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# ON CERTAIN TYPES OF CONVERGENCES

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*Summary.* Mappings preserving Cauchy sequences and certain types of convergences connected with these mappings are investigated.

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Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces. A sequence $S$ in $X$ is a mapping of the set $\mathbb{N}$ of all positive integers into $X$. Let $F_X$ denote the set of all Cauchy sequences in $X$ and let $F(X, Y)$ be the set of all mappings $f\colon X \to Y$ preserving Cauchy sequences, i.e.

$$F(X,Y) \doteq \{f\colon X \to Y\colon S \in F_X \Rightarrow f \circ S \in F_Y\}.$$

R. F. Snipes has shown in [2] that every uniformly continuous mapping belongs to $F(X, Y)$ and every mapping from $F(X, Y)$ is continuous.

**Definition 1.** (See [2].) Sequences $S$ and $T$ in a metric space $(X, d_X)$ are called parallel (written $S \parallel T$) if for every positive $\varepsilon$ there is a positive integer $k$ such that $d_X(S(n),\, T(n)) < \varepsilon$ for $n \geqslant k$.

Sequences $S$ and $T$ are called equivalent (written $S \sim T$) if for every positive integer $k$ such that $d_X(S(m), T(n)) < \varepsilon$ for $m,\, n \geqslant k$.

We recall some properties of these notions from [2]. Let $S$, $T$ and $P$ be sequences in a metric space $(X, d_X)$. Then we have

(1) $S \parallel T \Leftrightarrow T \parallel S, \quad S \sim T \Leftrightarrow T \sim S;$

(2) $S \sim T \Rightarrow S \parallel T;$

(3) $S \sim T \Rightarrow S \in F_X;$

(4) $S \parallel T \& S \in F_X \Leftrightarrow S \sim T;$

(5) $(S \parallel T \& T \parallel P) \Rightarrow S \parallel P, \quad (S \sim T \& T \sim P) \Rightarrow S \sim P;$

(6) if $S \in F_X$ and $T$ is a subsequence of $S$, then $S \sim T$.

In [1] it is shown that the uniform limit of mappings from $F(X,Y)$ belongs to $F(X,Y)$. Example 1 shows that a similar assertion for quasiuniform, continuous and uniform on compacta convergences is false.

Example 1. Let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$, $Y = \langle 0, 1 \rangle$, both with usual metric. Let $f(\frac{1}{n}) = 0$ for $n$ even and $f(\frac{1}{n}) = 1$ for $n$ odd. Further, let for $k$ even

$$f_k\left(\frac{1}{n}\right) = f\left(\frac{1}{n}\right) \text{ for } n \leqslant k \text{ and } f_k\left(\frac{1}{n}\right) = 0 \text{ for } n > k$$

and let for $k$ odd

$$f_k\left(\frac{1}{n}\right) = f\left(\frac{1}{n}\right) \text{ for } n \leqslant k \text{ and } f_k\left(\frac{1}{n}\right) = 1 \text{ for } n > k.$$

Then we observe that $f_k \in F(X,Y)$ for all $k \in \mathbb{N}$, the sequence $(f_k)$ converges quasiuniformly, continuously and uniformly on compacta to $f$, however $f \notin F(X,Y)$.

Now we shall define certain convergences.

**Definition 2.** Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces and let $f$, $f_n : X \to Y$ be mappings ($n = 1, 2, \ldots$). The sequence $(f_n)$ we shall denote $F$, i.e. $F(n) = f_n$ for $n \in \mathbb{N}$. We shall denote $F \square S$ (for $S \in X^{\mathbb{N}}$) the sequence, the $n$-th member of which is $(F(n) \circ S)(n)$, i.e. $(F \square S)(n) = f_n(S(n))$.

