{\lambda} - \frac{\partial T_{(3)}}{\partial q^{(1)}_{(1)}}\right) \frac{\partial}{\partial q^{(1)}_{(1)}} \otimes dq^{\lambda},
\]

which becomes important below.

6. INTEGRALS OF 2-CONNECTIONS

As already announced, the search for integrable connections on \( \pi_{1,0} \) with a common characteristic 2-connection on \( \pi \) might be of interest for the integration of the corresponding second-order ODE system. In this respect, if \( T_{(2)} \) is a 2-connection on \( \pi \) then any \( \Xi \) on \( \pi_{1,0} \) whose characteristic connection is \( T_{(2)} \) will be called associated to \( T_{(2)} \) and each integrable \( \Xi \) associated to \( T_{(2)} \) will be called an integral of \( T_{(2)} \).

First we recall a local result of [17]. Suppose we are given a system \( \{a^{1}, \ldots, a^{m}\} \) of independent first integrals of \( H_{T_{(2)}} \) on some open \( W \subset J^{1}\pi \), \( W \subset \pi_{1,0}(V) \), where \((V,\psi)\) is a fibered chart on \( Y \). If in these coordinates \( \det(\partial a^{\sigma}/\partial q^{(1)}_{(1)}) \neq 0 \) on \( W \), then by \( H_{\Xi} = \text{Anih}(\{da^{1}, \ldots, da^{m}\}) \) an integral \( \Xi \) on \( T_{(2)} \) on \( W \) is defined whose components are

\[
\Xi^{\sigma} = -A_{\pi}^{\sigma} \frac{\partial a^{\sigma}}{\partial t}, \quad \Xi^{\lambda} = -A_{\pi}^{\lambda} \frac{\partial a^{\lambda}}{\partial q^{\lambda}},
\]

where \( (A_{\pi}^{\lambda}) \) is the inverse matrix to \( (A_{\pi}^{\sigma}) = (\partial a^{\sigma}/\partial q^{(1)}_{(1)}) \).

In case of global integrals, the task splits into two parts; first, global connections on \( \pi_{1,0} \) associated to \( T_{(2)} \) must be determined and secondly, their integrability should be studied. In what follows we proceed analogously to [9]. We have to start with a natural vector bundle morphism

\[
\tilde{F}_{0} : V_{\pi_{1,0}}J^{1}\pi \otimes \pi_{*,0}(T^{*}Y) \to V_{\pi_{1,0}}J^{1}\pi \otimes \pi_{*,0}(T^{*}R)
\]
over $\mathcal{J}^1\pi$ induced by (5.10) on the associated vector bundles. This morphism maps deformations of connections on $\pi_{1,0}$ to deformations of 2-connections on $\pi$ in coordinates

$$\phi_{(2)} \circ \tilde{h}_0 = \phi^\pi + \phi^\lambda_{(1)}. \tag{6.1}$$

If $\Xi_0$ is a connection on $\pi_{1,0}$ associated to $\Gamma^{(2)}$ then evidently by $h_{\Xi_0} + \phi$ another such connection is defined if and only if

$$\phi \in \ker \tilde{h}_0, \tag{6.2}$$

and any soldering form $\phi$ on $\pi_{1,0}$ satisfying (6.2) will be called admissible. Due to (5.6), all natural admissible soldering forms on $\pi_{1,0}$ are of the form $\phi = kS, k \in \mathcal{F}(\mathbb{R})$. According to [6],

$$h_{\Xi_0} = \frac{1}{2} \left( h_{\Gamma^{(2)}} + I - \mathcal{L}_{\mathcal{U}_{\Gamma^{(2)}}} S \right) \tag{6.3}$$

is a horizontal form of a connection $\Xi_0$ on $\pi_{1,0}$ associated to $\Gamma^{(2)}$. Following the above ideas we can state that the family of connections on $\pi_{1,0}$ naturally associated to a 2-connection $\Gamma^{(2)}$ on $\pi$ is defined by

$$h_\Xi = h_{\Xi_0} + kS, \quad k \in \mathcal{F}(\mathbb{R}). \tag{6.4}$$

In coordinates, the components of $\Xi$ are

$$\Xi^\lambda = \frac{1}{2} \frac{\partial h_{\Xi_0}^\lambda}{\partial \phi^\lambda_{(1)}} + k(t) \delta^\lambda_\mu, \quad \Xi^\mu = \Gamma^\mu_{(2)} = \Xi^\lambda_{(1)}. \tag{6.5}$$

