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ON MONOTONE-LIKE MAPPINGS IN ORLICZ-SOBOLEV SPACES

VESA MUSTONEN, MATTI TIENARI, Oulu

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We study the mappings of monotone type in Orlicz-Sobolev spaces. We introduce a new class \((S_m)\) as a generalization of \((S_+)\) and extend the definition of quasimonotone map. We also prove existence results for equations involving monotone-like mappings.

Keywords: pseudomonotone, quasimonotone, Orlicz-Sobolev space, almost solvability

MSC 1991: 47H15, 35J40

1. INTRODUCTION

Since the pioneering work of Minty in 1962 the theory of monotone mappings from a real reflexive Banach space \(X\) into its dual space \(X^*\) has been extensively generalized by Brezis, Browder, Hess, Leray and Lions, Visik and many others. In its original form the theory considers mappings \(T\) which satisfy the condition

\[(u - v, T(u) - T(v)) \geq 0\]

for all \(u\) and \(v\) in \(X\),

where \((u, v)\) denotes the duality pairing between the element \(u\) in \(X\) and \(w\) in \(X^*\). In order to treat efficiently the solvability problems for nonlinear elliptic and parabolic equations and corresponding variational inequalities within the same framework, various generalizations of the concept of monotone maps have been introduced. Most important of these extensions turned out to be the mappings of class \((S_+)\), pseudomonotone mappings \((PM)\), mappings of the type \((M)\) and quasimonotone mappings \((QM)\). The fact that the classical topological degree can be constructed for the
class \((S_+)\) and for the class of pseudomonotone mappings in the weak form indicates that the classes are well-defined (see [2, 3, 19]).

A motivation for the definition of various classes of mappings of monotone type comes from the study of the existence of solutions for the problems associated to elliptic differential operators in divergence form

\[
Au(x) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha a_\alpha (x, u(x), \nabla u(x), \ldots, \nabla^m u(x)), \quad x \in \Omega
\]

where \(\Omega\) is an open set in \(\mathbb{R}^N\) and \(m \geq 1\). If the coefficients \(a_\alpha (x, \xi)\) satisfy a polynomial growth condition with respect to \(|\xi|\) and suitable analytical conditions, the differential operator (1.1) generates a nonlinear mapping \(T\) from the Sobolev space \(W^{m,p}(\Omega)\) to its dual space \(W^{-m,p}(\Omega)\) belonging to the class \((S_+), (PM)\) or \((QM)\), respectively.

Differential operator (1.1) is called strongly nonlinear if the coefficients \(A_\alpha\) do not satisfy any polynomial growth condition. The study of strongly nonlinear elliptic problems was initiated by Browder in 1973. Since then many contributions have been published into this direction. Browder's original idea was to consider operators of the form

\[
Au(x) + Bu(x),
\]

where \(A\) is a polynomial operator as above and \(B\) is a lower order operator having no growth restrictions. This approach led to the concept of generalized pseudomonotone mapping, which is, in general, neither everywhere defined nor bounded in the Sobolev space associated with the operator \(A\). Further contributions in this direction are due to Hess [10] and Landes [15] and many others. Browder was able to show that a degree theory can be extended also for a particular class of mappings where \(Bu(x) = g(x, u(x))\). A further extension for a more general lower order part was obtained by Kittilä [12].

Another line of development for treating strongly nonlinear elliptic boundary value problems is to employ Orlicz spaces in place of reflexive Lebesgue spaces \(L^p(\Omega)\). By this change the polynomial growth condition can be replaced by a more liberal condition associated with an Orlicz function. The theory of mappings of monotone type can be extended also for complementary systems of Orlicz-Sobolev spaces which are not reflexive in general, and existence theorems can be produced accordingly. The study along these lines was initiated by Donaldson [4] and continued by Gomez [6–8]. Further contributions in this direction include [9], [17] and [20], where a degree theory is constructed.

In this paper we continue the study of mappings of monotone type in Orlicz-Sobolev spaces. Our main task is to give a more complete characterization of relevant
classes and produce corresponding refined solvability theorems for equations. We introduce a new class \((S_m)\) which can be seen as a generalization of the class \((S_e)\) to the Orlicz-Sobolev space setting. We also extend the definition of quasimonotone mapping and prove that the class \((S_m)\) stands quasimonotone perturbations.

Our paper is organized as follows. In Section 2 we present the basic properties of Orlicz and Orlicz-Sobolev spaces. In the next section we study the classes of monotone-like operators in the complementary system formed by Orlicz-Sobolev spaces. In Section 4 we deal with the conditions for differential operators in divergence form in order to generate mappings of the type described in Section 3. In the last section we generalize the basic existence theorem for equations involving quasimonotone mappings.

2. Notations and definitions

We begin with some preliminaries on Orlicz-Sobolev spaces. Let \(\Omega\) be a bounded open subset in \(\mathbb{R}^N\) and let \(M: \mathbb{R} \to \mathbb{R}\) be an \(N\)-function, i.e., even, convex and continuous with \(M(t) > 0\) for \(t > 0\), \(M(t)/t \to 0\) as \(t \to 0\) and \(M(t)/t \to +\infty\) as \(t \to +\infty\). \(M\) is an \(N\)-function if and only if it can be represented in the form

\[
M(t) = \int_0^t m(s) \, ds
\]

where \(m: [0, \infty) \to [0, \infty]\) is increasing, right continuous, \(m(t) = 0\) if and only if \(t = 0\) and \(m(t) \to +\infty\) as \(t \to +\infty\). We extend \(m\) to \(\mathbb{R}\) by \(m(t) = -m(-t)\) for \(t < 0\) (odd continuation). The Orlicz class \(CM(\Omega)\) is defined as the set of all real-valued measurable functions \(u\) defined in \(\Omega\) such that

\[
\int_\Omega M(u) \, dx < \infty.
\]