Further, we shall denote

(A) $f_n \xrightarrow{A} f$, if $F$ converges uniformly to $f$;

(B) $f_n \xrightarrow{B} f$, if $S \in F_X$ implies $F \square S \parallel f \circ S$;

(C) $f_n \xrightarrow{C} f$, if $S \in F_X$ implies $F \square S \sim f \circ S$;

(D) $f_n \xrightarrow{D} f$, if $f \circ S \in F_Y$ (for $S \in X^{\mathbb{N}}$) implies $F \square S \parallel f \circ S$;

(E) $f_n \xrightarrow{E} f$, if $F$ converges uniformly on totally bounded sets to $f$;

(F) $f_n \xrightarrow{F} f$, if $F$ converges uniformly on countable totally bounded sets to $f$;

(G) $f_n \xrightarrow{G} f$, if $F$ converges pointwise to $f$ and $S \in F_X$ implies $F \square S \in F_Y$;

(H) $f_n \xrightarrow{H} f$, if $f \circ S \in F_Y$ (for $S \in X^{\mathbb{N}}$) implies $F \square S \sim f \circ S$;

(J) $f_n \xrightarrow{J} f$, if $F$ converges pointwise to $f$ and $f \circ S \in F_Y$ ( for $S \in X^{\mathbb{N}}$) implies $F \square S \in F_Y$.

R e m a r k 1. The assumption of pointwise convergence in the definition of $G$-convergence and $J$-convergence is needed. If namely $X = Y = (0,1 >$, with the usual metric, $f(x) = 1$ and $f_n(x) = \frac{1}{n}$ for all $x \in X$, then $F \Box S \in F_Y$ for every $S \in X^{\mathbb{N}}$, however $(f_n)$ does not converge.

Until further notice we shall assume that $(X, d_X)$ and $(Y, d_Y)$ are arbitrary metric spaces and the mappings belong to $Y^X$.

**Lemma 1.** *Every convergence $(A) - (J)$ implies pointwise convergence.*

P r o o f. The assertion is obvious for convergences $(A)$, $(E)$, $(F)$, $(G)$ and $(J)$. Let $f_n \xrightarrow{B} f$. The constant convergence $S$, $S(n) = x$ for $n \in \mathbb{N}$, is Cauchy in $X$, hence $F \Box S \parallel f \circ S$. This implies $d_Y(f_n(x), f(x)) \to 0$. Analogously for convergences $(C)$, $(D)$ and $(H)$. □

**Lemma 2.** *For convergences $(A)$–$(J)$ it holds: if $(f_n)$ converges to $f$, then every subsequence of $(f_n)$ converges to $f$ if the same sense, too.*

P r o o f. We shall prove the assertion only for $B$-convergence; proofs for the other convergences are similar. If $u\colon \mathbb{N} \to \mathbb{N}$ is an increasing mapping and $S \in X^{\mathbb{N}}$, we define $T \in X^{\mathbb{N}}$ as: $T(1) = T(2) = \ldots = T(u(1)) = S(1)$ and $T(u(n-1)+1) = \ldots = T(u(n)) = S(n)$ for $n > 1$. Then we have $T \circ u = S$ and $T \in F_X$ for $S \in F_X$. If $f_n \xrightarrow{B} f$ and $S \in F_X$, then $F \Box T \parallel f \circ T$ and hence also $(F \Box T) \circ u \parallel (f \circ T) \circ u$. Since $(F \Box T) \circ u = (F \circ u) \Box (T \circ u)$, we have $(F \circ u) \Box (T \circ u) \parallel f \circ (T \circ u)$ and $(F \circ u) \Box S \parallel f \circ S$. Therefore $f_{u(n)} \xrightarrow{B} f$. □

R e m a r k 2. Constant sequences converge for all these convergences but convergences $(C)$ and $(G)$ (for example, let $X = \{\frac{1}{n} : n \in \mathbb{N}\}$, $Y = \langle 0, 1 \rangle$ with the usual metric, $f(\frac{1}{n}) = 0$ for $n$ even, $f(\frac{1}{n}) = 1$ for $n$ odd and $f_k = f$ for all $k \in \mathbb{N}$).

