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SOLDERED DOUBLE LINEAR MORPHISMS

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Summary. Our aim is to show a method od finding all natural transformations of a functor TT^* into itself. We use here the terminology introduced in [4, 5]. The notion of a soldered double linear morphism of soldered double vector spaces (fibrations) is defined. Differentiable maps $f\colon C_0 \to C_0$ commuting with TT^*-soldered automorphisms of a double vector space $C_0 = V^* \times V \times V^*$ are investigated. On the set $Z_s(C_0)$ of such mappings, appropriate partial operations are introduced. The natural transformations $TT^* \to TT^*$ are bijectively related with the elements of $Z_s((TT^*)_0\mathbf{R}^n)$.

Keywords: Double vector space, double vector fibration, soldering, natural transformation

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1. $\mathcal{DL}$-SPACES (FIBRATIONS) WITH SOLDERING

As usual, let T denote the tangent functor; T is a lifting functor, i.e. a functor from the category of n-dimensional manifolds and their local diffeomorphisms into the category of fibred manifolds and morphisms. Similarly, the construction of a cotangent bundle and cotangent map can be interpreted as a covariant lifting functor, [2]. Further, TT, TT^*, T^*T, and T^*T^* are second order lifting functors, [2].

In [4, 5], double vector spaces ($\mathcal{DL}$-spaces), double vector fibrations and their morphisms were studied. For example, the tangent bundle TE of a vector bundle E has the structure of a double vector fibration. Other important examples are the cotangent bundle T^*E and the spaces TTM, TT^*M, T^*TM and T^*T^*M of a smooth manifold M.

The Cartesian product $C^0 = A \times B \times V$ of three finite-dimensional vector spaces can be regarded as a trivial double vector space $A \times B \times V \to A \times B$. Its $\mathcal{DL}$-automorphisms group $\mathrm{Aut}(C^0)$ is identified with $\mathrm{Aut}(A) \times \mathrm{Aut}(B) \times \mathrm{Aut}(V) \times \mathrm{Hom}(A \times B, V)$ where $\mathrm{Hom}(A \times B, V)$ denotes the vector space of all bilinear maps of $A \times B$ to V, [4]. Further, any $\mathcal{DL}$-space C is $\mathcal{DL}$-isomorphic with a suitable trivial

$\mathcal{DL}$-space C° (of the same dimension). Consequently, any automorphism $\varphi \in \mathrm{Aut}(C)$ can be written as a quadruple $(\varphi_1, \varphi_2, \varphi_3, \sigma)$.

J. Pradines introduced a 1-soldering of a $\mathcal{DL}$-object C as a linear isomorphism $\sigma_C : A \to V$, and a 1-soldered morphism $\varphi : C \to C'$ as a $\mathcal{DL}$-morphism satisfying $\varphi_3 \sigma_C = \sigma_{C'} \varphi_1$, [3, 1]. For our purpose, given a $\mathcal{DL}$-space C, $\pi : C \to A \times B$, we define

Definition 1. We say that C is a $\mathcal{DL}$-space with a
TE-soldering
or T^*E-soldering
or TT-soldering
or TT^*-soldering
or T^*T-soldering,
if we are given an isomorphism (or isomorphisms)

$$\begin{array}{ll} \chi_1 : V \to A & \\ \text{or } \chi_3 : A \to B^* & \\ \text{or } \chi_1 : V \to A, & \chi_2 : V \to B \\ \text{or } \chi_1 : V \to A, & \chi_2 : V \to B^* \\ \text{or } \chi_1 : V \to A^*, & \chi_2 : V \to B, \text{ respectively.} \end{array}$$

A $\mathcal{DL}$-morphism $\varphi : C \to C'$ of two $\mathcal{DL}$-spaces with a TE-soldering (or T^*E-soldering etc.) will be called soldered (more precisely, TE-soldered etc.) if its underlying linear morphisms φ_1, φ_2, φ_3 satisfy

$$\begin{array}{ll} \chi'_1 \varphi_3 = \varphi_1 \chi_1 & \\ \text{or } \varphi_2^* \chi'_3 \varphi_1 = \chi_3 & \\ \text{or } \chi'_1 \varphi_3 = \varphi_1 \chi_1, & \chi'_2 \varphi_3 = \varphi_2 \chi_2 \\ \text{or } \chi'_1 \varphi_3 = \varphi_1 \chi_1, & \varphi_2^* \chi'_2 \varphi_3 = \chi_2 \\ \text{or } \varphi_1^* \chi'_1 \varphi_3 = \chi_1, & \chi'_2 \varphi_3 = \varphi_2 \chi_2, \text{ respectively.} \end{array}$$

In this way, we obtain a category of TE-soldered $\mathcal{DL}$-spaces and morphisms, etc. Obviously, TT- and TT^*-solderings are special cases of the TE-soldering, and the T^*T-soldering induces a T^*E-soldering.

