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ON MAXIMAL OVERDETERMINED HARDY'S INEQUALITY OF SECOND ORDER ON A FINITE INTERVAL

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(Received December 1, 1998)

Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. A characterization of the weighted Hardy inequality

$$\| F u \|_2 \leq C \| F'' v \|_2, \quad F(0) = F'(0) = F(1) = F'(1) = 0$$

is given.

Keywords: weighted Hardy's inequality

MSC 1991: 26D10, 34B05, 46N20

INTRODUCTION

Let $I = [0, 1]$, $1 < p < \infty$, let $k \geq 1$ be an integer and let $AC_p^k$ denote the space of all functions on $I$ with absolutely continuous $(k - 1)$-th derivative $F^{(k-1)}(x)$ and such that

$$\| F \|_{AC_p^k} := \| F^{(k)} v \|_p < \infty, \quad F(0) = F'(0) = \ldots = F^{(k-1)}(0) = F(1) = \ldots = F^{(k-1)}(1) = 0,$$

where $v(x)$ is a locally integrable weight function and $\| g \|_p := (\int_0^1 |g(x)|^p \, dx)^{1/p}$.

1 The research work of the authors was partially supported by the Russian Fund of Basic Researches (Grant 97-01-00604) and the Ministry of Education of Russia (Grants 10.98GR and K-0560). The work of the second author was supported in part by INTAS Project 94-881.
We consider the characterization problem for the inequality

\[ \|Fv\|_q \leq C\|F^{(k)}v\|_p, \quad F \in AC_p^k. \]

The case \( k = 1 \) has been solved by P. Gurka [2] (see also [13]) and many works have been performed in this area by A. Kufner [6] and by A. Kufner with co-authors [1], [5], [7–10]. In particular, following Kufner’s terminology we call the inequality (1) “maximal overdetermined Hardy’s inequality”, that is when a function \( F \) and its derivatives vanish at both ends of the interval up to \((k - 1)\)-th order. A part of analysis related to the weighted Hardy inequality for functions vanishing at both ends of an interval was also given by G. Sinnamon [15] and the authors [11], [12]. In particular, the maximal inequality (1) on semiaxis was characterized in [11], [12].

The aim of the present paper is twofold. At first we prove an alternative version of (1) (see Theorem 1) and it allows, using the results of [4], to characterize the inequality (1), when \( p = q = 2, \ k = 2 \) (Theorem 3).

Without loss of generality we assume throughout the paper that the undeterminates of the form \( 0 \cdot \infty, 0/0, \infty/\infty \) are equal to zero.

AN ALTERNATE VERSION

Denote \( I_k f(x) \) and \( J_k f(x) \) the Riemann-Liouville operators of the form

\[ I_k f(x) = \frac{1}{\Gamma(k)} \int_0^x (x - y)^{k-1} f(y) \, dy, \quad x \in I, \]

\[ J_k f(x) = \frac{1}{\Gamma(k)} \int_x^1 (y - x)^{k-1} f(y) \, dy, \quad x \in I. \]

Then the maximal inequality (1) is equivalent either to

\[ \|(I_k f)u\|_q \leq C\|fv\|_p, \quad f \in P_{k-1}^\bot \]

or to

\[ \|(J_k f)u\|_q \leq C\|fv\|_p, \quad f \in P_{k-1}^\bot, \]

where \( P_{k-1} \) is the \( k \)-dimensional space of all polynomials \( q(t) = c_0 + c_1t + \ldots + c_{k-1}t^{k-1}, \ t \in I, \) and \( P_{k-1}^\bot \subset L_{p,v} := \{f: \|fv\|_p < \infty\} \) denotes the closed subspace of \( L_{p,v} \) of functions “orthogonal” to \( P_{k-1} \) in the sense that

\[ \int_0^1 f(x)g(x) \, dx = 0 \text{ for all } g \in P_{k-1}, \ f \in P_{k-1}^\bot. \]
In particular, \( f \in P_{k-1}^\perp \) if, and only if,
\[
\int_0^1 f(x) \, dx = \int_0^1 x f(x) \, dx = \ldots = \int_0^1 x^{k-1} f(x) \, dx = 0
\]
and, obviously,
\[
I_k f(x) = J_k f(x), \quad f \in P_{k-1}^\perp.
\]

