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EXACT 2-STEP DOMINATION IN GRAPHS

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Summary. For a vertex \( v \) in a graph \( G \), the set \( N_2(v) \) consists of those vertices of \( G \) whose distance from \( v \) is 2. If a graph \( G \) contains a set \( S \) of vertices such that the sets \( N_2(v), v \in S \), form a partition of \( V(G) \), then \( G \) is called a 2-step domination graph. We describe 2-step domination graphs possessing some prescribed property. In addition, all 2-step domination paths and cycles are determined.

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1. INTRODUCTION

Two vertices \( u \) and \( v \) in a graph \( G \) for which the distance \( d(u, v) = 2 \) are said to 2-step dominate each other. The set of vertices of \( G \) that are 2-step dominated by \( v \) is denoted by \( N_2(v) \); that is,

\[
N_2(v) = \{ u \in V(G) \mid d(v, u) = 2 \}.
\]

A set \( S \) of vertices of \( G \) is called a 2-step domination set if \( \bigcup_{v \in S} N_2(v) = V(G) \). A 2-step domination set \( S \) such that the sets \( N_2(v), v \in S \), are pairwise disjoint is called an exact 2-step domination set. If a graph \( G \) has an exact 2-step domination set, then \( G \) is called an exact 2-step domination graph or, for brevity, a 2-step domination graph. Each of the graphs \( G_1, G_2, \) and \( G_3 \) of Figure 1 is a 2-step domination graph.

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with an exact 2-step domination set $S_1 = \{u_1, u_2, u_3, u_4\}$, $S_2 = \{v_1, v_2, v_3, v_4\}$, and $S_3 = \{w_1, w_2, w_3, w_4\}$, respectively. We adopt the convention of drawing a vertex with a solid circle if the vertex belongs to the exact 2-step domination set under discussion. In general we follow the graph theoretic notation and terminology of the books [1], [2].

\[ G_1 : \hspace{1cm} G_2 : \hspace{1cm} G_3 : \]

Figure 1. Three 2-step domination graphs.

2. CONSTRUCTION 2-STEP DOMINATION GRAPHS

Our primary problem is to determine which graphs are 2-step domination graphs. If $G$ is a graph of order $p$ containing a vertex $v$ of degree $p - 1$, then no vertex of $G$ 2-step dominates $v$. This observation yields the next result. We denote the radius and diameter of a graph $G$ by $\text{rad} G$ and $\text{diam} G$, and the maximum degree of $G$ by $\Delta(G)$.

\textbf{Lemma 1.} If $G$ is a 2-step domination graph, then $\text{rad} G \leq 2$.

According to Lemma 1 then, $\Delta(G) \leq p - 2$ for every 2-step domination graph $G$ of order $p$. No further reduction in the bound for $\Delta(G)$ is possible. For example, if $p = 2n$, the graph $nK_2$ is a $(p - 2)$-regular 2-step domination graph in which the only exact 2-step domination set consists of the entire vertex set. The path $P_4$ (the graph $G_3$ of Figure 1) also has the property that it is a 2-step domination graph whose unique exact 2-step domination set is the vertex set of the graphs. In fact, these are the only connected graphs with this property.

\textbf{Theorem 2.} A connected graph $G$ is a 2-step domination graph with exact 2-step domination set $V(G)$ if and only if $G \simeq P_4$ or $G \simeq nK_2$ for some $n \geq 2$.

\textbf{Proof.} First, the graphs $nK_2$, $n \geq 2$, and $P_4$ have the desired property. Conversely, suppose that $G$ is a connected 2-step domination graph with exact 2-step
domination set $V(G)$. Necessarily, every vertex $v$ of $G$ has a unique vertex at distance 2 from $v$. Hence, $\text{diam } G \geq 2$. If $\text{diam } G \geq 4$, then $G$ contains an induced subgraph isomorphic to $P_5$, the central vertex of which is at distance 2 from two vertices; so this is impossible. There remain two cases.

Case 1. $\text{diam } G = 2$. Then, for every vertex $v$ of $G$ there is a unique vertex distinct from $v$ and not adjacent to $v$. Hence $p$ is even, say $p = 2n \geq 4$, and $G \cong \overline{K}_{2n}$.