The Orlicz space \(LM(\Omega)\) is the linear hull of \(CM(\Omega)\). Then \(LM(\Omega)\) is a Banach space with respect to the Luxemburg norm

\[
\|u\|_{LM} = \inf \left\{ k > 0 : \int_\Omega \frac{M(u)}{k} \, dx \leq 1 \right\}.
\]

One has \(LM(\Omega) = L_M(\Omega)\) if and only if \(M\) satisfies the \(\Delta_2\)-condition: there exist \(\alpha > 0\) and \(t_0 > 0\) such that

\[
M(2t) \leq \alpha M(t)
\]
for all \( t \geq t_0 \). The closure in \( L_M(\Omega) \) of all bounded measurable functions is denoted by \( E_M(\Omega) \). Then \( E_M(\Omega) \subset L_M(\Omega) \) and \( E_M(\Omega) = L_M(\Omega) \) if and only if \( M \) satisfies the \( \Delta_2 \)-condition. The conjugate \( N \)-function \( \overline{M} \) is defined by

\[
\overline{M}(t) = \sup \{ ts - M(s) : s \in \mathbb{R} \}.
\]

\( \overline{M} \) is also an \( N \)-function and \( \overline{\overline{M}} = M \). The space \( L_{\overline{M}}(\Omega) \) is the dual space of \( E_M(\Omega) \).

It is well-known that \( L_M(\Omega) L_{\overline{M}}(\Omega) \subset L^1(\Omega) \). We recall also Young’s inequality:

\[
M(x) + \overline{M}(y) \geq xy \quad \text{for all } x, y \in \mathbb{R}
\]

with equality if and only if \( x = M(y) \) or \( y = M(x) \). A sequence \( \{ u_n \} \) in \( L_M(\Omega) \) converges modularly to \( u \) if there exists \( \lambda > 0 \) such that

\[
\int_\Omega M(\frac{|u_n - u|}{\lambda}) \, dx \to 0,
\]

when \( n \to \infty \). Modular convergence coincides with norm convergence if and only if \( M \) satisfies the \( \Delta_2 \)-condition. If \( M_1 \) and \( M_2 \) are \( N \)-functions satisfying

\[
\lim_{t \to \infty} \frac{M_1(ct)}{M_2(t)} = 0 \quad \text{for all } c > 0,
\]

then \( M_1 \) grows essentially more slowly than \( M_2 \) and we denote \( M_1 \ll M_2 \).

Remark 2.1. Typical examples of \( N \)-functions satisfying the \( \Delta_2 \)-condition are \( (1 + |t|^\alpha) \log(1 + |t|) - |t|^\beta \) and \( |t|^p \) for \( p > 1 \). On the other hand, functions \( e^{\theta t} - |t| - 1 \) and \( e^{\theta |t|} - 1 \) for \( p > 1 \) are \( N \)-functions not satisfying the \( \Delta_2 \)-condition.

The Orlicz-Sobolev space of functions in \( L_M(\Omega) \) with all distributional derivatives up to the order \( m \) in \( L_M(\Omega) \) is denoted by \( W^m_{L_M}(\Omega) \). The space \( W^m_{\overline{E}_M}(\Omega) \) is defined analogously. These spaces are identified, as usual, to subspaces of the product \( \prod L_M(\Omega) \). The spaces \( W^m_{L_M}(\Omega) \) and \( W^m_{\overline{E}_M}(\Omega) \) are defined as the \( \sigma(\prod L_M, \prod \overline{E}_M) \) closure of \( D(\Omega) \) in \( W^m_{L_M}(\Omega) \) and as the norm closure of \( D(\Omega) \) in \( W^m_{\overline{E}_M}(\Omega) \), respectively. We recall that there exists an \( N \)-function \( Q \gg M \) such that the embedding \( W^1_{L_M}(\Omega) \to E_Q(\Omega) \) is compact (see [5, 6]).

The following spaces of distributions will also be used:

\[
W^{-m}_{L_M}(\Omega) = \left\{ f \in D'(\Omega) : f = \sum_{|\alpha| < m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_M(\Omega) \right\}
\]

\[
W^{-m}_{\overline{E}_M}(\Omega) = \left\{ f \in D'(\Omega) : f = \sum_{|\alpha| < m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in \overline{E}_M(\Omega) \right\}.
\]

They are endowed with their usual quotient norms. It is shown in [6] that if \( \Omega \) has the segment property, then

\[
\left( \begin{array}{c} Y \ Z \ X \end{array} \right) = \left( \begin{array}{cc} W^n_{L_M}(\Omega) & W^{-m}_{L_M}(\Omega) \\ W^n_{\overline{E}_M}(\Omega) & W^{-m}_{\overline{E}_M}(\Omega) \end{array} \right).
\]
constitutes a complementary system, i.e., $Y$ and $Z$ are real Banach spaces in duality with respect to a continuous pairing $(\cdot, \cdot)$ and $Y_0$ and $Z_0$ are closed subspaces of $Y$ and $Z$ respectively such that, by means of $(\cdot, \cdot)$, the dual of $Y_0$ can be identified to $Z$ and that of $Z_0$ to $Y$. The pairing between $u \in Y$ and $f = \sum_{\alpha \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \in Z$ is given by

$$(u, f) = \sum_{\alpha \leq m} \int_D (D^\alpha u) f_\alpha \, dx.$$ 

A sequence $(u_n) \subset Y$ converges modularly to $u$ in $Y$ if $D^\alpha u_n \to D^\alpha u$ modularly in $L_M(\Omega)$ for each $|\alpha| \leq m$. Standard references on Orlicz and Orlicz-Sobolev spaces include [1, 13, 14].

