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SEQUENTIAL CONVERGENCES ON FREE LATTICE ORDERED GROUPS

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Summary. In this paper the partially ordered set Conv G of all sequential convergences on G is investigated, where G is either a free lattice ordered group or a free abelian lattice ordered group.

Keywords: free lattice ordered group, free abelian lattice ordered group, sequential convergence

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J. Novák [16] proved that every free group admits a nontrivial sequential convergence such that the group operation is sequentially continuous. Compatible sequential convergences on free groups were dealt with by Frič and Zanolin [7] (cf. also further references quoted there).

Let G be a lattice ordered group. The partially ordered set $\operatorname{Conv} G$ of all compatible sequential convergences on G was studied by Harminc [10]. The questions dealing with $\operatorname{Conv} G$ were investigated also in the papers [8], [9], [12], [13], [14].

In what follows, we will apply the shorter term "convergence" rather than "compatible sequential convergence".

Let α be a cardinal. The free (abelian) lattice ordered group with α free generators will be denoted by $G(\alpha)$ (or $A(\alpha)$, respectively).

A natural question arises whether $G(\alpha)$ and $A(\alpha)$ admit a nontrivial convergence.

In the present paper the following results will be proved:

(A) If $\alpha = 1$, then $G(\alpha) = A(\alpha)$ has no nontrivial convergence.

(B) If $\alpha \geqslant 2$, then $G(\alpha)$ admits a nontrivial convergence.

(C) If $\alpha \geqslant 2$, then $A(\alpha)$ admits $2^{2^{\aleph_0}}$ nontrivial convergences.

(D) If $\alpha \geqslant 2$, then the partially ordered set $\operatorname{Conv} A(\alpha)$ has no atom.

The question whether the assertions of (C) and (D) are valid for $G(\alpha)$ remains open.

1. Preliminaries

In the whole paper the symbol α denotes a cardinal. The group operation in a lattice ordered group will be denoted additively.

The free abelian lattice ordered group $A(\alpha)$ of rank α has benn investigated by Weinberg [17], [18], Bernau [2] and Conrad [4]. For the non-abelian case, the free lattice ordered group $G(\alpha)$ with α free generators was studied by Conrad [5] (cf. also the monographs [1], [6], [15]).

The following two results will be applied below.

Lemma 1.1. *(Cf. [17], p. 197.) Let $\alpha \geqslant 2$, $0 < h \in A(\alpha)$. Then there are elements g_1 and g_2 in $A(\alpha)$ such that $0 < g_i < h$ is valid for $i = 1, 2$ and $g_1 \wedge g_2 = 0$.*

Proposition 1.2. *(Cf. [5].) Let X be the ℓ-ideal of $G(\alpha)$ generated by the set $x + y - x - y$, where x and y run over $G(\alpha)$. Then the factor lattice ordered group $G(\alpha)/X$ is isomorphic to $A(\alpha)$.*

Next let us recall, for the sake of completeness, the basic definitions concerning convergences in a lattice ordered group G. The notation from [12] will be applied.

Let N be the set of all positive integers. The direct product $\prod_{n \in N} G_n$, where $G_n = G$ for each $n \in N$, will be denoted by G^N. If $(g_n) \in G^N$, $g \in G$, and if $g_n = g$ is valid for each $n \in N$, then we write $(g_n) = \operatorname{const} g$. The elements of G^N are called sequences in G; the notion of a subsequence has the usual meaning.

A subset β of the positive cone $(G^N)^+$ of G^N is said to be a convergence in G if β is a convex subsemigroup of $(G^N)^+$ such that the following conditions are satisfied:

(I) If $(g_n) \in \beta$, then each subsequence of (g_n) belongs to β.

(II) Let $(g_n) \in (G^N)^+$. If each subsequence of (g_n) has a subsequence belonging to (β), then (g_n) belongs to β.

(III) Let $g \in G$. Then $\operatorname{const} g$ belongs to β if and only if $g = 0$.

The system of all convergences in G will be denoted by $\operatorname{Conv} G$; this system is partially ordered by inclusion.

For $(g_n) \in (G^N)^+$ and $g \in G$ we put $g_n \to_\beta g$ if and only if $\big(|g - g_n|\big) \in \beta$.

Proposition 1.3. *(Cf. [10].) The partially ordered set $\operatorname{Conv} G$ is a $\wedge$-semilattice having a least element. Each interval of $\operatorname{Conv} G$ is a complete Brouwerian lattice.*

The least element of $\operatorname{Conv} G$ is the trivial convergence on G; its definition is obvious. It will be denoted by β_0.