**Theorem 1.** *The convergences $(D)$, $(H)$ and $(J)$ are equivalent.*

P r o o f. $(D) \Rightarrow (H)$: Let $f_n \xrightarrow{D} f$. Let $S \in X^{\mathbb{N}}$ and $f \circ S \in F_Y$. Then $F \Box S \parallel f \circ S$ and by (4) $F \Box S \sim f \circ S$, i.e. $f_n \xrightarrow{H} f$.

$(H) \Rightarrow (J)$: Let $f_n \xrightarrow{H} f$. Then by Lemma 1 $(f_n)$ converges pointwise to $f$. Let $S \in X^{\mathbb{N}}$ and $f \circ S \in F_Y$. Then $F \Box S \sim f \circ S$ and by (3) $F \Box S \in F_Y$, i.e. $f_n \xrightarrow{J} f$.

$(J) \Rightarrow (D)$: Let us assume that there are mappings $g_n$ $(n \in \mathbb{N})$ and $f$ such that $g_n \xrightarrow{J} f$ and $(g_n)$ does not $D$-converge to $f$. Therefore there is $P \in X^{\mathbb{N}}$ such that $f \circ P \in F_Y$, however sequences $G \Box P$ (where $G(n) = g_n$) and $f \circ P$ are not parallel. Therefore there is $\eta > 0$ and an increasing mapping $u\colon \mathbb{N} \to \mathbb{N}$ such that $d_Y(((G \circ u) \Box (P \circ u))(n), (f \circ P \circ u)(n)) \geqslant \eta$ for all $n \in \mathbb{N}$. Put $F = G \circ u$, $S = P \circ u$. Then $f \circ S \in F_Y$, by Lemma 2 $f_n \xrightarrow{J} f$ $(f_n = F(n))$ and

(7) $$d_Y(f_n(S(n)), f(S(n))) \geqslant \eta \quad \text{for all } n \in \mathbb{N}.$$

Since $f \circ S \in F_Y$, so $F \,\square\, S \in F_Y$. Therefore

$$(8) \qquad \exists n_1 \in \mathbf{N} \quad \forall i, m \geqslant n_1 \colon d_Y\big(f_i(S(i)), f_m(S(m))\big) < \frac{\eta}{8}.$$

Since $f_n \xrightarrow{J} f$, by definition $(f_n)$ converges pointwise to $f$. Therefore there is an increasing mapping $k \colon \mathbf{N} \to \mathbf{N}$ such that

$$(9) \qquad k(n) \geqslant k(n-1) + 2$$

and

$$(10) \qquad d_Y\big(f_{k(n)}(S(n)), f(S(n))\big) < \frac{\eta}{4}.$$

Define a sequence $T$ as follows:

$$(11) \qquad T(n) = S(p), \quad \text{if } n = k(p) \text{ and } T(n) = S(n) \text{ otherwise.}$$

Then $T \circ k = S$ and $f \circ T \in F_Y$. Hence $F \,\square\, T \in F_Y$. Therefore

$$(12) \qquad \exists n_2 \in \mathbf{N} \quad \forall i, m \geqslant n_2 \colon d_Y\big(f_i(T(i)), f_m(T(m))\big) < \frac{\eta}{8}.$$

By virtue of (9) there is $r \geqslant \max\{n_1, n_2\}$ such that $r \neq k(p)$ for each $p \in \mathbf{N}$. Therefore according to (11) $T(r) = S(r)$. Then for each $i \geqslant r$ in view of (12) and (8) we have

$$\begin{aligned}(13)\ d_Y\big(f_i(T(i)), f_i(S(i))\big) &\leqslant d_Y\big(f_i(T(i)), f_r(T(r))\big) + \\ &+ d_Y\big(f_r(T(r)), f_r(S(r))\big) + d_Y\big(f_r(S(r)), f_i(S(i))\big) < \\ &< \frac{\eta}{8} + \frac{\eta}{8} = \frac{\eta}{4}.\end{aligned}$$

Since $f \circ T \in F_Y$, so according to (6) $f \circ T \sim f \circ T \circ k$. Hence $f \circ T \sim f \circ S$ and thus

$$(14) \qquad \exists s \in \mathbf{N} \quad \forall i \geqslant s \colon d_Y\big(f(T(i)), f(S(i))\big) < \frac{\eta}{4}.$$