Given a weak $\mathcal{DL}$-fibration $\mathfrak{C}$, [5], we say that $\mathfrak{C}$ is *TE-soldered* (or *T^*E-soldered*, etc.) if each fibre of $\mathfrak{C}$ is endowed with a TE-soldering (T^*E-soldering, etc.). Given two weak $\mathcal{DL}$-fibrations with a soldering of the same type, their morphism will be called *soldered* if its restriction to each fibre is a soldered $\mathcal{DL}$-morphism.

We say that a weak $\mathcal{DL}$-fibration $(\mathfrak{C}, p, M)$ with a soldering is a *soldered $\mathcal{DL}$-fibration* if there exists a $\mathcal{DL}$-space C with a soldering of the same type such that for $x \in M$, there exists an open neighborhood U of x and a soldered isomorphism of weak $\mathcal{DL}$-fibrations of the form $f : (\mathfrak{C}_U, p_U, U) \to (U \times C, pr_1, U)$ over identity.

Again, TE-soldered (or T^*E-soldered, etc) fibrations and their morphisms form a category.

A TE-soldering (T^*E-, or TT-, or TT^*-, or T^*T-soldering) of a $\mathcal{DL}$-fibration $(\mathfrak{C}, p, M)$ induces the following isomorphisms of the underlying fibrations:

$$\begin{array}{ll} \mathcal{X}_1 \colon \mathcal{V} \to \mathcal{A} & \\ \text{or } \mathcal{X}_3 \colon \mathcal{A} \to \mathcal{B} & \\ \text{or } \mathcal{X}_1 \colon \mathcal{V} \to \mathcal{A}, & \mathcal{X}_2 \colon \mathcal{V} \to \mathcal{B} \\ \text{or } \mathcal{X}_1 \colon \mathcal{V} \to \mathcal{A}, & \mathcal{X}_2 \colon \mathcal{V} \to \mathcal{B}^* \\ \text{or } \mathcal{X}_2 \colon \mathcal{V} \to \mathcal{A}^*, & \mathcal{X}_2 \colon \mathcal{V} \to \mathcal{B}, \text{ respectively.} \end{array}$$

2. The TT^*-soldered $\mathcal{DL}$-space $C_0 \colon V^* \times V \times V^* \to V^* \times V$

We will consider a trivial $\mathcal{DL}$-space $C_0 = V^* \times V \times V^*$, $\pi \colon C_0 \to V^* \times V$ with a TT^*-soldering $\chi_1 = id$, $\chi_2 = id$. Its $\mathcal{DL}$-automorphism $(\varphi_1, \varphi_2, \varphi_3, \sigma)$ is soldered if and only if

$$\varphi_1 = \varphi_2^{*-1} = \varphi_3.$$

Our main goal is to investigate differentiable maps $f \colon C_0 \to C_0$ which commute with all TT^*-soldered automorphisms of C_0. First, let us make some preliminary considerations.

Given a continuous $f \colon V^* \times V \to V^*$ such that

(1) $\quad \varphi^{*-1} f(a,v) = f(\varphi^{*-1}(a), \varphi(v))$ for any $\varphi \in \mathrm{Aut}(V)$, $a \in V^*$, $v \in V$,

it can be proved:

Lemma 1. *Let $a \in V^*$, $a \neq 0$; $v \in V$, $v \neq 0$. Then there exists a real number $\lambda(a,v)$ such that $f(a,v) = \lambda(a,v).a$.*

Proof. If $\langle v, a\rangle \neq 0$, choose a basis $\{v_1, \dots, v_m\}$ in V such that $v_1 = \frac{1}{\langle v,a\rangle} v$, $v_1^* = a$. Then $f(a,v) = \sum_{k=1}^{m} f_k(a,v).v_k^*$ where $\{v_1^*, \dots, v_m^*\}$ is a dual basis. Setting

$$\varphi(v_1) = v_1, \qquad \varphi(v_k) = -v_k \text{ for } k \geqslant 2,$$

(1) yields $f(a,v) = f_1(a,v).a$. In the case $\langle v, a\rangle = 0$, let us choose a basis with $v_2 = v$, $v_1^* = a$, and

(2) $\quad \varphi \in \mathrm{Aut}(V)$ with $\varphi(v_1) = v_1$, $\varphi(v_2) = v_2$, $\varphi(v_k) = -v_k$ for $k \geqslant 3$.

By (1), $f(a,v) = f_1(a,v).v_1^* + f_2(a,v).v_2^*$. Let $\varphi' \in \mathrm{Aut}(V)$ be given by $\varphi'(v_k) = v_k$ for $k \neq 2$, $\varphi'(v_2) = \varepsilon v_2$ with $\varepsilon \neq 0$. An application of (1) and the previous equality yields $\varepsilon^{-1} f_2(a,v) = f_2(a, \varepsilon v)$. By continuity of f, there exists $\lim_{\varepsilon \to 0} f_2(a, \varepsilon v) = f_2(a,v)$. Thus there exists also $\lim_{\varepsilon \to 0} \varepsilon^{-1} f_2(a,v)$, which implies $f_2(a,v) = 0$. In both cases, $\lambda(a,v) = f_1(a,v)$. $\square$