This result can be compared with [30] and [9]; in fact, if we view $K^\ast(t) = 2k(t)$ as a component of a linear connection $K^\ast$ on $\tau_0^\ast$, it is easy to verify that the connection (6.4) coincides with the so-called natural dynamical connection of type $\Omega$ [30] for any volume form $\Omega$ on $\mathbb{R}$ which is a (global) integral section of $K^\ast$. Moreover, denoting by $K$ the dual connection to $K^\ast$ on $\tau_0^\ast$ (i.e. $K(t) = -K^\ast(t)$), we get (6.4) as the only connection naturally assigned to the pair $\Gamma^{(2)}$, $K$ in the sense of [9].

Next, the role of torsions and related structures can be discussed. Denote by $\mathcal{H}_0$ and $\mathcal{T}_0$ the torsion and the tension of the connection $\Xi_0$ given by (6.3). By (5.7),
(5.8) and (5.9) one gets

\[
T_0 = \frac{1}{2} \frac{\partial^2 \Gamma^\sigma_{\gamma\lambda}}{\partial q^\alpha(1) \partial q^\alpha(1)} \frac{\partial}{\partial q^\alpha(1)} \otimes dq^\sigma \wedge dq^\lambda = 0,
\]

\[
\mathcal{H}_0 = \left( \frac{\partial \Gamma^\sigma_{\gamma\lambda}}{\partial q^\alpha(1)} q^{\sigma(1)} - \Gamma^\sigma_{\gamma\lambda(1)} \right) \frac{\partial}{\partial q^\alpha(1)} \otimes dt
\]

\[
+ \frac{1}{2} \left( \frac{\partial \Gamma^\sigma_{\gamma\lambda(2)}}{\partial q^\alpha(1) \partial q^\alpha(1)} q^{\sigma(1)} - \frac{\partial \Gamma^\sigma_{\gamma\lambda(1)}}{\partial q^\alpha(1)} \right) \frac{\partial}{\partial q^\alpha(1)} \otimes dq^\lambda.
\]

Let now \( \varphi \) be an arbitrary admissible soldering form on \( \pi_{1,0} \). Then for the connection \( \Xi \) on \( \pi_{1,0} \) defined by

\[
h_\Xi = h_{\Xi_0} + \varphi
\]

we have \( T = T_0 + [\varphi, S], \) \( i_\gamma T = 2\varphi, \) \( \mathcal{H} = \mathcal{H}_0 + \mathcal{L}C \varphi. \) Hence, the following assertion holds.

**Proposition 6.1.** Let \( \Gamma^{[2]} \) be a 2-connection on \( \pi \) and \( \varphi \) an admissible soldering form on \( \pi_{1,0} \). Then there is a unique connection \( \Xi \) on \( \pi_{1,0} \) associated to \( \Gamma^{[2]} \) such that its torsion \( T \) satisfies \( i_\gamma T = 2\varphi. \)

By (6.6), this connection is defined just by

\[
h_\Xi = \frac{1}{2} \left( h_{\Gamma^{[2]}} + I - \mathcal{L}_{\Pi^{[2]}} S \right) + \varphi.
\]

Now it is easy to see that if the torsion of a connection \( \Xi \) on \( \pi_{1,0} \) vanishes, then \( \Xi = \Xi_0 \) for \( \Gamma^{[2]} \) being the characteristic connection to \( \Xi. \)

Let now \( \psi = q^{\alpha(1)} \partial / \partial q^\alpha + w^\sigma(q^\sigma, q^{\alpha(1)}) \partial / \partial q^\alpha \) be a semispray on \( TM \) and \( D_{T^\psi} = \partial / \partial t + \psi \) a semispray on \( \mathbb{R} \times TM \) defining a 2-connection \( \Gamma^{[2]} \) on \( \pi. \) Following the above approach, a search for a connection \( \Lambda \) on \( TM \) associated to \( \psi \) is equivalent to a search for a connection \( \Xi \) of type (5.4) associated to \( \Gamma^{[2]}. \) From (6.6) we easily deduce that \( \varphi \) must satisfy

\[
\varphi^\sigma = \frac{1}{2} \frac{\partial w^\sigma}{\partial q^{\alpha(1)}} q^{(1)} - w^\sigma
\]

or equivalently

\[
\varphi^\alpha q^{\alpha(1)} = w^\sigma - \frac{1}{2} \frac{\partial w^\sigma}{\partial q^{\alpha(1)}} q^{(1)},
\]

which coincides with (3.6).
Following [29], certain 'homogeneous' considerations can be presented. First, in accordance with the autonomous situation and due to the underlying structures, a spray connection on $\pi$ can be defined as a 2-connection $\Gamma^{(2)}$ on $\pi$ whose components are homogeneous of order two in $q^\lambda_{(1)}$, i.e.