Now, let $t \in \mathbf{N}$ be such that $k(t) \geqslant \max\{r, s\}$. Then $(T \circ k)(t) = S(t)$ and according to (13), (10) and (14) we have

$$\begin{aligned} d_Y\big(f_{k(t)}(S(k(t))), f(S(k(t)))\big) &\leqslant d_Y\big(f_{k(t)}(S(k(t))), f_{k(t)}(S(t))\big) \\ &\quad + d_Y\big(f_{k(t)}(S(t)), f(S(t))\big) + d_Y\big(f(S(t)), f(S(d(t)))\big) \\ &< \frac{\eta}{4} + \frac{\eta}{4} + \frac{\eta}{4} < \eta.\end{aligned}$$

However, this contradicts (7). □

**Theorem 2.** *The convergences (B), (E) and (F) are equivalent.*

Proof. $(B) \Rightarrow (E)$: Let us assume that the assertion does not hold. Therefore there are a totally bounded set $M$ and mappings $h_n$ ($n = 1, 2, \dots$) and $f$ such that $h_n \xrightarrow{B} f$, however $(h_n)$ does not converge uniformly to $f$ on $M$. Thus there is $\varepsilon > 0$, a sequence $P$ in $M$ and an increasing mapping $v\colon \mathbf{N} \to \mathbf{N}$ such that

$$(15) \qquad d_Y\bigl(h_{v(n)}(P(n)), f(P(n))\bigr) \geqslant \varepsilon \quad \text{for all } n \in \mathbf{N}$$

Let us denote $G = H \circ v$, where $H(n) = h_n$. Then $d_Y\bigl(g_n(P(n)), f(P(n))\bigr) \geqslant \varepsilon$ for all $n \in \mathbf{N}$ and by Lemma 2 $g_n \xrightarrow{B} f$. The set $M$ is totally bounded, hence the sequence $P$ has a Cauchy subsequence. Therefore there is an increasing mapping $u\colon \mathbf{N} \to \mathbf{N}$ such that $P \circ u \in F_X$. We denote $S = P \circ u$, $F = G \circ u$. Then $S \in F_X$ and according to Lemma 2 $f_n \xrightarrow{B} f$ (where $f_n = F(n)$). According to (15) we have

$$(16) \qquad d_Y\bigl(f_n(S(n)), f(S(n))\bigr) \geqslant \varepsilon \quad \text{for all } n \in \mathbf{N}.$$

Since $S \in F_X$, we have from $B$-convergence $F \square S \parallel f \circ S$. Therefore there is $k \in \mathbf{N}$ such that for $n \geqslant k$ we have $d_Y\bigl(f_n(S(n)), f(S(n))\bigr) < \varepsilon$. However this contradicts (16). $(E) \Rightarrow (F)$: This is obvious. $(F) \Rightarrow (B)$: Let $f_n \xrightarrow{F} f$. Let $S \in F_X$ and $\varepsilon > 0$. Then the set $M = \{S(n) : n \in \mathbf{N}\}$ is countable and totally bounded. Thus $f_{n|M} \rightrightarrows f|_M$. Hence there is $k \in \mathbf{N}$ such that for $n \geqslant k$ we have $d_Y\bigl(f_n(S(n)), f(S(n))\bigr) < \varepsilon$ and therefore $F \square S \parallel f \circ S$, i.e. $f_n \xrightarrow{B} f$. □

**Lemma 3.** *If $f_n \xrightarrow{G} f$, then $f \in F(X,Y)$.*

Proof. Let $S \in F_X$. Since $f_n \xrightarrow{G} f$, so $(f_n)$ converges pointwise to $f$. Therefore there is an increasing mapping $k\colon \mathbf{N} \to \mathbf{N}$ such that $d_Y\bigl(f_{k(n)}(S(n)), f(S(n))\bigr) < \frac{1}{n}$ for all $n \in \mathbf{N}$. Therefore $(F \circ k) \square S \parallel f \circ S$. By Lemma 2 $f_{k(n)} \xrightarrow{G} f$, too. Therefore $(F \circ k) \square S \in F_Y$ and hence according to (4) $f \circ S \in F_Y$. Thus $f \in F(X,Y)$. □