Case 2. $\text{diam } G = 3$. In this case, $G$ contains an induced path $P_4: v_1, v_2, v_3, v_4$ and hence $d(v_1, v_4) = 3$. Thus each of $v_1$ and $v_4$ is the unique vertex at distance 2 from the other, as is the case for $v_2$ and $v_3$. We claim that $v_1$ is an end-vertex of $G$. If this is not the case, then $G$ contains a vertex $x$ distinct from $v_2$ and $v_3$; but then $d(v_1, x) = 2$, which is impossible. Consequently, $G \cong P_4$. 

The fact that the graphs $\overline{K}_{2n}$, $n \geq 2$, are $(2n - 2)$-regular 2-step domination graphs shows that $r$-regular 2-step domination graphs exist for every even integer $r > 2$. We next show that such is the case for odd values of $r$ as well.

Let $S$ consist of $2n$ vertices of the graph $nC_4$, two adjacent vertices from each component. Then $S$ is an exact 2-step domination set in the complement $\overline{nC_4}$. Since $\overline{nC_4}$ is $(4n - 3)$-regular, $r$-regular 2-step domination graphs exist for $r \equiv 1 \pmod{4}$. It remains to show the existence of $r$-regular 2-step domination graphs, where $r \equiv 3 \pmod{4}$.

For $n \geq 0$, define the vertex set of the graph $G'_n$ (as shown in Figure 2) by

$$V(G'_n) = \{u, u'\} \cup \{v, v'\} \cup \{w, w'\} \cup V \cup V',$$

where $V = \{v_1, v_2, \ldots, v_{n+2}\}$ and $V' = \{v'_1, v'_2, \ldots, v'_{n+2}\}$ and the edge set of $G'_n$ by

$$E(G'_n) = \{uu', vv', ww', uv, u'v', uw, w'u' | x \in V\} \cup \{xx, wx \mid x \in V\} \cup \{w'x, w'x \mid x \in V'\}.$$

Next let $F \cong K_1 \cup (2n + 1)K_2$, where $V(F) = V \cup \{v\}$ and $V'(F) = V \cup \{v'\}$, such that $\deg F v = \deg F v' = 4n + 2$. Now define the graph $G_n$ by $V(G_n) = V(G'_n)$ and

$$E(G_n) = E(G'_n) \cup E(F) \cup E(F').$$
The graph $G_n$ is a $(4n + 3)$-regular 2-step domination graph with exact 2-step domination set $\{u, u', w, w'\}$. We now summarize these observations.

**Theorem 3.** For every integer $r > 2$, there exists an $r$-regular 2-step domination graph.

The composition $G[H]$ of graphs $G$ and $H$ is constructed by replacing each vertex of $G$ by a copy of $H$ and each edge $v_i v_j$ of $G$ by the join $H_i + H_j$ of these respective copies of $H$. This operation has been often extended to the generalized composition $G[H_1, H_2, \ldots, H_p]$ of the labeled graph $G$ with $V(G) = \{v_1, v_2, \ldots, v_p\}$ determined by any $p$ graphs $H_1$. Again, each vertex $v_i$ of $G$ is replaced by $H_i$ and each edge $v_i v_j$ by the join $H_i + H_j$. This is illustrated in Figure 3.

With the aid of the generalized composition, we can construct new 2-step domination graphs from given 2-step domination graphs.

**Theorem 4.** Let $G$ be a 2-step domination graph with $V(G) = \{v_1, v_2, \ldots, v_p\}$. For positive integers $n_1, n_2, \ldots, n_p$, the generalized composition $G[K_{n_1}, K_{n_2}, \ldots, K_{n_p}]$ is a 2-step domination graph.

**Proof.** Since $G$ is a 2-step domination graph, there exists an exact 2-step domination set $S$, say, without loss of generality, $S = \{v_1, v_2, \ldots, v_k\}$. For $i = 1, 2, \ldots, k$, let $H_i$ be a graph such that $H_i \cong K_{n_i}$ and let $v'_i$ be a vertex of $H_i$. Then $S' = \{v'_1, v'_2, \ldots, v'_k\}$ is a 2-step domination set of $G[K_{n_1}, K_{n_2}, \ldots, K_{n_p}]$. Therefore, $G[K_{n_1}, K_{n_2}, \ldots, K_{n_p}]$ is a 2-step domination graph.
{v_1', v_2', ..., v_k'} is an exact 2-step domination set of the graph $G[H_1, H_2, ..., H_4]$. 