$$
\frac{\partial \Gamma_{(2)}^\sigma}{\partial q^\lambda_{(1)}} q^\lambda_{(1)} = 2\Gamma_{(2)}^\sigma.
$$

(6.7)

Throughout the paper, the smoothness on the zero section is assumed, hence (6.7) means that the functions $\Gamma_{(2)}^\sigma$ are quadratic in $q^\lambda_{(1)}$:

$$
\Gamma_{(3)}^\lambda = \frac{1}{2} \frac{\partial^2 \Gamma_{(2)}^\sigma}{\partial q^\lambda_{(1)} \partial q^\mu_{(1)}} q^\mu_{(1)} q^\lambda_{(1)}.
$$

It is easy to verify that if $E$ is linear then its characteristic $\Gamma^{(2)}$ is a spray connection if and only if $\Xi^\sigma = 0$ and conversely, the connection (6.3) associated to a spray connection is linear with

$$
\Xi^\sigma = 0, \quad \Psi_{(2)}^\sigma = \frac{1}{2} \frac{\partial^2 \Gamma_{(2)}^\sigma}{\partial q^\lambda_{(1)} \partial q^\mu_{(1)}} q^\mu_{(1)} q^\lambda_{(1)}.
$$

Consequently, if $\Gamma^{(2)}$ is a spray connection on $\pi$ then (6.3) is the unique linear connection on $\pi_{1,0}$ associated to $\Gamma^{(2)}$.

The integrability of a connection on an arbitrary fibered manifold (and thus of the corresponding equations) means equivalently the involutiveness of the corresponding horizontal distribution. Accordingly, the integrability conditions can be expressed among other by the vanishing of the Lie bracket of the absolute derivatives with respect to the connection. For a connection $\Xi$ on $\pi_{1,0}$ it means

$$
[D_{\Xi^\sigma}, D_{\Xi^\lambda}] = 0
$$

(6.8)

$$
[D_{\Xi^\sigma}, D_{\Xi^\lambda}] = 0.
$$

(6.9)
7. Symmetries

In what follows we deal with (infinitesimal) symmetries of the ODE studied, i.e. with vector fields as generators of groups of transformations invariant with respect to solutions. For the general case of PDE represented by connections we refer to [31] and references therein.

An (infinitesimal) symmetry of a connection $\Gamma$ on $\pi$ (or equivalently of the time dependent vector field (4.2)) can be defined as a vector field

$$\zeta = \zeta^t \frac{\partial}{\partial t} + \zeta^x \frac{\partial}{\partial x}$$

on $Y$ which is $\Gamma$-related to its first prolongation

$$\mathcal{J}^1 \zeta = \zeta^t \frac{\partial}{\partial t} + \zeta^x \frac{\partial}{\partial x} + \left(D(\zeta^t) - D(\zeta^x)\frac{\partial}{\partial x}\right) \frac{\partial}{\partial q^1(1)}$$

on $\mathcal{J}^1 \pi$, i.e. $\mathcal{J}^1 \zeta \circ \Gamma = T \zeta \circ \zeta$. In other words, $\mathcal{J}^1 \zeta$ is a contact vector field (a symmetry of the Cartan distribution) tangent to $\Gamma(Y)$ and thus an exterior symmetry of the equation $\Gamma(Y) \subset \mathcal{J}^1 \pi$.