2. The proofs of (A) – (D)

Proof of (A): Let N_0 be the additive group of all integers with the natural linear order. It is well-known (cf. [3], Chap. XIII) that the lattice ordered group $G(1)$ is isomorphic to $N_0 \times N_0$; thus $A(1) = G(1)$.

In view of [9], Corollary 2.10 we have $\operatorname{card} \operatorname{Conv} N_0 = 1$. According to [9], Theorem 4.5, the partially ordered set $\operatorname{Conv}(N_0 \times N_0)$ is isomorphic to $\operatorname{Conv} N_0 \times \operatorname{Conv} N_0$. Hence $\operatorname{card} \operatorname{Conv}(N_0 \times N_0) = 1$. Therefore (A) is valid. □

Let us consider the following condition for a lattice ordered group G:

(*) For each $0 < h \in G$ there exist g_1 and g_2 in G such that $g_1 \wedge g_2 = 0$ and $0 < g_i < h$ $(i = 1, 2)$.

A system $\{g_j\}$ $(j \in J)$ of elements of a lattice ordered group will be called disjoint if $g_j > 0$ for each $j \in J$ and $g_{j(1)} \wedge g_{j(2)} = 0$ whenever $j(1)$ and $j(2)$ are distinct elements of J.

Lemma 2.1. *Let G be a lattice ordered group, $G \neq \{0\}$. Assume that G satisfies the condition (*). Then there is an infinite disjoint system in G.*

Proof. We define by induction elements x_{1n} and x_{2n} $(n = 1, 2, \ldots)$ of G such that

(i) $0 < x_{n1}$, $0 < x_{n2}$ and $x_{n1} \wedge x_{n2} = 0$ for each $n \in N$,

(ii) if $1 < n \in N$, then $x_{n+1,1}$ and $x_{n+1,2}$ belong to the interval $[0, x_{n,2}]$ of G.

Since $G \neq \{0\}$, there is $0 < h \in G$. Because G satisfies the condition (*), there are elements x_{11} and x_{12} in G such that $0 < x_{1i} < h$ $(i = 1, 2)$ and $x_{11} \wedge x_{12} = 0$.

Assume that we have constructed x_{k1} and x_{k2} for $k = 1, 2, \ldots, n$ such that (i) is valid for $k = 1, 2, \ldots, n$ and (ii) is valid for $k = 1, 2, \ldots, n-1$. Put $h' = x_{n,2}$. According to (*) there are $x_{n+1,1}$ and $x_{n+1,2}$ in G such that $0 < x_{n+1,i} < h'$ for $i = 1, 2$, and $x_{n+1,1} \wedge x_{n+1,2} = 0$. Thus (i) is valid for $k = 1, 2, \ldots, n+1$, and (ii) holds for $k = 1, 2, \ldots, n$.

In view of (i) and (ii) we infer that $\{x_{n1}\}$ $(n \in N)$ is an infinite disjoint system in G. □

Proof of (C): Let $\alpha \geqslant 2$. Put $A(\alpha) = G$. In view of 1.1, G satisfies the condition (*). Thus, according to 2.1, there is an infinite disjoint set in G. Now it follows from [9], Theorem 7.7 that

$$\operatorname{card} \operatorname{Conv} G = 2^{2^{\aleph_0}}.$$

Thus (C) holds. □

Lemma 2.2. *Let G and H be lattice ordered groups such that G is a homomorphic image of H. Let $n \in N$ and assume that there is a disjoint subset S_1 in G with $\operatorname{card} S_1 = n$. Then there exists a disjoint S_2 in H with $\operatorname{card} S_2 = n$.*

Proof. Without loss of generality we can suppose that there is an ℓ-ideal X in H such that $G = H/X$. For $h \in H$ we denote $\bar{h} = h + X$. Let us verify by induction that the following assertion $a(n)$ is valid for each $n \in N$:

$(a(n))$ If $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n$ is a disjoint subset of G, then there are elements $b_1, \ldots, b_n$ in H such that $b_i \in \bar{a}_i$ for $i = 1, 2, \ldots, n$, and $\{b_1, b_2, \ldots, b_n\}$ is a disjoint subset of H.

Let $n = 1$. Then $\bar{a}_1 > \bar{0}$, hence there is $0 < b_1 \in \bar{a}_1$, and $\{b_1\}$ is a disjoint subset of H.

Assume that the above assertion holds for some $n \in N$. Let $\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n+1}\}$ be a disjoint subset of G. Thus $\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n\}$ is disjoint subset of G as well; hence there exist $b'_i \in \bar{a}_i$ $(i = 1, 2, \ldots, n)$ such that $\{b'_1, b'_2, \ldots, b'_n\}$ is a disjoint subset of H.