We need the following

**Lemma 1.** ([14], Chapter 4, Exercise 19). Let \( X \) be a Banach space and \( Y \subseteq X \) the closed subspace. Let \( X^* \) be the dual space and
\[
Y^\perp = \{ \varphi \in X^* : \varphi(y) = 0 \text{ for all } y \in Y \}.
\]

Then
\[
\text{dist}(e, Y) := \inf_{y \in Y} \|e - y\|_X = \sup_{\varphi \in Y^\perp} \frac{|\varphi(e)|}{\|\varphi\|_{X^*}}
\]
for all \( e \notin Y \).

**Proof.** Let \( y \in Y, \varphi \in Y^\perp \). Then
\[
\varphi(e) = \varphi(e) - \varphi(y) = \varphi(e - y)
\]
and
\[
|\varphi(e)| = |\varphi(e - y)| \leq \|\varphi\|_{X^*} \|e - y\|.
\]
Consequently,
\[
\sup_{\varphi \in Y^\perp} \frac{|\varphi(e)|}{\|\varphi\|_{X^*}} \leq \|e - y\|
\]
and
\[
\sup_{\varphi \in Y^\perp} \frac{|\varphi(e)|}{\|\varphi\|_{X^*}} \leq \text{dist}(e, Y).
\]

Now suppose \( e \notin Y, y \in Y \). Then \( e - y \notin Y \) and by the Hahn-Banach theorem there exists \( \varphi \in X^* \) such that \( \varphi(y) = 0 \) for all \( y \in Y \), \( \|\varphi\|_{X^*} = 1 \) and \( \varphi(e - y) = \|e - y\| \). This implies that \( \varphi \in Y^\perp \) and
\[
|\varphi(e)| = |\varphi(e - y)| = \|e - y\| \geq \text{dist}(e, Y).
\]
Therefore,

\[
\sup_{\varphi \in Y} \frac{|\varphi(e)|}{\|\varphi\|_X} \geq \text{dist}(e,Y).
\]

Combining the estimates (5) and (6) we obtain (4). \qed

Put

\[
M_k(p,q) := \sup_{A \in C^b_{p,q} \setminus F \neq 0} \frac{\|F u\|_q}{\|F^{(k)} v\|_p}.
\]

Because of (2) and (3) we have

\[
M_k(p,q) = \sup_{f \in P_{k-1}^L} \frac{\|(J_k f) u\|_q}{\|f\|_p} = \sup_{f \in P_{k-1}^L} \frac{\|(I_k f) u\|_q}{\|f\|_p}.
\]

Denote \( p' = p/(p-1) \) and \( q' = q/(q-1) \) for \( 1 < p,q < \infty \) and observe that \( (L,p,v)^* = L_{p',1/v} \) if and only if \( v \in L_{p,1/\text{loc}} \) and \( 1/v \in L_{p',1/\text{loc}} \).

The following result gives an alternative version of the problems to characterize (1), (2), (3) and helps us to realise the desired solution for \( p = q = k = 2 \).