We end this section by presenting some useful convergence results.

**Lemma 2.2.**

(i) If $u_n \to u$ a.e. in $\Omega$, $u_n \to u$ in $L_M(\Omega)$ for $\sigma(L_M, E_M)$ and $v_n \to v$ in $E_M(\Omega)$ strongly, then $u_n v_n \to u v$ in $L^1(\Omega)$ strongly.

(ii) If $u_n \to u$ in $E_M(\Omega)$ strongly and $P << M$, then $P^{-1}(M(u_n)) \to P^{-1}(M(u))$ in $E_M(\Omega)$ strongly.

(iii) $u_n \to u$ in $L_M(\Omega)$ modularly if and only if $u_n \to u$ in measure and there exist a convergent sequence $(f_n)$ in $L^1(\Omega)$ and $\epsilon > 0$ such that $M(\epsilon u_n) \leq f_n$ a.e. in $\Omega$.

**Proof.** The proofs of (i) and (ii) can be found in [6] and [20], for example. To prove (iii), assume first that $u_n \to u$ in measure and $M(c u_n) \leq f_n$ a.e. in $\Omega$, where $c > 0$ and $f_n \to f$ in $L^1(\Omega)$. Then

$$M(\frac{1}{2} (u_n - u)) \leq \frac{1}{2} M(c u_n) + \frac{1}{2} M(c u) \leq \frac{1}{2} f_n + \frac{1}{2} f \quad \text{a.e. in } \Omega.$$ 

By the dominated convergence theorem

$$M(\frac{1}{2} (u_n - u)) \to 0 \text{ in } L^1(\Omega).$$

Hence $u_n \to u$ in $L_M(\Omega)$ modularly.

Assume next that $u_n \to u$ in $L_M(\Omega)$ modularly. Hence $M(2 \epsilon (u_n - u)) \to 0$ in $L^1(\Omega)$ for some $\epsilon > 0$ implying $u_n \to u$ in measure. Moreover, choosing $0 < \epsilon_2 < \min(\epsilon_1, \frac{1}{2M(2 \epsilon_1)})$ we get

$$M(\epsilon_2 u_n) < \frac{1}{2} M(2 \epsilon_2 (u_n - u)) + \frac{1}{2} M(2 \epsilon_2 u) \quad \text{a.e. in } \Omega$$

where the right hand side converges in $L^1(\Omega)$. $\square$
3. Classes of monotone-like mappings

The original definitions for various classes of mappings of monotone type were given for mappings acting from a real reflexive Banach space $X$ into its dual space $X^*$. The norm convergence in $X$ and in $X^*$ is denoted by $\to$ and the weak convergence by $\rightharpoonup$. We recall the following classical notions. A mapping $T$ from $X$ to $X^*$ is said to be

- monotone, denote $T \in (\text{MON})$, if $(u - v, T(u) - T(v)) \geq 0$ for all $u, v \in X$

- of class $(S_\ast)$ if for any sequence $\{u_n\}$ in $X$ with $u_n \to u$ and $\limsup (u_n - u, T(u_n)) \leq 0$ we have $u_n \to u$ in $X$

- pseudomonotone, denote $T \in (\text{PM})$, if for any sequence $\{u_n\}$ in $X$ with $u_n \to u$ and $\limsup (u_n - u, T(u_n)) \leq 0$ we have $T(u_n) \to T(u)$ and $\{u_n, T(u_n)\} \to (u, T(u))$

- quasimonotone, $T \in (\text{QM})$, if for any sequence $\{u_n\}$ in $X$ with $u_n \to u$ we have $\limsup (u_n - u, T(u_n)) \leq 0$

- of class $(M)$, if for any sequence $\{u_n\}$ in $X$ with $u_n \to u$, $T(u_n) \to \chi$ and $\limsup (u_n, T(u_n)) \leq \langle \chi, u \rangle$ we have $\chi = T(u)$

- bounded if it takes bounded sets of $X$ into bounded sets of $X^*$

- demicontinuous if $u_n \to u$ in $X$ implies $T(u_n) \to T(u)$ in $X^*$

For bounded demicontinuous mappings we have $(S_\ast) \subset (\text{PM}) \subset (\text{QM})$ and $(\text{QM}) \cap (M) = (\text{PM})$. Also the perturbation result $(S_\ast) + (\text{QM}) = (S_\ast)$ is useful in applications. Note that the above condition of quasimonotony can be written also in the form: for any sequence $\{u_n\}$ in $X$ with $u_n \to u$ and $\limsup (u_n - u, T(u_n)) \leq 0$ we have $\lim (u_n - u, T(u_n)) \to 0$.

Our task now is to study the corresponding classification of monotone-like mappings in the complementary system of Orlicz-Sobolev spaces

$$
\left( \begin{array}{c}
Y \\
Y_0 \\
Z \\
Z_0
\end{array} \right) = \left( \begin{array}{c}
W^{\alpha}_\infty L^m_\Omega \\
W^{\alpha}_\infty E^m_\Omega \\
W^{\alpha}_\infty L^w_\Omega \\
W^{\alpha}_\infty E^w_\Omega
\end{array} \right)
$$

where $\Omega \subset \mathbb{R}^N$ is an open and bounded subset with the segment property. Essential modifications are needed in the definitions above since Orlicz-Sobolev spaces are not reflexive, in general, and the differential operators in divergence form with natural
growth conditions are neither bounded nor everywhere defined. Moreover, the duality
map in Orlicz-Sobolev spaces is not a single-valued \((S_f)\)-mapping, in general.