We have $\bar{0} < \bar{a}_{n+1}$, hence there is $b'_{n+1} \in \bar{a}_{n+1}$ with $0 < b'_{n+1}$. For $i = 1, 2, \ldots, n$ we put

$$c_i = b'_i \wedge b'_{n+1}, \quad b_i = b'_i - c_i.$$

Then $c_i \in X$ for $i = 1, 2, \ldots, n$. Next, if $b_i = 0$ for some $i \in \{1, 2, \ldots, n\}$ then $b'_i \in \bar{0}$, which is a contradiction. Thus $b_i > 0$ for $i = 1, 2, \ldots, n$. Clearly $b_i \in \bar{a}_i$ for $i = 1, 2, \ldots, n$.

Denote

$$c = c_1 \vee c_2 \vee \ldots \vee c_n, \quad b_{n+1} = b'_{n+1} - c.$$

We have $0 \leqslant c \leqslant b'_{n+1}$, hence $0 \leqslant b_{n+1}$. Clearly $c \in C$. Thus $b_{n+1} \in \bar{a}_{n+1}$. If $b_{n+1} = 0$, then $b'_{n+1} \in \bar{0}$, which is impossible; therefore $b_{n+1} > 0$.

Now from the relation $b_i \leqslant b'_i$ for $i = 1, 2, \ldots, n$ we infer that $\{b_1, b_2, \ldots, b_n\}$ is a disjoint of H. Let $i \in \{1, 2, \ldots, n\}$. Then

$$\begin{aligned} 0 \leqslant b_i \wedge b_{n+1} &= b_i \wedge (b'_{n+1} - c) \leqslant b_i \wedge (b'_{n+1} - c_i) = \\ &= (b'_i - c_i) \wedge (b'_{n+1} - c_i) = (b'_i \wedge b'_{n+1}) - c_i = 0. \end{aligned}$$

Thus $b_i \wedge b_{n+1} = 0$. Therefore $\{b_1, b_2, \ldots, b_{n+1}\}$ is a disjoint subset of H. This completes the proof of the lemma. □

Lemma 2.3. *Let $\alpha \geqslant 2$ and $n \in N$. Then there exists a disjoint set with n elements in $G(\alpha)$.*

Proof. We have already proved above that there is an infinite disjoint set in $A(\alpha)$. According to 1.2, $A(\alpha)$ is a homomorphic image of $G(\alpha)$. Hence in view of 2.2, there is a disjoint subset with n elements in $G(\alpha)$. □

Lemma 2.4. *Let $\alpha \geqslant 2$. Then there is an infinite disjoint subset in $G(\alpha)$.*

Proof. This is a consequence of 2.3 and of [6], Theorem 3.9.

The following lemma generalizes Theorem 7.3 of [9]. □

Lemma 2.5. *Let $\{b_n\}$ $(n \in N)$ be a disjoint subset of a lattice ordered group G. Then there exists $\beta \in \operatorname{Conv} G$ such that the sequence (b_n) belongs to β.*

P r o o f. By way of contradiction, suppose that there exists no β with the desired properties.

Thus (cf. [10], Theorem 2.2) there exist $k \in N$, $g, g_1, g_2, \ldots, g_k \in G$ and subsequences (y_n^m) $(m = 1, 2, \ldots, k)$ of the sequence (b_n) such that for each $n \in N$ the relation

$$0 < g \leqslant \sum_{m=1}^{k} (g_m + y_n^m - g_m) \tag{1}$$

is valid.

Assume that k is the least positive integer with the just mentioned property.

Since the sequence (b_n) is disjoint it follows that each its subsequence is disjoint and therefore for each $m = 1, 2, \ldots, k$ the sequence

$$(g_m + y_n^m - g_m)_{n \in N}$$

is disjoint as well. This implies that we cannot have $k = 1$; hence $k > 1$.

Consider the relation (1) for $n = 1$. Hence there are elements $h_1, h_2, \ldots, h_k$ in G^+ such that

$$g = h_1 + h_2 + \ldots + h_k \tag{2}$$

and

$$h_m \leqslant g_m + y_1^m - g_m \text{ for } m = 1, 2, \ldots, k. \tag{3}$$

In view of (2) there exists $m \in \{1, 2, \ldots, k\}$ such that $h_m > 0$; without loss of generality we can suppose that $m = 1$.