**Lemma 4.** *If $f_n \xrightarrow{G} f$ then $f_n \xrightarrow{B} f$.*

Proof. Let us assume that the assertion does not hold. Therefore there are mappings $g_n (n \in \mathbf{N})$ and $f$ such that $g_n \xrightarrow{G} f$ but $(g_n)$ does not $B$-converge to $f$. Therefore there is a sequence $P \in F_X$ such that sequences $G \square P$ (where $G(n) = g_n$) and $f \circ P$ are not parallel. Thus there is $\delta > 0$ and an increasing mapping $u\colon \mathbf{N} \to \mathbf{N}$ such that

$$d_Y\bigl(((G \circ u) \square (P \circ u))(n), (f \circ P \circ u)(n)\bigr) \geqslant \delta \quad \text{for all } n \in \mathbf{N}.$$

Denote $F = G \circ u$, $S = P \circ u$. Then $S \in F_X$, by Lemma 2 $f_n \xrightarrow{G} f$ and

$$(17) \qquad d_Y\bigl(f_n(S(n)), f(S(n))\bigr) \geqslant \delta \quad \text{for all } n \in \mathbf{N}.$$

Since $f_n \xrightarrow{G} f$, so $(f_n)$ converges pointwise to $f$. Hence there is an increasing mapping $k\colon \mathbf{N} \to \mathbf{N}$ such that

$$k(n) \geqslant 2^n \tag{18}$$

and

$$d_Y\big(f_{k(n)}(S(n)), f(S(n))\big) < \frac{\delta}{4}. \tag{19}$$

Define a sequence $T$ as:

$$T(n) = S(p) \ \text{ if } \ n = k(p) \quad \text{and} \quad T(n) = S(n) \ \text{ otherwise.} \tag{20}$$

Then $T \circ k = S$. It is easy to see that $T \in F_X$. With respect to (18) and (20) there is an increasing mapping $j\colon \mathbf{N} \to \mathbf{N}$ such that $j(n) \neq k(p)$ for each $p$, $n \in \mathbf{N}$. Then $f_{j(n)}(T(j(n))) = f_{j(n)}(S(j(n)))$, therefore $(F \square T) \circ j = (F \square S) \circ j$. Since $S$, $T \in F_X$, so from $G$-convergence we have $F \square S \in F_Y$ and $F \square T \in F_Y$. Hence by (6) $(F \square T) \circ j \sim F \square T$, $(F \square S) \circ j \sim F \square S$ and by (5) $F \square T \sim F \square S$ and hence by (2) $F \square T \parallel F \square S$. Therefore there is $n_1 \in \mathbf{N}$ such that $d_Y\big(f_m(T(m)), f_m(S(m))\big) < \frac{\delta}{4}$ for $m \geqslant n_1$. Since $f_n \xrightarrow{G} f$, so according to Lemma 3 $f \in F(X,Y)$. Hence $f \circ T \in F_Y$. According to (6) $f \circ S = (f \circ T) \circ k \sim f \circ T$. Hence $f \circ S \parallel f \circ T$ and therefore there is $n_2 \in \mathbf{N}$ such that $d_Y\big(f(T(m)), f(S(m))\big) < \frac{\delta}{4}$ for $m \geqslant n_2$. Now, let $s \in \mathbf{N}$ be such that $k(s) \geqslant \max\{n_1, n_2\}$. Then

$$d_Y\big(f_{k(s)}(T(k(s))), f_{k(s)}(S(k(s)))\big) < \frac{\delta}{4},$$
$$d_Y\big(f(T(k(s))), f(S(k(s)))\big) < \frac{\delta}{4}.$$