**Definition 3.1.** A mapping \(T : Y_0 \subset D(T) \subset Y \to Z\) is

- quasimonotone (denote \(T \in (QM)\)), if

\[
\begin{align*}
\{u_n\} & \subset D(T) \\
u_n & \to u \in Y \quad \text{for } \sigma(Y, Z_0) \\
T(u_n) & \to \chi \in Z \quad \text{for } \sigma(Z, Y_0) \\
\lim \sup (u_n, T(u_n)) & \leq (u, \chi) \\
\end{align*}
\]

- pseudomonotone \((T \in (PM))\), if

\[
\begin{align*}
\{u_n\} & \subset D(T) \\
u_n & \to u \in Y \quad \text{for } \sigma(Y, Z_0) \\
T(u_n) & \to \chi \in Z \quad \text{for } \sigma(Z, Y_0) \\
\lim \sup (u_n, T(u_n)) & \leq (u, \chi) \\
\end{align*}
\]

- of class \((S_m)\), if

\[
\begin{align*}
\{u_n\} & \subset D(T) \\
u_n & \to u \in Y \quad \text{for } \sigma(Y, Z_0) \\
T(u_n) & \to \chi \in Z \quad \text{for } \sigma(Z, Y_0) \\
\lim \sup (u_n, T(u_n)) & \leq (u, \chi) \\
\end{align*}
\]

- quasibounded with respect to \(u \in Y_0\), if \(T(u)\) remains bounded in \(Z\) whenever \(u \in D(T)\) remains bounded in \(Y\) and \((u - \tilde{u}, T(u))\) remains bounded from above.

- finitely continuous, if \(T\) is continuous from each finite dimensional subspace of \(Y_0\) into \(Z\) for \(\sigma(Z, Y_0)\)

- satisfies the condition \((M_m)\), if

\[
\begin{align*}
\{u_n\} & \subset D(T) \\
u_n & \to u \in Y \quad \text{modularly} \\
T(u_n) & \to \chi \in Z \quad \text{for } \sigma(Z, Y_0) \\
\lim \sup (u_n, T(u_n)) & \leq (u, \chi) \\
\end{align*}
\]

Clearly any pseudomonotone mapping and mapping of class \((S_m)\) satisfies the condition \((M_m)\). Quasimonotone mappings satisfying the condition \((M_m)\) are denoted
by $(QM_m)$. It is straightforward to check that the sum of two quasibounded mappings with respect to the same $u \in Y_0$ is also quasibounded with respect to $u$. Zero map belongs to each of the classes defined above except $(S_m)$. In the sequel we shall denote the restriction of any class $(\cdot)$ defined above to the class of quasibounded mappings with respect to $u$ by a subscript $(\cdot)_u$.

For the classes in Definition 3.1, we have the following inclusions and perturbation result.

**Theorem 3.2.**

(i) $(S_m) \subset (PM) \subset (QM_m)$

(ii) $(S_m)_u + (QM_m)_u = (S_m)_u$

**Proof.** The first assertion follows immediately from the definitions. To prove (ii), assume $T = T_1 + T_2$, where $T_1$ and $T_2$ are quasibounded with respect to $u \in Y_0$, $T_1 \in (S_m)$ and $T_2 \in (QM_m)$. Clearly $Y_0$ is a subset of $D(T) = D(T_1) \cap D(T_2)$ and $T$ is quasibounded with respect to $u$. Suppose

$$
\{(u_n) \subset D(T)
\quad u_n \rightarrow u \in Y
\quad T(u_n) \rightarrow \chi \in Z
\quad \limsup (u_n, T(u_n)) \leq \langle u, \chi \rangle.
$$

Since $T_1$ and $T_2$ are quasibounded with respect to $u$, we may deduce that

$$
T_1(u_n) \rightarrow \chi_1 \in Z \text{ and } T_2(u_n) \rightarrow \chi_2 \in Z
$$

for a subsequence with $\chi = \chi_1 + \chi_2$. Since $T_2 \in (QM)$, we have

$$
\limsup (u_n, T_2(u_n)) \leq \langle u, \chi_2 \rangle.
$$

In view of $T_1 \in (S_m)$, we get $u \in D(T_1)$, $\chi_1 = T_1(u)$, $(u_n, T_1(u_n)) \rightarrow \langle u, \chi_1 \rangle$ and $u_n \rightarrow u$ modularly in $Y$. Hence

$$
\limsup (u_n, T_1(u_n)) \leq \langle u, \chi_1 \rangle
$$

implying, on account of $T_2 \in (QM_m)$, that $u \in D(T_2)$, $T_2(u) = \chi_2$ and