Since the path $P_4$ is a 2-step domination graph (in which every vertex belongs to a 2-step domination set), by varying the orders of four complete graphs, we have the following.

**Corollary 5.** For every integer $n \geq 4$, there exists a 2-step domination graph of order $n$.

Furthermore, the proof of Theorem 4 shows that the graph $P_4[K_n, K_n, K_n, K_n]$ illustrates the fact that for every positive integer $n$, there exists a 2-step domination graph whose vertex set can be partitioned into $n$ subsets, each of which is an exact 2-step domination set.

We now describe some additional examples of 2-step domination graphs. First we present some other terms, whose definitions are expected. A set $S$ of vertices of a graph $G$ is an exact 1-step domination set if the union $\bigcup N(v)$ of the open neighborhoods of the vertices $v$ of $S$ is $V(G)$ and the sets $N(v)$, $v \in S$, are pairwise disjoint. A graph then is a 1-step domination graph if it contains an exact 1-step domination set. The graphs shown in Figure 4 are 1-step domination graphs. So the complete bipartite graphs $K_{m,n}$, for any pair $m, n$ of positive integers, are 1-step domination graphs.

Our special interest is in disconnected 1-step domination graphs.

**Theorem 6.** A disconnected graph $G$ is a 1-step domination graph if and only if its complement $\overline{G}$ is a 2-step domination graph.
Proof. Let $G$ be a disconnected graph. Suppose first that $G$ is a 1-step domination graph. Then $\text{diam} \ G = 2$ and the vertices adjacent to a vertex $v$ of $G$ are precisely the vertices at distance 2 from $v$ in $\overline{G}$. Thus if $S$ is an exact 1-step domination set of $G$, then $S$ is an exact 2-step domination set of $\overline{G}$. Conversely, if $\overline{G}$ is a 2-step domination graph, then $G$ is a 1-step domination graph. •

If $G$ is a disconnected graph whose four components $G_i, 1 \leq i \leq 4$, are given in Figure 4, then by Theorem 6, $G$ is a 2-step domination graph. We already observed in Theorem 2 that $nK_2, n \geq 2$, is a 2-step domination graph. We have now seen several examples of 2-step domination graphs. If $S$ is an exact 2-step domination set of a 2-step domination graph $G$, then, of course, $S \subseteq V(G)$, but there need not be any relationship between the numbers $|S|$ and $|V(G)|$.

Theorem 7. For any rational number $a/b$, with $0 < a/b < 1$, there exists a 2-step domination graph $G$ with an exact 2-step domination set $S$ such that $|S|/|V(G)| = a/b$.

Proof. Since we have already characterized those 2-step domination graphs $G$ with $|S|/|V(G)| = 1$, we assume that $0 < a/b < 1$. We have already noted that the graph $H \simeq 3aK_2$ is a 2-step domination graph. Let $G$ be the generalized composition obtained by replacing some vertex of $H$ by the graph $K_{a+b+1}$ (and replacing all other vertices by $K_1$). By Theorem 4, $G$ is a 2-step domination graph with $|S| = 4a$ and $|V(G)| = 4b$. Consequently, $|S|/|V(G)| = a/b$. □
3. 2-STEP DOMINATION PATHS AND CYCLES

We now determine all those paths and cycles that are 2-step domination graphs. We begin by showing that if \( m \equiv 1, 2, \) or 3 (mod 8), then \( P_m \) is not a 2-step domination graph.

**Theorem 8.** For every nonnegative integer \( n \), none of the paths \( P_{8n+1}, P_{8n+2}, \) and \( P_{8n+3} \) are 2-step domination graphs.

**Proof.** Suppose that the result is false. Since none of \( P_1, P_2, \) and \( P_3 \) are 2-step domination graphs, there is a smallest positive integer \( m \) (of the form \( 8n + 1, 8n + 2, \) or \( 8n + 3 \)) such that \( P_m \) is a 2-step domination graph. Suppose that \( P_m \) is the path \( v_1, v_2, \ldots, v_m \). Let \( S \) be an exact 2-step domination set of \( P_m \). We consider three cases.