Since $T(k(s)) = S(s)$, so by virtue of (19) we have

$$\begin{aligned} d_Y\big(f_{k(s)}(S(k(s))), f(S(k(s)))\big) &\leqslant d_Y\big(f_{k(s)}(S(k(s))), f_{k(s)}(S(s))\big) \\ &\quad + d_Y\big(f_{k(s)}(S(s)), f(S(s))\big) + d_Y\big(f(S(s)), f(S(k(s)))\big) \\ &< \frac{\delta}{4} + \frac{\delta}{4} + \frac{\delta}{4} < \delta. \end{aligned}$$

However this contradicts (17). □

**Lemma 5.** *Let $f \in F(X,Y)$ and let $f_n \xrightarrow{B} f$. Then $f_n \xrightarrow{C} f$.*

Proof. Let $S \in F_X$. Then $F \square S \parallel f \circ S$. Since $f \in F(X,Y)$, so $f \circ S \in F_Y$ and hence by (4) $F \square S \sim f \circ S$, i.e. $f_n \xrightarrow{C} f$. □

**Theorem 3.** *The convergences $(C)$ and $(G)$ are equivalent.*

Proof. $(C) \Rightarrow (G)$: Let $f_n \xrightarrow{C} f$. The pointwise convergence follows from Lemma 1. Let $S \in F_X$. Then $F \square S \sim f \circ S$ and hence by (3) $F \square S \in F_Y$, i.e. $f_n \xrightarrow{G} f$. $(G) \Rightarrow (C)$: This follows from Lemma 4, Lemma 3 and Lemma 5.

Therefore we have only four convergence (i.e. $(A)$, $(B)$, $(C)$, $(D)$). $\square$

**Lemma 6.** *If $f_n \xrightarrow{A} f$, then $f_n \xrightarrow{D} f$.*

Proof. Let $S \in X^{\mathbf{N}}$ and $f \circ S \in F_Y$. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$, there is $n_0 \in \mathbf{N}$ such that for $n \geqslant n_0$ and each $x \in X$: $d_Y\big(f_n(x), f(x)\big) < \varepsilon$. Therefore for each $n \geqslant n_0$ we have $d_Y\big(f_n(S(n)), f(S(n))\big) < \varepsilon$. Hence $F \square S \parallel f \circ S$ and $f_n \xrightarrow{D} f$. $\square$

**Lemma 7.** *If $f_n \xrightarrow{A} f$, then $f_n \xrightarrow{B} f$.*

Proof. It follows from Theorem 2. $\square$

**Theorem 4.** *Let $(X, d_X)$, $(Y, d_Y)$ be metric spaces. Then for convergences $(A)$, $(B)$, $(C)$, $(D)$ we have the following diagram:*

$$\begin{array}{ccc} A & \Rightarrow & D \\ \Downarrow & & \\ B & \Leftarrow & C \end{array}$$

Proof. It follows from Theorem 3, Lemmas 4, 6 and 7 and Examples 2, 3 and 4. $\square$

Example 2. Let $X = \mathbf{N}$, $Y = \langle 0, 1\rangle$ with the usual metric. Let $f_k(n) = 0$ for $n \leqslant k$, $f_k(n) = 1$ for $n > k$ and $f(n) = 0$ for all $n \in \mathbf{N}$. Then $Y$ is a totally bounded space, $f_k$, $f \in F(X, Y)$, $(f_k)$ converges to $f$ in the sense $(B)$ and $(C)$ and $(f_k)$ does not converge to $f$ in the sense $(A)$ and $(D)$.

Example 3. Let $X = \{\frac{1}{n} \colon n \in \mathbf{N}\}$, $Y = \langle 0, 1\rangle$ with the usual metric. Let $f(\frac{1}{n}) = 0$ for $n$ odd, $f(\frac{1}{n}) = 1$ for $n$ even and $f_k = f$ for all $k \in \mathbf{N}$. Then $X$ and $Y$ are totally bounded spaces, $(f_k)$ converges to $f$ in the sense $(A)$, $(B)$ and $(D)$ and $(f_k)$ does not converge to $f$ in the sense $(C)$.