$$
\langle u_n, T_2(u_n) \rangle \rightarrow \langle u, \chi_2 \rangle.
$$

\[\square\]
4. Differential operators in divergence form

Let \( \Omega \) be an open and bounded subset in \( \mathbb{R}^N \) with the segment property and denote
\[
\begin{pmatrix}
Y \\
Y_0
\end{pmatrix} = \begin{pmatrix}
W^m_0 L_{ij}(\Omega) & W^{-m} L_{ij}(\Omega) \\
W^m_0 E_{ij}(\Omega) & W^{-m} E_{ij}(\Omega)
\end{pmatrix}.
\]
We shall consider differential operators in divergence form
\begin{align}
(4.1) \quad A^{(1)} u(x) &= \sum_{|\alpha| = m} (-1)^{|\alpha|} D^\alpha a_{\alpha}(x, u, \nabla u, \ldots, \nabla^m u), \quad x \in \Omega \\
(4.2) \quad A^{(0)} u(x) &= \sum_{|\alpha| < m} (-1)^{|\alpha|} D^\alpha a_{\alpha}(x, u, \nabla u, \ldots, \nabla^m u), \quad x \in \Omega
\end{align}
and the corresponding mappings \( T_1: D(T_1) \to Z \) and \( T_0: D(T_0) \to Z \) defined by the formulas
\[
\langle v, T_1(u) \rangle = \int_\Omega \sum_{|\alpha| = m} a_{\alpha}(x, \xi(u)) D^\alpha v \, dx, \quad u \in D(T_1), v \in Y
\]
and
\[
\langle v, T_0(u) \rangle = \int_\Omega \sum_{|\alpha| < m} a_{\alpha}(x, \xi(u)) D^\alpha v \, dx, \quad u \in D(T_0), v \in Y
\]
where
\[
D(T_1) = \{ u \in Y : a_{\alpha}(x, \xi(u)) \in L^p_{\Gamma}(\Omega) \text{ for } |\alpha| = m \} \\
D(T_0) = \{ u \in Y : a_{\alpha}(x, \xi(u)) \in L^p_{\Gamma}(\Omega) \text{ for } |\alpha| < m \}.
\]
We use the following notations: If \( \xi = \{ \xi_\alpha : |\alpha| \leq m \} \in \mathbb{R}^N \) is an \( m \)-jet, then \( \zeta = \{ \zeta_\alpha : |\alpha| = m \} \in \mathbb{R}^N \) denotes its top order part and \( \eta = \{ \eta_\alpha : |\alpha| < m \} \in \mathbb{R}^N \) its lower order part. For a differentiable function \( u \), \( \xi(u) \) denotes \( \{ D^\alpha u : |\alpha| \leq m \} \).

Now we introduce the conditions on the differential operators \( A^{(0)} \) and \( A^{(1)} \) which give mappings \( T_0 \) and \( T_1 \) the properties described in Definition 3.1.

\((A_1)\) Each \( a_{\alpha}(x, \xi) : \Omega \times \mathbb{R}^N \to \mathbb{R} \) is measurable for any fixed \( \xi \in \mathbb{R}^N \) and continuous in \( \xi \) for a.e. fixed \( x \).

\((A_2)\) There exist constants \( c_1, c_2 > 0 \) and functions \( k_{\alpha} \) in \( E_{\Gamma^p}(\Omega) \) for all \( |\alpha| = m \), \( k_{\alpha} \in L^p(\Omega) \) for \( |\alpha| < m \) and an \( N \)-function \( P << M \) such that for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^N \)
\[
|a_{\alpha}(x, \xi)| \leq k_{\alpha}(x) + c_1 \sum_{|\beta| = m} M^{-1}(M(c_2) \xi_{\beta}) + c_1 \sum_{|\beta| < m} P^{-1}(M(c_3) \xi_{\beta})
\]

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if $|\alpha| = m$,

$$|a_0(x, \xi)| \leq k_0(x) + c_1 \sum_{|\beta| = m} \overline{M}^{-1}(P(\varphi_\beta(\xi))) + c_1 \sum_{|\beta| < m} \overline{M}^{-1}(M(\varphi_\beta(\xi)))$$

if $|\alpha| < m$.

(A3) For a.e. $x$ in $\Omega$, all $\eta$ in $\mathbb{R}^{N_1}$, $\zeta$ and $\zeta'$ in $\mathbb{R}^{N_1}$ with $\zeta \neq \zeta'$,

$$\sum_{|\alpha| = m} (a_0(x, \eta, \zeta) - a_0(x, \eta, \zeta')) (\zeta - \zeta') > 0$$

(A3)$_{e}$ For a.e. $x$ in $\Omega$, all $\eta$ in $\mathbb{R}^{N_1}$, $\zeta$ and $\zeta'$ in $\mathbb{R}^{N_1}$,

$$\sum_{|\alpha| = m} (a_0(x, \eta, \zeta) - a_0(x, \eta, \zeta')) (\zeta - \zeta') \geq 0$$

(A4) There exist functions $b_0(x)$ in $E'_\Omega$ for $|\alpha| = m$, $b(x)$ in $L^1(\Omega)$, constants $d_1, d_2 > 0$ and some fixed element $\psi \in W^{1,n}_0(\Omega)$ such that

$$\sum_{|\alpha| = m} a_0(x, \xi) (a_0 - D^\alpha \psi(x)) \geq d_1 \sum_{|\alpha| = m} M(d_2 \xi_\alpha) - \sum_{|\alpha| = m} b_0(x) \xi_\alpha - b(x)$$

for a.e. $x$ in $\Omega$ and all $\xi$ in $\mathbb{R}^{N_1}$.

These conditions are generalizations of the classical Leray-Lions conditions to Orlicz-Sobolev space setting (cf. [9, 17, 18]).

We shall study first the properties of $T_0$.

**Proposition 4.1.** If (A1) and (A3)$_{e}$ hold, then $T_0$ is finitely continuous, $D(T_0) = Y$, $T_0$ is bounded and belongs to $(QM_n)$.

**Proof.** It is proved ([9]) that if $\{u_n\}$ remains bounded in $Y$, then $\{a_0(x, \xi(u_n))\}$ remains bounded in $L^1(\Omega)$ for $|\alpha| < m$, which proves that $D(T_0) = Y$ and $T_0$ is bounded. Finite continuity follows as in [6].