Lemma 2. *Let a, $a' \in V^* - \{0\}$, v, $v' \in V - \{0\}$. There exists $\varphi \in \operatorname{Aut}(V)$ satisfying $\varphi^{*-1}(a) = a'$, $\varphi(v) = v'$ if and only if $\langle v, a\rangle = \langle v', a'\rangle$.*

Lemma 3. *There exists a unique continuous function $\xi\colon \mathbf{R} \to \mathbf{R}$ such that $f(a, v) = \xi(\langle v, a\rangle).a$ for any $a \in V^*$, $v \in V$. If f is differentiable, then ξ is also differentiable.*

Now assume a fixed continuous $f\colon V^* \times V \times V^* \to V^*$ such that

$$\varphi^{*-1} f(a, v, b) = f(\varphi^{*-1}(a), \varphi(v), \varphi^{*-1}(b)) \tag{3}$$

for any $\varphi \in \operatorname{Aut}(V)$, $a, b \in V^*$, $v \in V$. Suppose $\dim V \geqslant 2$.

Lemma 4. *Given two linearly independent forms $a, b \in V^*$, and $v \in V$, there exist uniquely determined real numbers $\lambda(a, v, b)$, $\mu(a, v, b)$ such that*

$$f(a, v, b) = \lambda(a, v, b).a + \mu(a, v, b).b.$$

Proof. Suppose $\langle v, a\rangle \neq 0$ or $\langle v, b\rangle \neq 0$, and choose a basis with $v_1^* = a$, $v_2^* = b$, $\langle v, v_k\rangle = 0$ for $k \geqslant 3$. Then $v = \alpha v_1 + \beta v_2$ where α, $\beta \in \mathbf{R}$, $\alpha \neq 0$, $\beta \neq 0$. We can write $f(v_1^*, \alpha v_1 + \beta v_2, v_2^*) = \sum_{k=1}^{m} f_k(v_1^*, \alpha v_1 + \beta v_2, v_2^*).v_k^*$. Using (2) and (3) we obtain

$$f_k(v_1^*, \alpha v_1 + \beta v_2, v_2^*) = 0 \quad \text{for} \quad k \geqslant 3.$$

By continuity, the numbers

$$\lambda(a, v, b) = f_1(v_1^*, \alpha v_1 + \beta v_2, v_2^*) \quad \text{and} \quad \mu(a, v, b) = f_2(v_1^*, \alpha v_1 + \beta v_2, v_2^*)$$

satisfy the above equality even in the case $\langle v, a\rangle = \langle v, b\rangle = 0$. Since a, b are independent, λ and μ are unique. □

Lemma 5. *Let $U \subset V^* \times V \times V^*$ denote an open subset consisting of all triples (a, v, b) such that a, b are independent. There exist uniquely determined continuous functions λ, $\mu\colon U \to \mathbf{R}$ such that for any two independent a, $b \in V^*$ and any $v \in V$ we have*

$$f(a, v, b) = \lambda(a, v, b).a + \mu(a, v, b).b.$$

Lemma 6. *Let a, b and a', b' be two couples of linearly independent forms, and v, $v' \in V$. There exists $\varphi \in \operatorname{Aut}(V)$ such that*

$$\varphi^{*-1}(a) = a', \qquad \varphi(v) = v', \qquad \varphi^{*-1}(b) = b'$$

if and only if

$$\langle v, a\rangle = \langle v', a'\rangle \quad \text{and} \quad \langle v, b\rangle = \langle v', b'\rangle.$$

Lemma 7. *There are uniquely determined continuous functions ξ, $\eta\colon \mathbf{R}^2 \to \mathbf{R}$ such that for any a, $b \in V^*$ independent and $v \in V$, we have*

$$f(a, v, b) = \xi(\langle v, a\rangle, \langle v, b\rangle).a + \eta(\langle v, a\rangle, \langle v, b\rangle).b. \tag{4}$$

Proposition 1. *Let* $\dim V \geqslant 2$. *Then there are unique continuous functions ξ, $\eta\colon \mathbf{R}^2 \to \mathbf{R}$ such that for arbitrary two forms a, $b \in V^*$ and any $v \in V$, (4) is valid.*

The proof follows by the previous lemma and by continuity of f, ξ, η. If f is differentiable, we can find differentiable functions ξ, η. In the case $\dim V = 1$, Proposition 1 is not true. Nontheless, we prove:

Proposition 2. *Let* $\dim V = 1$. *Let $f\colon V^* \times V \times V^* \to V^*$ be a differentiable map satisfying (3). Then there exist (not unique) differentiable functions $\xi\colon \mathbf{R} \to \mathbf{R}$, $\eta\colon \mathbf{R}^2 \to \mathbf{R}$ such that for any a, $b \in V^*$ and $v \in V$ we have*

$$f(a, v, b) = \xi(\langle v, a\rangle).a + \eta(\langle v, a\rangle, \langle v, b\rangle).b.$$