Example 4. Let $X = \{\frac{1}{n} \colon n \in \mathbf{N}\}$, $Y = \mathbf{N}$ with usual metric. Let $f_k(\frac{1}{n}) = \min\{k, n\}$ and $f(\frac{1}{n}) = n$. Then $X$ is totally bounded space, $f_k \in F(X, Y)$ for all $k \in \mathbf{N}$, $f \notin F(X, Y)$, $(f_k)$ converges to $f$ in the sense $(D)$ and $(f_k)$ does not converge to $f$ in the sense $(A)$, $(B)$ and $(C)$.

From Theorems 2 and 4 and Examples 3 and 4 we get

**Theorem 5.** *Let $(X, d_X)$ be a totally bounded metric space and let $(Y, d_Y)$ be a metric space. Then we have*

$$\begin{array}{ccc} A & \Rightarrow & D \\ \Updownarrow & & \\ B & \Leftarrow & C \end{array}$$

**Lemma 8.** *Let $(Y, d_Y)$ be a totally bounded metric space. Then $D$-convergence implies $A$-convergence.*

P r o o f. Let us assume that the assertion does not hold. Therefore there are mappings $g_n$, $f$ such that $g_n \xrightarrow{D} f$, but $(g_n)$ does not uniformly converge to $f$. Thus there is $\varepsilon > 0$, $S \in X^{\mathbf{N}}$ and an increasing mapping $k \colon \mathbf{N} \to \mathbf{N}$ such that

$$d_Y\big(g_{k(n)}(S(n)), f(S(n))\big) \geqslant \varepsilon \quad \text{for all } n \in \mathbf{N}.$$

Denote $F = G \circ u$, where $G(n) = g_n$. Then we have

$$d_Y\big(f_n(S(n)), f(S(n))\big) \geqslant \varepsilon \quad \text{for all } n \in \mathbf{N}. \tag{21}$$

By Lemma 2 we have $f_n \xrightarrow{D} f$. Since $Y$ is a totally bounded space, there is and increasing mapping $m \colon \mathbf{N} \to \mathbf{N}$ such that $(f \circ S) \circ m \in F_Y$. Since $f_n \xrightarrow{D} f$, so according to Lemma 2 $f_{m(n)} \xrightarrow{D} f$. Hence $(F \circ m) \,\square\, (S \circ m) \parallel f \circ (S \circ m)$. Thus there is $n_0 \in \mathbf{N}$ such that

$$d_Y\big(f_{m(n)}(S(m(n))), f(S(m(n)))\big) < \varepsilon \quad \text{for } n \geqslant n_0.$$

However this contradicts (21). □

From Theorem 4, Lemma 8 and Examples 2 and 3 we obtain

**Theorem 6.** *Let $(Y, d_Y)$ be a totally bounded metric space. Then we have*

$$\begin{array}{ccc} A & \Leftrightarrow & D \\ & & \Downarrow \\ C & \Rightarrow & B \end{array}$$

From Theorems 5 and 6 and Example 3 we get

**Theorem 7.** *Let $(X, d_X)$ and $(Y, d_Y)$ be totally bounded metric spaces. Then we have*

$$\begin{array}{ccccc} & & C & & \\ & & \Downarrow & & \\ A & \Leftrightarrow & B & \Leftrightarrow & D \end{array}$$

Now we shall investigate the case when the mappings $f_n$ belong to $F(X,Y)$.