Suppose $u_n \to u$ in $Y$ for $\sigma(Y, Z_0)$ and $T_0(u_n) \to \chi$ in $Z$ for $\sigma(Z, Y_0)$. By compact embedding, $D^\alpha u_n \to D^\alpha u$ in $E'_\Omega(\Omega)$ for $|\alpha| < m$. Since $\{a_0(x, \xi(u_n))\}$ is bounded in $L^1(\Omega)$ for $|\alpha| < m$, we may assume

$$a_0(x, \xi(u_n)) \to h_0 \in L^1(\Omega) \quad \text{for } \sigma(L^1, E'_\Omega)$$

for a subsequence. Clearly

$$\langle \phi, x \rangle = \sum_{|\alpha| < m} \int_{\Omega} h_0 D^\alpha \phi \, dx \quad \text{for all } \phi \in Y$$

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\[
\langle a_n, T_0(u_n) \rangle = \sum_{|\alpha| \leq m} \int_\Omega a_n(x, \xi(u_n)) D^\alpha u_n \, dx \rightarrow \sum_{|\alpha| \leq m} \int_\Omega h_n D^\alpha u \, dx = \langle u, \chi \rangle
\]
proving that \( T_0 \) is quasimonotone. If \( u_n \rightharpoonup u \) in \( Y \) modularity in the above, then \( a_n(x, \xi(u_n)) \rightarrow a_n(x, \xi(u)) \) a.e. for a subsequence implying \( h_n = a_n(x, \xi(u)) \). Hence \( T_0 \) satisfies the condition \( (M_m) \) and the proof is complete. \( \square \)

For the operator \( T_1 \) we adopt the following continuity and boundedness property from [9].

**Proposition 4.2.** If \( (A_1), (A_2) \) and \( (A_3)_x \) hold, then \( T_1 \) is finitely continuous and quasibounded with respect to any \( u \in Y_0 \).

Next we have the following extensions of the previous results of [6,9] for the mapping \( T_1 \).

**Theorem 4.3.**

a) If \( (A_1), (A_2) \) and \( (A_3)_x \) hold, then \( T_1 \) is pseudomonotone.

b) If \( (A_1), (A_2), (A_3) \) and \( (A_4) \) hold, then \( T_1 \) is of class \( (S_m) \).

**Proof.** First we prove part a). Suppose \( (A_1), (A_2) \) and \( (A_3)_x \) hold and

\[
\begin{aligned}
\{u_n\} \subset D(T_1) \\
u_n \rightharpoonup u & \in Y \quad \text{for } \sigma(Y, Z_0) \\
T_1(u_n) & \rightarrow \chi \in Z \quad \text{for } \sigma(Z, Y_0) \\
\limsup (u_n, T_1(u_n)) & \leq (u, \chi).
\end{aligned}
\]

By the argument used in the proof of [9, Proposition 5.1], we may assume that \( \{a_n(x, \xi(u_n))\} \) remains bounded in \( L^\infty(\Omega) \). Consequently,

\[
a_n(x, \xi(u_n)) \rightarrow h_n \in L^\infty(\Omega) \quad \text{for } \sigma(L^\infty, E_\Omega)
\]

for a subsequence and

\[
\langle \phi, \chi \rangle = \sum_{|\alpha| \leq m} \int_\Omega h_n D^\alpha \phi \, dx \quad \text{for all } \phi \in Y_0.
\]

By \( \sigma(Y, Z) \) density of \( Y_0 \) in \( Y' \), (4.3) holds for all \( \varphi \in Y \). Next we prove that \( a_n(x, \xi(u)) = h_n \) a.e. in \( \Omega \) for all \( |\alpha| = m \). By the compact embedding, \( D^\beta u_n \rightarrow D^\beta u \) in \( E_\Omega(\Omega) \) for \( |\beta| < m \). The condition \( (A_3)_x \) implies

\[
\sum_{|\alpha| \leq m} \int_\Omega \left( a_n(x, \eta(u_n)) - a_n(x, \xi(u_n)) \right) (v_n - D^\alpha u_n) \, dx \geq 0
\]
for all \( v = (v_a) \in \left( L^\infty(\Omega) \right)^N \). Therefore

\[
\langle u_n, T_1(u_n) \rangle \geq \sum_{|a|=m} \int_\Omega a_a(x, \xi(u_n)) v_a \, dx
\]

\[
+ \sum_{|a|=m} \int_\Omega a_a(x, \eta(u_n), \theta) \{ D^a u_n - v_a \} \, dx.
\]

The condition \((A_2)\) and the compact embedding imply

\[ a_a(x, \eta(u_n), \theta) \rightarrow a_a(x, \eta(u), \theta) \]

in \( E_\Omega(\Omega) \) (see [6]). Hence

\[
\langle u, \chi \rangle \geq \sum_{|a|=m} \int_\Omega h_a v_a \, dx + \sum_{|a|=m} \int_\Omega a_a(x, \eta(u), \theta) \{ D^a u - v_a \} \, dx
\]

and consequently

\[
\sum_{|a|=m} \int_\Omega \left( a_a(x, \eta(u), \theta) - h_a \right) \{ v_a - D^a u \} \, dx \geq 0
\]

for \( v = (v_a) \in \left( L^\infty(\Omega) \right)^N \). Let \( 0 < j < i \) be arbitrary integers and \( t > 0 \). Denote

\[ \Omega_i = \{ x \in \Omega : |D^a u(x)| \leq i \ \text{a.e. in} \ \Omega \ \text{for all} \ |a| = m \} \]

and

\[ \theta = (\nabla u) \chi_1 + t \theta \chi_1, \]

where \( \theta \in \left( L^\infty(\Omega) \right)^N \) is arbitrary. By (4.5),

\[
- \sum_{|a|=m} \int_{\Omega \setminus \Omega_i} a_a(x, \eta(u), \theta) D^a u \, dx
\]

\[
+ t \sum_{|a|=m} \int_{\Omega_i} \left( a_a(x, \eta(u), \xi(u) + t\theta) - h_a \right) w_a \, dx \geq 0.
\]