Proof. We can suppose $V = \mathbf{R}$. A map $f(-,-,0)\colon V^* \times V \to V^*$ satisfies (1). By Lemma 3, there is a differentiable function $\xi\colon \mathbf{R} \to \mathbf{R}$ such that $f(a, v, 0) = \xi(\langle v, a\rangle).a$ for any $a \in V^*$, $v \in V$. Let a map $g\colon V^* \times V \times V^* \to V^*$ be given by

$$g(a, v, b) = f(a, v, b) - \xi(\langle v, a\rangle).a.$$

Clearly, g satisfies (3) and $g(a, v, 0) = 0$. There exists a differentiable function $\mu'\colon V^* \times V \times (V^* - 0) \to \mathbf{R}$ such that for any $a \in V^*$, $v \in V$, $b \in V^* - \{0\}$, we have $g(a, v, b) = \mu'(a, v, b).b$. Let us define $\mu\colon V^* \times V \times V^* \to \mathbf{R}$ as follows:

$$\mu(a, v, b) = \mu'(a, v, b) \text{ for } b \neq 0, \quad \mu(a, v, 0) = \frac{\partial g(a, v, 0)}{\partial b},$$

where μ is differentiable and $g(a, v, b) = \mu(a, v, b).b$ for a, $b \in V^*$, $v \in V$. If $b \neq 0$, then $\mu(a, v, b) = \mu({\varphi^*}^{-1}(a), \varphi(v), {\varphi^*}^{-1}(b))$ for any $\varphi \in \mathrm{Aut}(V)$. Since μ is continuous, the equality holds even for $b = 0$. It can be verified that there is a function $\eta\colon \mathbf{R}^2 \to \mathbf{R}$ such that

$$\mu(a, v, b) = \eta(\langle v, a\rangle, \langle v, b\rangle) \text{ for } a, b \in V^*, \quad v \in V - \{0\}.$$

If we choose a basis $\{v_1\}$ of V we have $\eta(x, y) = \mu(xv_1^*, v_1, yv_1^*)$. Therefore μ is differentiable. By continuity of μ as well as η, the above equality holds even in the case $v = 0$. □

In the next part, consider a continuous map $f\colon V^* \times V \times V^* \to V$ such that

$$\varphi f(a,v,b) = f(\varphi^{*-1}(a), \varphi(v), \varphi^{*-1}(b)). \tag{5}$$

Lemma 8. *Assume a, $b \in V^*$, $v \in V - \{0\}$. Then there is a single real number $\lambda(a,v,b)$ such that $f(a,v,b) = \lambda(a,v,b).v$.*

Proof. (a) First let $\langle v,a\rangle \neq 0$, $\langle v,b\rangle \neq 0$. If a, b are independent, then there is a basis in V such that

$$a = \langle v,a\rangle.v_1^*, \quad v = v_1, \quad b = \langle v,b\rangle.v_1^* + v_2^*.$$

In the expression of f with respect to the basis, $f_k(a,v,b) = 0$ for $k \geqslant 3$. This follows by (5) if we use φ introduced in (2). Choose

$$\varphi' \in \operatorname{Aut}(V)\colon \quad \varphi'(v_k) = v_k \ \text{ for } k \neq 2, \qquad \varphi'(v_2) = \varepsilon^{-1} v_2 \tag{6}$$

where $\varepsilon \neq 0$. By (5),

$$\varepsilon^{-1} f_2(a,v,b) + f_2(a,v,\langle v,b\rangle v_1^* + \varepsilon v_2^*).$$

Since $\lim\limits_{\varepsilon\to 0} f_2(a,v,\langle v,b\rangle v_1^* + \varepsilon v_2^*) = f_2(a,v,\langle v,b\rangle v_1^*)$ we have $f_2(a,v,b) = 0$. Therefore

$$f(a,v,b) = f_1(a,v,b).v, \qquad \lambda(a,v,b) = f_1(a,v,b). \tag{7}$$

If a, b are linearly dependent then there is a basis $\{v_1,\dots,v_m\}$ of V with $a = \langle v,a\rangle v_1^*$, $v = v_1$, $b = \langle v,b\rangle v_1^*$. Choose $\varphi(v_1) = v_1$, $\varphi(v_k) = -v_k$ for $k \geqslant 2$. The condition (5) gives $f_k(a,v,b) = 0$ for $k \geqslant 2$, $f(a,v,b) = f_1(a,v,b).v$ as above.

(b) Assume $\langle v,a\rangle \neq 0$, $\langle v,b\rangle = 0$. The symmetric case is similar. If $b = 0$ we can proceed as above. If $b \neq 0$ we choose a basis with $a = \langle v,a\rangle v_1^*$, $v = v_1$, $b = v_2^*$. We obtain $f_k(a,v,b) = 0$ for $k \geqslant 3$. Using (6), (5) gives $\varepsilon^{-1} f_2(a,v,b) = f_2(a,v,\varepsilon b)$ and consequently, $f_2(a,v,b) = 0$, i.e. λ is given by (7).