**Case 1.** Suppose that \( m = 8n + 1 \). We now consider two subcases.

**Subcase 1.1.** Assume that four consecutive vertices among \( v_1, v_2, v_3, v_4, v_5, v_6 \) belongs to \( S \). If \( v_i, v_{i-1}, v_{i+1}, v_{i+2} \) belong to \( S \), then the vertices \( v_1, v_2, \ldots, v_6 \) of \( P_{8n+1} \) are 2-step dominated by the vertices \( v_1, v_2, v_3, v_4 \). Consequently, \( P_{8n-5} = P_{8(n-1)+3} \) is a 2-step domination graph, contrary to assumption.

Suppose next that \( v_2, v_3, v_4, v_5 \) belong to \( S \). Then the vertices \( v_1, v_2, \ldots, v_7 \) of \( P_{8n+1} \) are 2-step dominated by the vertices \( v_2, v_3, v_4, v_5 \). This implies that \( P_{8n-6} = P_{8(n-1)+2} \) is a 2-step domination graph, which is impossible. Similarly, we cannot have \( v_3, v_4, v_5, v_6 \) belong to \( S \) in order to have \( v_1 \) and \( v_2 \) 2-step dominated.

**Subcase 1.2.** Assume that \( v_1 \) belong to \( S \). Since \( v_1 \) and \( v_2 \) must be 2-step dominated by elements of \( S \), it follows that \( v_3, v_4 \) belong to \( S \). We can assume that \( v_5 \notin S \); otherwise, the situation is covered by Subcase 1.1. Since \( v_4 \) is 2-step dominated by some vertex, \( v_5 \) belongs to \( S \). Because \( v_5 \notin S \) and \( v_7 \) is 2-step dominated by some vertex, \( v_5 \) belong to \( S \). If \( n = 1 \), we have a contradiction; if \( n \geq 2 \), we are repeating this Subcase with the path \( P_{8(n-1)+1} \). Continuing in this manner, we see that \( v_{8n+1} \) belong to \( S \) but that \( v_{8n+1} \) is 2-step dominated by no vertex, producing a contradiction.

If neither \( v_1 \) nor four consecutive vertices among \( v_1, v_2, v_3, v_4, v_5, v_6 \) belong to \( S \), then we must still have \( v_3, v_4 \) belong to \( S \) in order to have \( v_1 \) and \( v_2 \) 2-step dominated. Now since \( v_3 \) must be 2-step dominated, \( v_5 \) belong to \( S \). In order for \( v_4 \) to be 2-step dominated, either \( v_2 \) belong to \( S \) or \( v_6 \) belong to \( S \), producing four consecutive vertices among \( v_1, v_2, v_3, v_4, v_5, v_6 \) in \( S \). That is, Subcases 1.1 and 1.2 are exhaustive.

The proofs of the cases where \( m = 8n + 2 \) and \( m = 8n + 3 \) are similar and are, therefore, omitted. 

\[ \square \]
We next complete the problem for paths by showing that all other paths are 2-step domination graphs.

**Theorem 9.** For every positive integer \( n \), \( P_{8n} \) is a 2-step domination graph, and for every nonnegative integer \( n \), \( P_{8n+4}, P_{8n+5}, P_{8n+6}, \) and \( P_{8n+7} \) are 2-step domination graphs.

**Proof.** Consider the path \( P_m: v_1, v_2, \ldots, v_m \), where \( m \) is of the form described in the statement of the theorem. For \( m < 8 \), Figure 5 shows that each path \( P_m \) is a 2-step domination graph. For \( j = 4, 5, 6, 7 \), denote by \( S_j \) the exact 2-step domination set of the path \( P_j \).

![Figure 5](image)

We now make some observations that will be useful to us later. For the path \( P_{8n} \), \( n \geq 1 \), an exact 2-step domination set \( S_1 = \{v_i \mid i \equiv 3, 4, 5, 6 \pmod{8}\} \) is described in Figure 6. The set \( S_2 = \{v_i \mid i \equiv 1, 2, 3, 4 \pmod{8}\} \) is also shown in Figure 6. It is not an exact 2-step domination set, but in this case, every vertex of \( P_{8n} \) is 2-step dominated except \( v_{8n-1} \) and \( v_{8n} \).