Letting \( i \rightarrow \infty \) and dividing by \( t \), we get

\[
\sum_{|a|=m} \int_{\Omega} \left( a_a(x, \eta(u), \xi(u) + t\theta) - h_a \right) w_a \, dx \geq 0.
\]
Since $D^a u + tu_\nu \to D^a u$ in $L^\infty(\Omega_j)$, when $t \to 0^+$, we have
\[a_a(x, \eta(u), \zeta(u) + t\eta) \to a_a(x, \eta(u), \zeta(u))\]
in $E_{\mathfrak{g}}(\Omega_j)$. Consequently,
\[\sum_{|\alpha| = m} \int_{\Omega_j} (a_a(x, \xi(u)) - h_\alpha) u_\nu \, dx \geq 0\]
for all $@ \in (L^\infty(\Omega))^m$ implying
\[(4.6)\quad a_a(x, \xi(u)) = h_\alpha \quad \text{a.e. in } \Omega_j\]
for all $|\alpha| = m$. Since $\nu$ was arbitrary, (4.6) holds a.e. in $\Omega$. Therefore $u \in D(T_1)$ and $\chi = T_1(u)$. Substituting $\nu = \zeta(u)\chi_{\Omega_0}$ into (4.4) we get
\[(u_n, T_1(u_n)) \geq \sum_{|\alpha| = m} \int_{\Omega_0} a_a(x, \xi(u_n)) (D^a u) \chi_{\Omega_0} \, dx
+ \sum_{|\alpha| = m} \int_{\Omega_0} a_a(x, \eta(u_n), \zeta(u_n)\chi_{\Omega_0}) (D^a u_n - (D^a u)\chi_{\Omega_0}) \, dx\]
implying
\[\liminf (u_n, T_1(u_n)) \geq \sum_{|\alpha| = m} \int_{\Omega_0} a_a(x, \xi(u)) D^a u \, dx + \sum_{|\alpha| = m} \int_{\Omega \setminus \Omega_0} a_a(x, u, 0) D^a u \, dx
+ \sum_{|\alpha| = m} \int_{\Omega_0} a_a(x, \xi(u)) D^a u \, dx,\]
when $i \to \infty$. Hence $(u_n, T_1(u_n)) \to (u, T_1(u))$ and the proof of part a) is complete.

To prove part b), suppose $(A_1), (A_2), (A_3)$ and $(A_4)$ hold and
\[\begin{cases}
\{u_n\} \subset D(T_1) \\
\limsup (u_n, T_1(u_n)) \leq (u, \chi).
\end{cases}\]

By the previous part, $u \in D(T_1), \chi = T_1(u)$ and $(u_n, T_1(u_n)) \to (u, \chi)$. As above, $D^a u_n \to D^a u$ in $E_{\mathfrak{g}}(\Omega)$ for $|\alpha| < m$. In view of strict inequality in $(A_3)$, we may
deduce as in [15] that \( D^\alpha u_n \to D^\alpha u \) a.e. for \(|\alpha| = m\), for a subsequence. This implies \( D^\alpha u_n \to D^\alpha u \) in measure for the original sequence. By (A2) and (A3),

\[
f_n := \sum_{|\alpha| = m} a_\alpha (x, \xi(u_n)) D^\alpha u_n \geq \sum_{|\alpha| = m} a_\alpha (x, \eta(u_n), 0) D^\alpha u_n
\]

\[
\geq - \sum_{|\alpha| = m} \left( |k_\alpha (x) D^\alpha u_n| + c_1 \sum_{|\beta| < m} P^{-1} \left( M(c_2 D^\beta u_n) \right) D^\beta u_n \right).
\]

By compact embedding and Lemma 2.2, the right hand side converges in \( L^1(\Omega) \). Denoting \( f = \sum_{|\alpha| = m} a_\alpha (x, \xi(u)) D^\alpha u \) we get for some \( h \in L^1(\Omega) \) that \( f_n \geq -h \), \( f_n \to f \) a.e. in \( \Omega \) and

\[
\int_{\Omega} f_n \, dx \to \int_{\Omega} f \, dx,
\]

for a subsequence. Using the result of [11, p. 206], \( f_n \to f \) in \( L^1(\Omega) \) for a subsequence, and hence, by standard contradiction argument, \( f_n \to f \) in \( L^1(\Omega) \) also for the original sequence. By condition (A4),

\[
d_1 \sum_{|\alpha| = m} M(d_2 D^\alpha u_n) \leq \sum_{|\alpha| = m} a_\alpha (x, \xi(u_n)) (D^\alpha u_n - D^\alpha \varphi(x))
\]

\[
+ \sum_{|\alpha| = m} b_\alpha (x) D^\alpha u_n + b(x).
\]

Using Lemma 2.2, we conclude that the right hand side of the inequality above converges in \( L^1(\Omega) \). Therefore \( D^\alpha u_n \to D^\alpha u \) in \( L_{M^1}(\Omega) \) modularly for \(|\alpha| = m\), by Lemma 2.2 (ii).

**Remark 4.4.** If \( \lim M(c) / M(t) = \infty \) for some \( c > 1 \), then any bounded sequence in \( L_{M^1}(\Omega) \) which converges a.e. converges also modularly and hence we may remove condition (A2) from Theorem 4.4 b) (see [20]). Note also that if \( M \) and \( \bar{M} \) satisfy the \( \Delta_2 \)-condition, then we may choose \( P = M \) in condition (A2).