**Theorem 1.** Let \( 1 < p,q < \infty \) and the weight functions \( u \) and \( v \) be such that \( (L_{p,v})^* = L_{p',1/u} \), \( (L_{q,u})^* = L_{q',1/u} \). Then

\[
M_k(p,q) = \sup_{f \in L_{q',1/u}} \|f/u\|_{q'}^{-1} \text{dist} \left( I_k f, P_{k-1} \right).
\]

**Proof.** Applying Lemma 1 and the duality of \( L_{p,u} \) and \( L_{p',1/u} \), \( L_{q,u} \) and \( L_{q',1/u} \), \( J_k \) and \( I_k \), we write

\[
M_k(p,q) = \sup_{g \in P_{k-1}^L} \frac{\|(J_k g) u\|_q}{\|g v\|_p} = \sup_{g \in P_{k-1}^L} \sup_{f \in L_{q',1/u}} \|f/u\|_{q'} \frac{\left| \int_{0}^{1} (J_k g) f \right|}{\|f/u\|_{q'} \|g v\|_p} = \sup_{f \in L_{q',1/u}} \|f/u\|_{q'}^{-1} \sup_{g \in P_{k-1}^L} \|g v\|_p \left| \int_{0}^{1} (I_k f) g \right| = \sup_{f \in L_{q',1/u}} \|f/u\|_{q'}^{-1} \text{dist} \left( I_k f, P_{k-1} \right).
\]

\[\square\]

**Remark.** The equality (8) holds for \( J_k f \) instead of \( I_k f \).
The case $p = 2$

The implicit formulae (8) becomes clearer when $p = 2$. Let $d\mu(x) = |v(x)|^{-2} \, dx$ and

$$F_k(x) = I_k(fu)(x) = \frac{1}{\Gamma(k)} \int_0^x (x - y)^{k-1} f(y)u(y) \, dy.$$  

Then

$$\text{dist}_{L_{2,\mu}}(F_k, P_{k-1}) = \left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} |F_k(x) - F_{k,0} - \sum_{i=1}^{k-1} F_{k,i}\omega_i(x)|^2 \, d\mu(x) \right|^{1/2} \right)^{1/2},$$

where $L_{2,\mu} = \{ f : \| f \|_{2,\mu} := (\int_{\mathbb{R}} |f|^2 \, d\mu)^{1/2} < \infty \}$ and

$$F_{k,0} = \frac{1}{\mu(I)} \int_{I} F_k \, d\mu,$$

$$F_{k,i} = \frac{1}{\mu_i(I)} \int_{I} F_k \omega_i \, d\mu, \quad i = 1, \ldots, k - 1$$

and polynomials $\{\omega_i(x)\}$, $i = 1, \ldots, k - 1$, appear from the Gram-Schmidt orthogonalization process of $\{1, t, \ldots, t^{k-1}\}$ in $L_{2,\mu}$ (see [4], Lemma 2).

Observe, that if $p \neq 2$, $p \in (1, \infty)$ and $k = 1$, then

$$\left( \int_{I} |F_1 - F_{1,0}|^p \, d\mu_p \right)^{1/p} \leq \text{dist}_{L_{p,\mu}}(F_1, P_0) \leq 2 \left( \int_{I} |F_1 - F_{1,0}|^p \, d\mu_p \right)^{1/p},$$

(see [3]), where $d\mu_p(x) = |v(x)|^{-p} \, dx$.

Thus, for $p = 2$ the characterization problems of (1), (2) and (3) are equivalent to the following Poincaré-type inequality

$$\left\| F_k - F_{k,0} - \sum_{i=1}^{k-1} F_{k,i}\omega_i \right\|_{2,\mu} \leq C \| f \|_{q'}.$$
THE CASE \( k = 2 \)

We need the following notation. Let \( k > 1 \), \( 1 < p, q < \infty \), \( 1/r = 1/q - 1/p \) if \( 1 < q < p < \infty \). Put

\[
A_{k,0} = A_{k,0;(a,b),u,v} = \left\{ \right. \\
= \sup_{a < t < b} \left( \int_{a}^{b} (x-t)^{q(k-1)} |u(x)|^{q} \, dx \right)^{1/q} \left( \int_{a}^{b} |v|^{-p'} \right)^{1/p'}, \quad p \leq q \\
= \left( \int_{a}^{b} (x-t)^{q(k-1)} |u(x)|^{q} \, dx \right)^{r/q} \left( \int_{a}^{b} |v(t)|^{-p'} \, dt \right)^{1/r}, \quad p > q,
\]