(c) Let $\langle a,v\rangle = \langle b,v\rangle = 0$. If a, b are independent we choose $\{v_1,\dots,v_m\}$ such that $a = v_2^*$, $v = v_1$, $b = v_3^*$. We obtain $f_k(a,v,b) = 0$ for $k \geqslant 4$; an automorphism φ given by $\varphi(v_k) = v_k$ for $k \neq 2,3$, $\varphi(v_2) = \varepsilon^{-1} v_2$, $\varphi(v_3) = \varepsilon^{-1} v_3$, $\varepsilon \neq 0$ yields $f_2(a,v,b) = f_3(a,v,b) = 0$. If a, b are dependent, $a \neq 0$ we use a basis with $a = v_2^*$, $v = v_1$, $b = \alpha v_2^*$, $\alpha \in \mathbf{R}$. Similarly for $b \neq 0$. The case $a = b = 0$ is clear. $\square$

Lemma 9. *Let* $\dim V \geqslant 3$. *There exists a unique continuous function $\vartheta\colon \mathbf{R}^2 \to \mathbf{R}$ such that for any two independent forms a, b and $v \neq 0$ we have*

$$f(a,v,b) = \vartheta(\langle v,a\rangle, \langle v,b\rangle).v. \tag{8}$$

Proof. By Lemma 6, 7 there exists a uniquely determined function $\vartheta\colon \mathbf{R}^2 \to \mathbf{R}$ such that for any two independent forms a, b and $v \neq 0$, (8) is true. In an arbitrary basis $\{v_1,\dots,v_m\}$ we have $\vartheta(x,y) = \lambda(xv_1^* + v_2^*, v_1, yv_1^* + v_3^*)$. The function λ described in Lemma 7 is continuous on its domain, hence ϑ is also continuous. $\square$

By continuity of f and ϑ we obtain

Proposition 3. *Let* $\dim V \geqslant 3$. *There exists a unique continuous function* $\vartheta\colon \mathbf{R}^2 \to \mathbf{R}$ *such that for* a, $b \in V^*$ *and* $v \in V$, *(8) holds. If* f *is differentiable then* ϑ *is also differentiable.*

Lemma 10. *Let* $\dim V = 2$, $a,b,a',b' \in V^*$, $v, v' \in V - \{0\}$. *If* $\langle v, a\rangle = \langle v', a'\rangle$ *and* $\langle v, b\rangle = \langle v', b'\rangle$ *then the corresponding real numbers introduced by Lemma 7 satisfy*

$$\lambda(a, v, b) = \lambda(a', v', b').$$

The proof uses continuity of λ *and a suitable choice of a basis and* $\varphi \in \mathrm{Aut}(V)$.

Proposition 4. *Let* $\dim V = 2$. *Then there is a unique continuous function* $\vartheta\colon \mathbf{R}^2 \to \mathbf{R}$ *such that*

$$f(a, v, b) = \vartheta(\langle v, a\rangle, \langle v, b\rangle).v \quad \text{for } a, b \in V^*, \quad v \in V.$$

If f *is differentiable then* ϑ *is also differentiable.*

Proposition 5. *Let* $\dim V = 1$ *and let* $f\colon V^* \times V \times V^* \to V$ *be a differentiable map satisfying (5). Then there exists a differentiable* $\vartheta\colon \mathbf{R}^2 \to \mathbf{R}$ *such that* $f(a, v, b) = \vartheta(\langle v, a\rangle, \langle v, b\rangle).v$.

Proof. We can suppose $V = \mathbf{R}$ and use the canonical isomorphism $\mathbf{R} \simeq \mathbf{R}^*$. A map $f(-,-,0)\colon V^* \times V \to V$ satisfies the assumptions of Lemma 3. Thus there exists a diffferentiable function $\vartheta'\colon \mathbf{R} \to \mathbf{R}$ such that $f(a, v, 0) = \vartheta'(\langle v, a\rangle).v$. Now consider the map $g\colon V^* \times V \times V^* \to V$ given by $g(a, v, b) = f(a, v, b) - \vartheta'(\langle v, a\rangle).v$. Again, g satisfies (5). Moreover, $g(a, v, 0) = 0$. There exists a differentiable $\mu\colon V^* \times V \times V^* \to \mathbf{R}$ such that $g(a, v, b) = \mu(a, v, b).v$ for a, $b \in V^*$, $v \in V$. If $v \neq 0$ then $\mu(\varphi^{*-1}(a), \varphi(v), \varphi^{*-1}(b)) = \mu(a, v, b)$ for any $\varphi \in \mathrm{Aut}(V)$. Since μ is continuous this equality holds even if $v = 0$. Further, there exists a function $\vartheta''\colon \mathbf{R}^2 \to \mathbf{R}$ with $\mu(a, v, b) = \vartheta''(\langle v, a\rangle, \langle v, b\rangle)$ for any a, $b \in V^*$, $v \in V - \{0\}$. Evaluation in a basis of V shows that ϑ'' is differentiable. By continuity of μ and ϑ'', the above equality holds even if $v = 0$. Hence $f(a, v, b) = (\vartheta'\langle v, a\rangle + \vartheta''(\langle v, a\rangle, \langle v, b\rangle)).\,v$ for a, $b \in V^*$, $v \in V$. The uniqueness of the function $\vartheta = \vartheta' + \vartheta''$ is obvious. □