It is easy to see that an arbitrary $\Gamma$-horizontal vector field $\zeta = f D\Gamma$, $f \in \mathcal{F}(Y)$, is a symmetry, which immediately means that $\zeta$ on $Y$ is a symmetry of $\Gamma$ if and only if

$$\mathcal{J}^1 \left( \psi^t \circ \zeta \right) \circ \Gamma = V \Gamma \circ \psi^t \circ \zeta^t.$$

In coordinates, (7.2) is represented by a system of PDE

$$\frac{\partial \varphi^t}{\partial t} + \frac{\partial \varphi^x}{\partial x} \Gamma^1 = \frac{\partial \Gamma^x}{\partial q^1} \varphi^1$$

for the family of generating functions $\varphi^x = \zeta^x - \Gamma^x \zeta^0$ on $Y$, $x = 1, \ldots, m$. Using the horizontal form of $\Gamma$, (7.3) can be expressed by

$$\mathcal{L}_{\psi^t(\zeta)} h^t = 0$$

and in terms of $D\Gamma$ it reads

$$[\zeta, D\Gamma] = -D\Gamma(\zeta^0) D\Gamma.$$

The last relation says that the symmetries of $\Gamma$ could be defined directly as the symmetries of the horizontal distribution $H_\Gamma$. 163
Since $\mathcal{L}_\pi(\zeta)^h = 0$ if and only if $\zeta$ is $\pi$-projectable, the symbol of $\tau_\pi$ in (7.4) can be omitted under the same assumption. In view of the fact that the vanishing of $\mathcal{L}_h\zeta$ is equivalent to $T_\alpha \circ h_\tau = h_\tau \circ T_\alpha$ for the flow $\{\alpha_t\}$ of $\zeta$, a projectable vector field on $Y$ is a symmetry of $\Gamma$ if and only if its flow permutes the integral sections of $\Gamma$. Moreover, if such a symmetry is horizontal then it moves integral sections along themselves while $\pi$-vertical symmetries (time-dependent fields on $M$, see Sec. 4) permute integral sections without changing their parametrization. In this case (7.5) reads $[\zeta, \partial/\partial t + v] = 0$.

An intrinsic role both of vertical prolongations of connections and of strong horizontal distributions appears in case of vertical symmetries, the set of which we denote by $\text{Sym}_v(\Gamma)$. Actually, applying the ideas of Sec. 4, (7.2) means

$$J^1 \zeta \circ \Gamma = \Gamma \circ \zeta$$

for vertical $\zeta$. Consequently, in terms of integral sections of $\Gamma$ and $\Gamma$ one gets that for any $\zeta \in \text{Sym}_v(\Gamma)$, a section $\gamma$ of $\pi$ is an integral section of $\Gamma$ if and only if $\xi = \zeta \circ \gamma$ is an integral section of $\Gamma$. Moreover, $\zeta \in \text{Sym}_v(\Gamma)$ if and only if $\xi$ is an integral section of $\Gamma$ for each integral section $\gamma$ of $\Gamma$.

Let $\Xi$ be a connection on $\pi_{1,0}$ and $\zeta$ a vertical vector field on $Y$. Then $J^1 \zeta \in H_{\Xi,0}$ locally means

$$D(\zeta^\zeta) = \Xi \zeta^\zeta,$$

and by (5.3), (7.3) and (7.7) the strong horizontal distribution $H_{\Xi,0}$ contains first prolongations of vertical symmetries of integral sections of $\Xi$.

Let $\Gamma^{(2)}$ be a 2-connection on $\pi$. Since it is a particular (holonomic) type of a connection on $\pi_1$, a vector field $\zeta^{(1)} = \zeta^\zeta \partial/\partial t + \zeta^\eta \partial/\partial q^\eta + \zeta^{(1)}_1 \partial/\partial q^{(1)}_1$ on $J^1 \pi$ will be called a first-order symmetry of $\Gamma^{(2)}$ (or of the corresponding semispray $D_{\Gamma^{(2)}}$) if

$$\mathcal{L}_{\zeta^{(1)}}(\zeta^{(1)} h_{\Gamma^{(2)}}) = 0$$

or equivalently

$$[\zeta^{(1)}, D_{\Gamma^{(2)}}] = -D_{\Gamma^{(2)}}(\zeta^\zeta) D_{\Gamma^{(2)}}.$$

Hence the first-order symmetries of $\Gamma^{(2)}$ are just the symmetries of $H_{\Gamma^{(2)}}$. In coordinates,

$$\zeta^{(1)}_{(1)} = D_{\Gamma^{(2)}}(\zeta^\zeta) - \zeta^{(1)}_1 D_{\Gamma^{(2)}}(\zeta^\zeta)$$

$$D_{\Gamma^{(2)}}(\phi^\zeta) = \frac{\partial \gamma^\zeta}{\partial q^\zeta} \phi^\zeta + \frac{\partial \gamma^\eta_1}{\partial q^{(1)}_1} D_{\Gamma^{(2)}}(\phi^\eta_1)$$
for \( \varphi = \zeta - q_1^{(1)} \). Again, by (7.8) \( \pi_1 \)-projectable fields are first-order symmetries of \( \Gamma^{(2)} \) if and only if their flows permute 1-jets of integral sections of \( \Gamma^{(2)} \).