**Theorem 8.** *Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces. Let $f_n \in F(X,Y)$ for all $n \in \mathbb{N}$. If $(f_n)$ converges to $f$ in the sense $(A)$, $(B)$ or $(C)$, then $f \in F(X,Y)$.*

Proof. According to Theorems 2 and 4 it is sufficient to prove for $E$-convergence. Let $S \in F_X$. Then the set $M = \{S(n)\colon n \in \mathbb{N}\}$ is totally bounded, hence $f_{n|M} \rightrightarrows f_{|M}$. Since $f_{n|M} \in F(M,Y)$ we have by [1] $f_{|M} \in F(M,Y)$. Therefore $f \circ S = f_{|M} \circ S \in F_Y$ and $f \in F(X,Y)$. Example 4 shows that this assertion is not true for $D$-convergence. □

From theorems 8 and 6 we obtain

**Theorem 9.** *If $(Y,d_Y)$ is a totally bounded metric space, then the class $F(X,Y)$ is closed for all convergences $(A)$, $(B)$, $(C)$, $(D)$.*

By Theorems 4 and 8, Lemma 5 and Examples 2 and 4 we get

**Theorem 10.** *Let $f_n \in F(X,Y)$ for all $n \in \mathbb{N}$. Then we have*

$$\begin{array}{c} A \Rightarrow D \\ \Downarrow \\ C \Leftrightarrow B \end{array}$$

From Theorems 10 and 5 and Example 4 we obtain

**Theorem 11.** *Let $(X,d_X)$ be a totally bounded metric space. Let $f_n \in F(X,Y)$ for all $n \in \mathbb{N}$. Then we have*

$$\begin{array}{c} A \Leftrightarrow B \Leftrightarrow C \\ \Downarrow \\ D \end{array}$$

From Theorems 10 and 6 and Example 2 we get

**Theorem 12.** *Let $(Y,d_Y)$ be a totally bounded metric space. Let $f_n \in F(X,Y)$ for all $n \in \mathbb{N}$. Then we have*

$$\begin{array}{c} A \Leftrightarrow D \\ \Downarrow \\ C \Leftrightarrow B \end{array}$$

By Theorems 11 and 12 we obtain

**Theorem 13.** *Let $(X, d_X)$, $(Y, d_Y)$ be totally bounded metric spaces. Let $f_n \in F(X,Y)$ for all $n \in \mathbf{N}$. Then all convergences $(A)$, $(B)$, $(C)$ and $(D)$ are equivalent.*

**Lemma 9.** *Let $f \in F(X,Y)$ and $f_n \xrightarrow{D} f$. Then $f_n \xrightarrow{C} f$.*

P r o o f. Let $S \in F_X$. Then $f \circ S \in F_Y$, by Theorem 1 $f_n \xrightarrow{H} f$ and hence $F \square S \sim f \circ S$, i.e. $f_n \xrightarrow{C} f$. □

From Theorem 10, Lemma 9 and Examples 2 and 5 we obtain

**Theorem 14.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Then in the class $F(X,Y)$ we have*

$$\begin{array}{ccccc} A & \Rightarrow & D & \Rightarrow & B \\ & & & & \Updownarrow \\ & & & & C \end{array}$$

E x a m p l e 5. Let $X = \mathbf{N}$, $Y = \mathbf{N}$ with the usual metric. Let $f_k(n) = \min\{n, k\}$, $f(n) = n$. Then $f_k, f \in F(X,Y)$, $(f_k)$ converges to $f$ in the sense $(B)$, $(C)$ and $(D)$ and $(f_k)$ does not converge to $f$ in the sense $(A)$.

From Theorem 11 and 14 we obtain

**Theorem 15.** *Let $(X, d_X)$ be a totally bounded metric space, let $(Y, d_Y)$ be a metric space. Then in the class $F(X,Y)$ all convergences $(A)$, $(B)$, $(C)$ and $(D)$ are equivalent.*

R e m a r k 3. We remark that $C$-convergence implies continuous convergence. Example 1 shows that continuous convergence does not imply $C$-convergence. If $X$ is a complete metric space, then both convergences are equivalent. Further, $B$-convergence implies convergence on compacta. Example 1 shows that the contrary assertion is not true. If $X$ is a complete metric space, then both convergences are equivalent.

The proofs are not difficult.

*References*


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### Súhrn

## O ISTÝCH TYPOCH KONVERGENCIÍ

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V práci je vyšetrovaný vzájomný vzťah niekoľkých typov konvergencií súvisiacich s triedou zobrazení zachovávajúcich fundamentálnosť postupností.

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