\[
A_{k,1} = A_{k,1;(a,b),u,v} = \left\{ \right. \\
= \sup_{a < t < b} \left( \int_{a}^{b} |u|^{q} \right)^{1/q} \left( \int_{a}^{b} (t-x)^{p'(k-1)} |v(x)|^{-p'} \, dx \right)^{1/p'}, \quad p \leq q \\
= \left( \int_{a}^{b} (x-t)^{q(k-1)} |u(x)|^{q} \, dx \right)^{r/q} \left( \int_{a}^{b} |v(t)|^{-p'} \, dt \right)^{1/r}, \quad p > q,
\]

\[
B_{k,0} = B_{k,0;(a,b),u,v} = \left\{ \right. \\
= \sup_{a < t < b} \left( \int_{a}^{b} (x-t)^{q(k-1)} |u(x)|^{q} \, dx \right)^{1/q} \left( \int_{a}^{b} |v|^{-p'} \right)^{1/p'}, \quad p \leq q \\
= \left( \int_{a}^{b} (x-t)^{q(k-1)} |u(x)|^{q} \, dx \right)^{r/q} \left( \int_{a}^{b} |v(t)|^{-p'} \, dt \right)^{1/r}, \quad p > q,
\]

\[
B_{k,1} = B_{k,1;(a,b),u,v} = \left\{ \right. \\
= \sup_{a < t < b} \left( \int_{a}^{b} |u|^{q} \right)^{1/q} \left( \int_{a}^{b} (t-x)^{p'(k-1)} |v(x)|^{-p'} \, dx \right)^{1/p'}, \quad p \leq q \\
= \left( \int_{a}^{b} (x-t)^{q(k-1)} |u(x)|^{q} \, dx \right)^{r/q} \left( \int_{a}^{b} |v(t)|^{-p'} \, dt \right)^{1/r}, \quad p > q,
\]

\[
A_{k} = A_{k;(a,b),u,v} = \max(A_{k,0}, A_{k,1}) , \\
B_{k} = B_{k;(a,b),u,v} = \max(B_{k,0}, B_{k,1}).
\]

The constants \( A_{k} \) and \( B_{k} \) are equivalent to the norms of the Riemann-Liouville operators \( I_{k} \) and \( J_{k} \), respectively, from \( L_{p,v}(a,b) \) into \( L_{q,u}(a,b) \) [16–17].

**Theorem 2.** Let \( 1 < p, q < \infty \), \( k = 2 \) and let the hypothesis of Theorem 1 be fulfilled. Then

\[
M_{2}(p, q) \leq \inf_{0 < \tau < \lambda < \sigma < 1} \left( A_{2; (0, \tau), u,v} + A_{1; (\tau, \lambda), u,(x-\tau)^{-1} v(x)} + B_{1; (\tau, \lambda), (x-\tau) u(x), v} + \right. \\
+ D_{\tau, \lambda}^{*} + D_{\tau, \lambda} + D_{2; (\tau, \sigma), u,v} + A_{1; (\lambda, \sigma), (\sigma-x) u(x), v} + \\
+ B_{1; (\lambda, \sigma), u,(\sigma-x)^{-1} v(x)} + D_{\lambda, \sigma} + D_{\lambda, \sigma}^{*}) ,
\]

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where

\[D_{\tau,\lambda} = \left( \int_{\tau}^{\lambda} |u|^q \right)^{1/q} \left( \int_{\tau}^{\lambda} (\tau - x)^{p'} |v(x)|^{-p'} \, dx \right)^{1/p'}, \]

\[D_{\lambda,\sigma} = \left( \int_{\lambda}^{\sigma} (\sigma - x)^q |u(x)|^q \, dx \right)^{1/q} \left( \int_{\lambda}^{\sigma} |v|^{-p'} \right)^{1/p'}, \]

\[D_{\tau,\lambda}^* = \left( \int_{\tau}^{\lambda} (x - \tau)^q |u(x)|^q \, dx \right)^{1/q} \left( \int_{\tau}^{1} (x - \tau)^{p'} |v(x)|^{-p'} \, dx \right)^{1/p'}, \]

\[D_{\lambda,\sigma}^* = \left( \int_{\lambda}^{\sigma} |u|^q \right)^{1/q} \left( \int_{\sigma}^{1} (x - \sigma)^{p'} |v(x)|^{-p'} \, dx \right)^{1/p'}. \]