According to (3) we have

$$h_1 \wedge (g_1 + y_n^1 - g_1) = 0 \text{ for } n = 2, 3, \ldots. \tag{4}$$

From (1) we obtain

$$0 < h_1 \leqslant \sum_{m=1}^{k} (g_m + y_n^m - g_m) \tag{5}$$

for each $n \in N$; let us consider the relation (5) for $n \geqslant 2$. By applying (4) we get

$$0 < h_1 \leqslant \sum_{m=2}^{k} (g_m + y_n^m - g_m) \text{ for each } n \geqslant 2.$$

In view of the minimality of k we have arrived at a contradiction. □

R e m a r k 2.5.1. The above lemma can be obtained also by applying [11], Section 6, Lemma 6.6. (In [11], Section 6 it is assumed that lattice oredered groups under consideration are abelian, but Lemma 6.6 is valid in the non-abelian case, too).

Corollary 2.6. *Let $\{b_n\}$ $(n \in N)$ be a disjoint subset of a lattice ordered group G. Then* $\operatorname{card} \operatorname{Conv} G > 1$.

P r o o f of (B): This is an immediate consequence of 2.4 and 2.6. □

Lemma 2.7. *Let G be a lattice ordered group and let $(x_n) \in (G^N)^+$ such that $x_n > 0$ for each $n \in N$. Assume that G satisfies the condition $(*)$. Then there are $(x'_n), (y_n), (z_n) \in (G^N)^+$ such that (x'_n) is a subsequence of (x_n), (z_n) is disjoint and $z_n \leqslant y_n \leqslant x'_n$ for each $n \in N$.*

P r o o f. We begin with the sequence $(x_n^1) = (x_n)$ and put $x'_1 = x_1 = y_1$. In view of $(*)$ there exist $a_1, a_2 \in G$ such that $0 < a_1$, $0 < a_2$, $a_1 \wedge a_2 = 0$ and $a_1, a_2 < y_1$. Put

$$N(1) = \{1 < n \in N : a_1 \wedge x_n^1 > 0\}.$$

Now we distinguish two cases.

a) Suppose that $N(1)$ is finite. Then we put $z_1 = a_1$, and in the next step we work with the sequence $(x_n^2) = (x_n^1)_{n \geqslant m}$, where m is the least positive integer such that $a_1 \wedge x_j = 0$ for each $j \geqslant m$. We set $x'_2 = x_m^1$.

b) Suppose that $N(1)$ is infinite. Then we put $z_1 = a_1$ and in the next step we work with the sequence $(x_n^2) = (a_1 \wedge x_n^1)_{1 < n \in N(1)}$. We set $x'_2 = x_2$.

By an obvious induction procedure we can verify that by repeating this process we obtain sequences (x'_n), (y_n) and (z_n) with the desired properties. □

Lemma 2.8. *Let G be a lattice ordered group and let $\beta \in \operatorname{Conv} G$, $\beta \neq \beta_0$. Assume that G satisfies the condition $(*)$. Then there exists a disjoint sequence in $(G^N)^+$ which belongs to β.*

P r o o f. Since $\beta \neq \beta_0$, there exists $(x_n) \in \beta$ such that $x_n > 0$ for each $n \in N$. Let (x'_n) and (z_n) be as in 2.7. Then (z_n) is disjoint and (x'_n) belongs to β. Since $z_n \leqslant x'_n$ for each $n \in N$, the sequence (z_n) belongs to β as well. □

Lemma 2.9. *Let G be an abelian lattice ordered group and let $\beta \in \operatorname{Conv} G$. Suppose that (u_n) and (v_n) are disjoint sequences belonging to β such that $u_n \wedge v_m = 0$ for each $n, m \in N$. Then there exist $\beta_1, \beta_2 \in \operatorname{Conv} G$ such that $(u_n) \in \beta_1$, $(v_n) \in \beta_2$, $\beta_1 \neq \beta_2$ and $\beta_1, \beta_2 < \beta$.*

P r o o f. This follows from [9], Theorem 7.3 and Corollary 7.6. □

P r o o f of (D): Let $\alpha \geqslant 2$. Put $A(\alpha) = G$. By way of contradiction, assume that there exists an atom β of Conv G. Thus there is $(x_n) \in \beta$ such that $x_n > 0$ for each $n \in N$. According to 1.1, G satisfies the condition (*). In view of 2.8 there exists a disjoint sequence (z_n) belonging to β. For each $n \in N$ we put $u_n = z_{2n-1}$, $v_n = z_{2n}$. Then $(u_n), (v_n) \in \beta$. Let β_1 and β_2 be as in 2.9. We have $\beta_0 < \beta_i < \beta$ for $i = 1, 2$; this contradicts the assumption that β is an atom in Conv G. □

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