Definition 2. Let $\Phi = (\varphi^{*-1}, \varphi, \varphi^{*-1}, \sigma)$ be a TT^*-soldered $\mathcal{DL}$-automorphism of a trivial $\mathcal{DL}$-space $C_0 = V^* \times V \times V^*$. We say that a $\mathcal{DL}$-automorphism Φ is strongly soldered if the bilinear map $\sigma\colon V^* \times V \to V^*$ is φ-symmetric, i.e. if it satisfies

$$\langle v, \sigma(a, \varphi^{-1}(w))\rangle = \langle w, \sigma(a, \varphi^{-1}(v))\rangle \quad \text{for } v, w \in V, \quad a \in V^*. \tag{9}$$

Now let a continuous map $f\colon C_0 \to C_0$ satisfy

$$\Phi\, f = f\, \Phi \tag{10}$$

for any soldered (or strongly soldered) $\Phi \in \mathrm{Aut}(C_0)$. We write $f = (f_1, f_2, f_3)$, and Φ as above. An evaluation of (10) gives for any $\varphi \in \mathrm{Aut}(V)$ and any bilinear (or symmetric bilinear) map σ

$$\varphi^{*-1} f_1(a, v, b) = f_1(\varphi^{*-1}(a), \varphi(v), \varphi^{*-1}(b) + \sigma(a, v)), \tag{11}$$

$$\varphi^{*-1} f_2(a, v, b) = f_2(\varphi^{*-1}(a), \varphi(v), \varphi^{*-1}(b) + \sigma(a, v)), \tag{12}$$

$$\begin{aligned} \varphi^{*-1} f_3(a, v, b) + \sigma(f_1(a, v, b), f_2(a, v, b)) = \\ f_3(\varphi^{*-1}(a), \varphi(v), \varphi^{*-1}(b) + \sigma(a, v)). \end{aligned} \tag{13}$$

Suppose $\dim V \geqslant 2$. By Propositions 1,3,4 (setting $\sigma = 0$) there are uniquely determined continuous functions ξ, η, ϑ, ι, κ: $\mathbf{R}^2 \to \mathbf{R}$ such that for any a, $b \in V^*$, $v \in V$ we have

$$\begin{aligned} f_1(a, v, b) &= \xi(\langle v, a \rangle, \langle v, b \rangle).a + \eta(\langle v, a \rangle, \langle v, b \rangle).b, \\ f_2(a, v, b) &= \vartheta(\langle v, a \rangle, \langle v, b \rangle).v, \\ f_3(a, v, b) &= \iota(\langle v, a \rangle, \langle v, b \rangle).a + \kappa(\langle v, a \rangle, \langle v, b \rangle).b. \end{aligned}$$

The map $f = (f_1, f_2, f_3)$ satisfies (10) for any $\mathcal{DL}$-automorphism of the form $\Phi = (\varphi^{*-1}, \varphi, \varphi^{*-1}, 0)$. It remains to find out under what conditions f satisfies (10) if $\Phi = (1_V^{*-1}, 1_V, 1_V^{*-1}, \sigma)$ with σ an arbitrary (or 1_V-symmetric) bilinear map. By (11),

$$\begin{aligned} \xi(\langle v, a \rangle, \langle v, b \rangle).a + \eta(\langle v, a \rangle, \langle v, b \rangle).b = \xi(\langle v, a \rangle, \langle v, b \rangle + \langle v, \sigma(a, v) \rangle).a \\ + \eta(\langle v, a \rangle, \langle v, b \rangle + \langle v, \sigma(a, v) \rangle).b \\ + \eta(\langle v, a \rangle, \langle v, b \rangle \quad + \langle v, \sigma(a, v) \rangle).\sigma(a, v). \end{aligned}$$

If $a \neq 0$, $v \neq 0$ we can choose a 1_V-symmetric σ such that $\sigma(a, v) \neq 0$, $\langle v, \sigma(a, v) \rangle = 0$. Then $\eta(\langle v, a \rangle, \langle v, b \rangle) = 0$, and $\eta = 0$ by continuity. Now it is obvious that $\xi(x, y)$ does not depend on y. Therefore $f_1(a, v, b) = \xi(\langle v, a \rangle).a$. By (12), $\vartheta(x, y)$ is independent of y, i.e. $f_2(a, v, b) = \vartheta(\langle v, a \rangle).v$. Finally, by (13), $\kappa(x, y)$ and $\iota(x, y)$ are also independent of y, and $\kappa = \xi\vartheta$. Thus $f_3(a, v, b) = \iota(\langle v, a \rangle).a + \xi(\langle v, a \rangle)\vartheta(\langle v, a \rangle).b$. So we have proved