![Figure 6](image)

The set \( S_1 \) shows that \( P_{8n}, n \geq 1 \), is a 2-step domination graph. Now label the vertices of the paths \( P_j (j = 4, 5, 6, 7) \) in Figure 5 from left to right as \( v_{8n+1}, v_{8n+2}, \ldots, v_{8n+j} \). The paths \( P_{8n+j} \) can be formed by taking the union of \( P_n \) (see Figure 6) and \( P_j \) and joining \( v_{8n} \) and \( v_{8n+1} \). The set \( S_j \cup S_1 \) is an exact 2-step domination set for \( P_{8n+j} \) for \( j = 4, 5, 6 \); while \( S_1 \cup S_1 \) is an exact 2-step domination set for \( P_{8n+7} \).

**Corollary 10.** The path \( P_m \) is a 2-step domination graph if and only if \( m = 0, 4, 5, 6, \) or \( 7 \pmod{8} \).

In order to characterize the 2-step domination cycles, we begin with a preliminary result.

**Lemma 11.** If a cycle \( C_n: v_1, v_2, \ldots, v_n, v_1 \) (\( n \geq 4 \)) is a 2-step domination graph with exact 2-step domination set \( S \), then there is an integer \( i \) (\( 1 \leq i \leq n \)) such that...
either (1) \( v_i, v_{i+1}, v_{i+2}, v_{i+3} \in S \) or (2) \( v_i, v_{i+2}, v_{i+3} \in S \) and \( v_{i+1} \notin S \) (where all addition is performed modulo \( n \)).

**Proof.** If \( n = 4 \), then \( S = \{ v_1, v_2, v_3, v_4 \} \) is the only exact 2-step domination set, and the result follows. Thus we may assume that \( n \geq 5 \). Suppose that there are no vertices \( v_1, v_{i+1}, v_{i+2}, v_{i+3} \) for which (1) or (2) holds.

Every vertex \( v_j \in S \) (1 \( \leq j \leq n \)) is 2-step dominated by either \( v_{j-1} \) or \( v_{j+2} \). Hence, without loss of generality, we may assume that \( v_1, v_3 \in S \). By our assumption, there are now two possibilities for \( v_2 \) and \( v_4 \).

**Case 1.** \( v_2, v_4 \notin S \). Hence \( v_n \in S \). (See Figure 7a.) If \( v_{n-1} \in S \), then (1) is satisfied; while if \( v_{n-1} \notin S \), (2) is satisfied, producing a contradiction.

**Case 2.** \( v_2 \in S \) and \( v_4 \notin S \). (See Figure 7b.) Since \( v_2 \) is not 2-step dominated by \( v_4 \), it follows that \( v_n \in S \). Thus, \( v_n, v_1, v_2, v_3 \in S \), producing a contradiction. \( \blacksquare \)

![Figure 7](image)

We can now describe all 2-step domination cycles.

**Theorem 12.** A cycle \( C_n \) is a 2-step domination graph if and only if \( n = 4 \) or \( n \equiv 0 \pmod{8} \).

**Proof.** We have already seen that \( C_4 \) is a 2-step domination graph. It is straightforward to see that for other values of \( m < 8 \), the cycle \( C_m \) is not a 2-step domination graph. Now let \( C_m : v_1, v_2, \ldots, v_m, v_1 \) (\( n \geq 1 \)) be a cycle. The set \( S = \{ v_i \mid i \equiv 1, 2, 3, 4 \pmod{8} \} \) is an exact 2-step domination set.

For the converse, assume that \( C_m : v_1, v_2, \ldots, v_m, v_1 \) is a 2-step domination graph with \( m \geq 8 \) and with exact 2-step domination set \( S \). By Lemma 11, we can assume, without loss of generality, that (1) \( v_1, v_2, v_3, v_4 \in S \) or (2) \( v_1, v_2, v_4 \in S \) and \( v_3 \notin S \). If (1) occurs, then \( v_5, v_6, v_7, v_8 \notin S \). If \( m > 8 \), then the vertices of \( P_m \) must repeat in this manner in groups of 8, that is, \( v_i \in S \) if \( i \equiv 1, 2, 3, 4 \pmod{8} \) and

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otherwise. Thus $m \equiv 0 \pmod{8}$. If (2) occurs, then $v_3, v_6, v_8 \notin S$ and $v_9 \in S$.
If $m > 8$, then the vertices of $P_m$ must repeat in this manner as well. In any case, $m \equiv 0 \pmod{8}$.

References


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