**Proof.** If \( f \in P_1^A \), then for all \( x \in [0, 1] \) we have

\[(11) \quad I_2f(x) = J_2f(x). \]

Let \( \lambda \in (0, 1) \) and for any \( \tau \in (0, \lambda) \) and \( x \in (\tau, \lambda) \) we find

\[I_2f(x) = \int_0^x \left( \int_0^y f \right) \, ds = \int_0^\tau \left( \int_0^y f \right) \, ds + \int_\tau^x \left( \int_0^y f \right) \, ds \]

\[= \int_0^\tau (\tau - y) f(y) \, dy \quad \int_\tau^x \left( \int_0^y f \right) \, ds \]

\[= \int_0^\tau (\tau - y) f(y) \, dy \quad \int_\tau^x f(y) \left( \int_y^\lambda ds \right) \, dy \]

\[\quad - \int_x^\lambda f(y) \left( \int_y^\tau ds \right) \, dy \quad \int_\tau^\lambda f(y) \left( \int_\tau^y ds \right) \, dy \]

\[= \int_0^\tau (\tau - y) f(y) \, dy \quad \int_\tau^\lambda (y - \tau) f(y) \, dy \]

\[\quad - (x - \tau) \int_x^\lambda f \quad (x - \tau) \int_\lambda^1 f. \]

Analogously, with \( \sigma \in (\lambda, 1) \) for \( x \in (\lambda, \sigma) \) we write

\[I_2f(x) = J_2f(x) = \int_x^1 \left( \int_0^1 f \right) \, ds \]

\[= \int_x^\lambda \left( \int_0^1 f \right) \, ds + \int_\lambda^1 \left( \int_0^1 f \right) \, ds \]

\[= \int_x^\lambda (y - \sigma) f(y) \, dy - \int_0^\lambda (\sigma - y) f(y) \, dy \]

\[\quad - (\sigma - x) \int_x^\lambda f \quad (\sigma - x) \int_0^\lambda f. \]
Now we estimate the norm of each term on the right hand side. Using \([16-17]\) we obtain
\[
\| \chi_{[0,T]} (I_2 f) u \|_q \leq A_{2;0,T},u,v \| \chi_{[0,T]} f v \|_p \leq A_{2;0,T},u,v \| f v \|_p.
\]
Plainly
\[
\| \chi_{[\tau,\lambda]} (I_2 f) u \|_q \leq \| \chi_{[\tau,\lambda]}(x)u(x) \int_0^\tau (\tau - y)f(y) \, dy \|_q
\]
\[
+ \| \chi_{[\tau,\lambda]}(x)u(x) \int_0^\lambda (y - \tau)f(y) \, dy \|_q + \| \chi_{[\tau,\lambda]}(x)(x - \tau) \int_\tau^\lambda f \|_q
\]
\[
+ \| \chi_{[\tau,\lambda]}(x)(x - \tau) \int_\tau^\lambda f \|_q
\]
(we use the Hölder inequality for the first and the fourth term and the upper estimates which follow from the weighted Hardy inequalities \([13]\) for the second and the third term)
\[
\leq (D_{\tau,\lambda} + A_1(\tau,\lambda),u,(x-\tau)^{-1}v(x) + B_1(\tau,\lambda),(x-\tau)u(x),v + D^*_\tau,\lambda) \| f v \|_p.
\]
Similarly, applying \((11)\),
\[
\| \chi_{[\sigma,\tau]} (I_2 f) u \|_q \leq (D_{\sigma,\tau} + B_1(\sigma,\tau),u,(\sigma-x)^{-1}v(x) + A_1(\sigma,\tau),(\sigma-x)u(x),v + D_{\sigma,\tau}) \| f v \|_p.
\]
Finally we obtain
\[
\| (I_2 f) u \|_q \leq \| \chi_{[0,T]} (I_2 f) u \|_q + \| \chi_{[\tau,\lambda]} (I_2 f) u \|_q
\]
\[
+ \| \chi_{[\sigma,\tau]} (I_2 f) u \|_q + \| \chi_{[\sigma,\lambda]} (I_2 f) u \|_q
\]
\[
\leq (A_{2;0,T},u,v + D_{\tau,\lambda} + A_1(\tau,\lambda),u,(x-\tau)^{-1}v(x) + B_1(\tau,\lambda),(x-\tau)u(x),v
\]
\[
+ D^*_\tau,\lambda + D^*_\sigma,\tau + B_1(\sigma,\lambda),(\sigma-x)^{-1}v(x)
\]
\[
+ A_1(\sigma,\lambda),(\sigma-x)u(x),v + D_{\lambda,\sigma} + B_2(\sigma,\lambda),u,v) \| f v \|_p.
\]
Since \(\tau, \lambda\) and \(\sigma\) were arbitrary the upper bound \((10)\) of \(M_2(p,q)\) follows.