Proposition 6. *Let $\dim V \geqslant 2$. Continuous (or differentiable) maps $f: C_0 \to C_0$ which commute with all soldered (or strongly soldered) automorphisms of C_0 are precisely all maps of the form*

$$\begin{aligned} f_1(a, v, b) &= \xi(\langle v, a \rangle).a, \\ f_2(a, v, b) &= \vartheta(\langle v, a \rangle).v, \\ f_3(a, v, b) &= \iota(\langle v, a \rangle).a + \xi(\langle v, a \rangle)\vartheta(\langle v, a \rangle).b, \end{aligned}$$

where $\xi, \vartheta, \iota: \mathbf{R} \to \mathbf{R}$ are arbitrary continuous differentiable functions.

In the case $\dim V = 1$, the previous proposition holds in its differentiable version only. The proof uses the morphism $\Phi = (1_V^{*\,-1}, 1_V, 1_V^{*\,-1}, \varepsilon\sigma)$. Here any bilinear σ is 1_V-symmetric, $\varepsilon \neq 0$.

Definition 3. On the set $Z(C_0)$ of all differentiable maps of the $\mathcal{DL}$-space $C_0 = V^* \times V \times V^*$ into itself, the following partial operations may be introduced:
if $f, g \in Z(C_0)$ and $\pi_1 f = \pi_2 g$ we define $f \underset{1}{+} g$,
for $f, g \in Z(C_0)$ satisfying $\pi_2 f = \pi_2 g$ we define $f \underset{2}{+} g$,
if $f, g \in Z(C_0)$ with $g(C_0) \subset V^*$ we define $f + g$,
for $f, g \in Z(C_0)$ we define a composition fg.

Denote by $Z_s(C_0)$ (or $Z_{ss}(C_0)$) the set of all $f \in Z(C_0)$ satisfying (10) for any soldered (or strongly soldered, respectively) $\Phi \in \mathrm{Aut}(C_0)$; $Z_s(C_0) = Z_{ss}(C_0)$ is closed under the above operations. The previous results yield:

Theorem 1. *By means of* $+$, *the set* $Z_s(C_0) = Z_{ss}(C_0)$ *is generated by maps of the following form:*

$$(a, v, b) \mapsto \xi(\langle v, a\rangle) \underset{2}{\cdot} (\vartheta(\langle v, a\rangle) \underset{1}{\cdot} (a, v, b)), \tag{14}$$

where $\xi, \vartheta \colon \mathbf{R} \to \mathbf{R}$ *are arbitrary differentiable functions;*

$$(a, v, b) \mapsto (0, 0, \iota(\langle v, a\rangle).a), \tag{15}$$

where $\iota \colon \mathbf{R} \to \mathbf{R}$ *is differentiable.*

3. Natural transformations of TT^* into TT^*

Since any two of the functors TT^*, T^*T and T^*T^* are naturally equivalent, [2], it suffices to investigate any one of them. We choose TT^* here. The case TT is essentially different, [2, 6].

TT^* is a second order lifting functor. Moreover, it assigns to any differentiable manifold M a $\mathcal{DL}$-fibration TT^*M and to a diffeomorphism $\varphi\colon M \to N$ a $\mathcal{DL}$-isomorphism $TT^*(\varphi^{-1})\colon TT^*M \to TT^*N$. The underlying vector fibrations of TT^*M are $\mathcal{A} = T^*M$, $\mathcal{B} = TM$, $\mathcal{V} = T^*M$ with projections $\Pi_1\colon TT^*M \to \mathcal{A}$, $\Pi_2\colon TT^*M \to \mathcal{B}$ given as follows. If $X \in T_\omega(T^*M)$ we set $\Pi_1 X = \omega$, $\Pi_2 X = Tq(X)$ where $q\colon T^*M \to M$ is a natural projection. TT^*M has a natural structure of a $\mathcal{DL}$-fibration with TT^*-soldering (similar statements hold for TTM, T^*TM, T^*T^*M, TE or T^*E which explains the terminology introduced in Definition 1 where the case T^*T^* was omitted).

It is known that the natural transformations $F \to G$ of two r-th order lifting functors F, G are bijectively related with the L^r_n-equivariant maps $F_0\mathbf{R}^n \to G_0\mathbf{R}^n$ where $F_0\mathbf{R}^n = (F\mathbf{R}^n)_0$ denotes a fibre over the origin $0 \in \mathbf{R}^n$, and $L^r_n = \operatorname{inv} J^2_0(\mathbf{R}^n, \mathbf{R}^n)_0$ is the group of all invertible r-jets on $\mathbf{R}^n$ with source and target 0, [2].