Within the context of the equations studied, we are mainly interested in symmetries acting on \( Y \). Consequently, a vector field \( \zeta \) (7.1) on \( Y \) is called a (zero-order) symmetry of \( \Gamma^{(2)} \) if its first prolongation \( J^1 \zeta \) is the first-order one. The relation (7.9) now reads

(7.11) \[ [J^1 \zeta, D_{\Gamma^{(2)}}] = -D(\zeta^2)D_{\Gamma^{(2)}}, \]

and (7.10) holds trivially. Accordingly, a \( \pi_1 \)-projectable first-order symmetry \( \zeta^{(1)} \) is just the prolongation \( J^1 \zeta \) of the symmetry \( \zeta = T\pi_1 \zeta^{(1)} \).

Let \( \zeta \) be \( \pi \)-projectable. Then it is a symmetry of \( \Gamma^{(2)} \) if and only if the flow of \( J^1 \zeta \) permutes first jets of integral sections of \( \Gamma^{(2)} \). Due to the definition of \( J^1 \zeta \), \( \zeta \) is a symmetry if and only if its flow permutes the integral sections of \( \Gamma^{(2)} \) in themselves.

Suppose the symmetries to be \( \pi \)-vertical and denote their set by \( \text{Sym}_\pi(\Gamma^{(2)}) \).

Analogously to (7.6) one sees that \( \zeta \in \text{Sym}_\pi(\Gamma^{(2)}) \) if and only if \( J^2 \zeta \circ \Gamma^{(2)} = \gamma T(\Gamma^{(2)}) \circ J^1 \zeta \),

where \( J^2 \zeta = J^2(\zeta, \text{id}_R) \) is the second prolongation of \( \zeta \). Consequently, \( \gamma \) is an integral section of \( \Gamma^{(2)} \) if and only if \( \xi = \gamma \circ \gamma \) is an integral section of \( \gamma T(\Gamma^{(2)}) \) and \( \zeta \in \text{Sym}_\pi(\Gamma^{(2)}) \) if and only if \( \xi \) is an integral section of \( \gamma T(\Gamma^{(2)}) \) for each integral section \( \gamma \) of \( \Gamma^{(2)} \).

The presented classification enables us to describe very naturally the interrelations between (vertical) symmetries of a 2-connection \( \Gamma^{(2)} \) (second-order ODE) and its field of geodesics \( \Gamma \) (first-order ODE) (see (5.12)). First, since any integral section of \( \Gamma \) is an integral section of \( \Gamma^{(2)} \), if \( \zeta \) is a symmetry of \( \Gamma^{(2)} \) then its restriction to the domain is a symmetry of \( \Gamma \). Secondly, by virtue of the fact that

\[ J^1(\Gamma, \text{id}_R) \circ \Gamma = \Gamma^{(2)} \circ \Gamma \quad \text{if and only if} \quad J^1(\gamma T, \text{id}_R) \circ \gamma T = \gamma T^{(2)} \circ \gamma T, \]

one easily deduces that a symmetry of \( \Gamma \) is a symmetry of its prolongation \( \Gamma^{(2)} \circ \Gamma \).

Recall the autonomous situation. In this case the considerations describe the generators of groups of transformations invariant with respect to the graphs of geodesics. For example, according to (7.9) a vector field \( \zeta^{(1)} \) on \( TM \) is a first-order symmetry of a semispray \( w \) on \( TM \) if \( [\zeta^{(1)}, w] = 0 \) while according to (7.11) a vector field \( \zeta \) on \( M \) is a (zeroth-order) symmetry of \( w \) if \( [\zeta, w] = 0 \).

Notice finally that comparing the above classification e.g. with [23] or [6], the first-order symmetries of a 2-connection are the so-called dynamical symmetries of the corresponding semispray while the zero-order ones are nothing but the Lie symmetries.
References


Author's address: Alexander Vondra, Department of Mathematics, Military Academy in Brno, PS 13, 61200 Brno, Czech Republic.