**Remark.** Theorem 2 gives the upper bound for \(M_k(p,q)\), when \(k = 2\). Obviously the similar upper estimates can be proved by the same method for \(k > 2\). We omit the details.

Denote \(\mathcal{E}\) the right hand side of \((10)\) when \(p = q = 2\). The following result brings the characterization of \((1)\) for \(p = q = k = 2\).
Theorem 3. Let the hypothesis of Theorem 1 be fulfilled for \( p = q = 2 \). Then

\[
\frac{1}{40} \kappa \mathcal{E} \leq M_2(2, 2) \leq \mathcal{E},
\]

where \( \kappa = \kappa(v) \).

Proof. The upper bound is an immediate corollary of Theorem 2. To prove the lower bound we use Theorem 1 and the arguments from Lemma 7 [4]. Let

\[
d\mu(x) = |v(x)|^{-2} \, dx; \quad \mu(I) = \int_I d\mu(y);
\]

\[
\omega(x) = \int_I (x - y) \, d\mu(y); \quad d\mu_1(x) = |\omega(x)|^2 \, d\mu(x); \quad \mu_1(1) = \int_I d\mu_1(y).
\]

If we take the point \( \lambda \in I \) such that \( \omega(\lambda) = 0 \) and choose \( \tau, \sigma \) so that

\[
0 < \tau < \lambda < \sigma < 1, \quad \mu(0, \tau) = \mu(\tau, \lambda) \text{ and } \mu(\lambda, \sigma) = \mu(\sigma, b),
\]

then there exist positive numbers \( \delta_i = \delta_i(v) \in (0, 1), \ i = 1, \ldots, 5 \) for which

\[
\mu(0, \lambda) = \delta_1 \mu(I), \quad \mu_1(\tau, \lambda) = \delta_2 \mu_1(I), \quad \mu_1(\lambda, \sigma) = \delta_3 \mu_1(I),
\]

\[
\int_0^\tau (\tau - s)^2 \, d\mu(s) = \delta_4 \frac{\mu_1(I)}{\mu(I)^2},
\]

\[
\int_\sigma^1 (s - \sigma)^2 \, d\mu(s) = \delta_5 \frac{\mu_1(I)}{\mu(I)^2}.
\]

Set \( \delta = \min_i \delta_i \) and \( \kappa = (\delta)^{3/2} \). Then Lemma 7 [4] gives us the required lower bound \( M_2(2, 2) \geq \frac{1}{40} \kappa \mathcal{E} \).

References


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