In our case, $(TT^*)_0\mathbf{R}^n$ is canonically $\mathcal{DL}$-isomorphic with the trivial $\mathcal{DL}$-space $\mathbf{R}^{n*} \times \mathbf{R}^n \times \mathbf{R}^{n*}$ so we can identify them. The Taylor decomposition yields a bijection $L^2_n \to L^1_n \times \operatorname{Hom}_s(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n)$ where Hom_s denotes the vector space of symmetric bilinear maps. In fact, a local diffeomorphism $\alpha\colon \mathbf{R}^n \to \mathbf{R}^n$ with $\alpha(0) = 0$ may be written as

$$\alpha(x) = (T\alpha)_0 + \sigma_\alpha(x, x) + R(x)$$

in some neighborhood of 0 (σ_α is a symmetric bilinear form on $\mathbf{R}^n$, $\lim\limits_{x\to 0} \frac{R(x)}{\|x\|^2} = 0$). The above identification is given by $j^2_0\alpha \mapsto ((T\alpha)_0, \sigma_\alpha)$.

It can be verified that L^2_n is a semidirect product of L^1_n and a commutative group $\operatorname{Hom}_s(\mathbf{R}^n \times \mathbf{R}^n, \mathbf{R}^n)$.

A diffeomorphism α of $\mathbf{R}^n$ with $\alpha(0) = 0$ induces an automorphism $(TT^*)_0\alpha^{-1}$ of a $\mathcal{DL}$-space $\mathbf{R}^{n*} \times \mathbf{R}^n \times \mathbf{R}^{n*}$, $(TT^*)_0\alpha^{-1} = ((T_0\alpha)^{*-1}, T_0\alpha, (T_0\alpha)^{*-1}, \sigma)$ where $T_0\alpha$ is a tangent map (differential) at $0 \in \mathbf{R}^n$, $\sigma\colon \mathbf{R}^{n*} \times \mathbf{R}^n \to \mathbf{R}^{n*}$ is a bilinear map given by

$$\langle (T_0\alpha)^{-1}\delta(v', (T_0\alpha)^{-1}v), a\rangle = -\langle v, \sigma(a, v')\rangle \text{ for } v, v' \in \mathbf{R}^n,\ a \in \mathbf{R}^{n*}; \tag{16}$$

δ denotes the second differential of α at 0.

Lemma 11. *The bilinear map σ is $T_0\alpha$-symmetric.*

Consequently, $(TT^*)_0\alpha^{-1}$ is a strongly soldered $\mathcal{DL}$-automorphism depending on $j^2_0\alpha$ only. This enables us to define a map

$$\nu\colon L^2_n \to \operatorname*{Aut}_{o}(\mathbf{R}^{n*} \times \mathbf{R}^n \times \mathbf{R}^{n*}), \quad \nu(j^2_0\alpha) = ((T_0\alpha)^{*-1}, T_0\alpha, (T_0\alpha)^{-1}, \sigma)$$

where Aut_o denotes the group of strongly soldered automorphisms. If we use an expression of L^2_n as a semidirect product we can rewrite ν in the form $\nu(f, \delta) = (f^{*-1}, f, f^{*-1}, \sigma)$ where the bilinear maps δ, σ are related by the condition (16). Therefore ν is a group isomorphism.

Proposition 7. *There is a bijective correspondence between all natural transformations $TT^* \to TT^*$ and the elements of $Z_{ss}((TT^*)_0\mathbf{R}^n)$.*

Theorem 2. *By means of $+$, the set of all natural transformations of the functor TT^* into itself is generated by the transformations*

$$X \in T_a(T^*M) \mapsto \xi(\langle T_{q_M}X, a\rangle) \underset{2}{\cdot} (\vartheta(\langle T_{q_M}X, a\rangle) \underset{1}{\cdot} X) \tag{17}$$

where ξ, ϑ are arbitrary differentiable functions and $q_M : T^*M \to M$ is a natural projection

$$X \in T_a(T^*M) \mapsto \iota(\langle T_{q_M}X, a\rangle).e_M(a) \tag{18}$$

where ι is differentiable, $q_M(a) = x$, and $e_M : T^*_xM \to T_0(T^*_xM)$ means a canonical isomorphism.

By Proposition 7, it suffices to show that the transformations (17), (18) correspond to the generators (14), (15) of Theorem 1. The proof in local coordinates is straightforward.

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Souhrn

VÁZANÉ DVOJNĚ LINEÁRNÍ MORFISMY

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Cílem článku je prezentovat invariantní postup pro nalezení všech přirozených transformací funktoru TT^* do sebe. Užíváme zde terminologie zavedené v [4, 5]. Definujeme zde pojem dvojně lineárního morfismu dvojně lineárních vektorových prostorů resp. fibrací. Dále vyšetřujeme diferencovatelná zobrazení $f : C_0 \to C_0$, která komutují s TT^*-vázanými automorfismy dvojně vektorového prostoru $C_0 = V^* \times V \times V^*$. Na množině $Z_s(C_0)$ takových zobrazení jsou zavedeny potřebné parciální operace a jejich žitím je vhodně nagenerována množina $Z_s((TT^*)_0\mathbf{R}^n)$. Její prvky jsou ve vzájemně jednoznačné korespondenci s přirozenými transformacemi funktoru TT^* do sebe.

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