### 5. Solvability results for equations

We shall close this paper by solvability and almost solvability results for monotone-like mappings in the complementary systems of Orlicz-Sobolev spaces. We adopt first a well-known existence result for pseudomonotone mappings in a complementary system from [9].

**Theorem 5.1.** Let \( (Y, Y_0, Z, Z_0) \) be a complementary system with \( Y_0 \) and \( Z_0 \) separable. Let \( T : Y_0 \subset D(T) \subset Y \to Z \) be pseudomonotone. Assume that the following conditions hold with respect to some elements \( u \in Y_0 \) and \( f \in Z_0 \):

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(i) \( T \) is finitely continuous
(ii) \( T \) is quasibounded with respect to \( \bar{u} \)
(iii) \( \langle u - \bar{u}, T(u) - f \rangle > 0 \) when \( u \in D(T) \) has sufficiently large norm in \( Y \).

Then \( f \in T(D(T)), \) i.e., the equation \( T(u) = f \) is solvable.

Let \( \Omega \) be an open and bounded subset in \( \mathbb{R}^N \) with the segment property and denote the complementary system of Orlicz-Sobolev spaces by

\[
\begin{pmatrix}
Y & Z
\end{pmatrix}
= \begin{pmatrix}
W_{p,q}^1 L^p(\Omega) & W^{-m} L^q(\Omega) \\
W_{p,q}^1 E^p(\Omega) & W^{-m} E^q(\Omega)
\end{pmatrix}
\]

For this complementary system we have the following generalization.

**Theorem 5.2.** Let \( T : Y_0 \subset D(T) \subset Y \rightarrow Z \) belong to class \( (QM_m) \). Assume the following conditions hold with respect to some elements \( \bar{u} \in Y_0 \) and \( f \in Z_0 \):

(i) \( T \) is finitely continuous
(ii) \( T \) is quasibounded with respect to \( \bar{u} \)
(iii) \( \langle u - \bar{u}, T(u) - f \rangle > 0 \) when \( u \in D(T) \) has sufficiently large norm in \( Y \).

Then \( f \in T(D(T)), \) i.e., the equation \( T(u) = f \) is almost solvable.

**Proof.** Define a mapping \( \hat{T} : Y_0 \subset D(\hat{T}) \subset Y \rightarrow Z \) by

\[
\hat{T}(u) = T(u + \bar{u})
\]

with \( D(\hat{T}) = D(T) - \bar{u} \). It is straightforward to check that also the mapping \( \hat{T} \) belongs to class \( (QM_m) \). Moreover, \( \hat{T} \) satisfies the following conditions:

(i) \( \hat{T} \) is finitely continuous
(ii) \( \hat{T} \) is quasibounded with respect to 0
(iii) \( \langle u, \hat{T}(u) - f \rangle \geq 0 \) when \( u \in D(\hat{T}) \) has sufficiently large norm in \( Y \).

Define \( J_n : D(J_n) \rightarrow Z \) by

\[
(5.1) \quad \langle v, J_n(u) \rangle = \frac{1}{n} \sum_{|\alpha| = m} \int_\Omega \tilde{M}^{-1}(M(\frac{1}{n} D^\alpha u)) D^\alpha v \, dx \quad \text{for } u \in Y
\]

with

\[
D(J_n) = \{ u \in Y \mid \tilde{M}^{-1}(M(\frac{1}{n} D^\alpha u)) \in L_p(\Omega) \text{ for all } |\alpha| = m \}.
\]

We can apply Theorem 4.2 and 4.3 to conclude that \( J_n \in (S_m) \), \( J_n \) is finitely continuous and quasibounded with respect to any \( \theta \in Y_0 \). According to Theorem 3.2, the mapping \( \hat{T}_n = J_n + \hat{T} \) with \( D(\hat{T}_n) = D(J_n) \cap D(\hat{T}) \) belongs to class \( (S_m) \) and is quasibounded with respect to 0. In particular, \( \hat{T}_n \) is pseudomonotone and satisfies
the conditions (i) and (ii) of Theorem 5.1. To prove (iii) with respect to 0 and f, we note that \( \langle u, J_n(u) \rangle > 0 \) for all \( u \in D(J_n) \) with \( u \neq 0 \). By (iii),

\[
\langle u, T_n(u) - f \rangle = \langle u, \tilde{T}(u) - f \rangle + \langle u, J_n(u) \rangle > 0,
\]

when \( u \in D(T_n) \) has sufficiently large norm in \( Y \). By Theorem 5.1, there exists \( u_n \in D(T_n) \) such that

\[
J_n(u_n) + \tilde{T}(u_n) = f
\]

for any \( n \). Therefore

\[
\langle u_n, \tilde{T}(u_n) - f \rangle = -\langle u_n, J_n(u_n) \rangle < 0 \quad \text{whenever } u_n \neq 0.
\]

In view of (iii), \( \{u_n\} \) remains bounded in \( Y \). Consequently, we may conclude from (5.1) that \( \|J_n(u_n)\|_Z \to 0 \) and \( T(u_n) = T(u_n + 0) \to f \) in \( Z \) strongly, when \( n \to \infty \). Therefore \( f \) belongs to the norm-closure of \( T(D(T)) \).

Remark 5.3. Let \( \Omega \) be an open bounded subset in \( \mathbb{R}^N \). To indicate the application of our solvability results we consider a boundary value problem

\[
(5.2) \quad \begin{cases}
A^{(1)}u(x) + A^{(0)}u(x) = h(x) & \text{in } \Omega \\
u(x) = 0 & \text{on } \partial \Omega
\end{cases}
\]

where \( A^{(1)} \) and \( A^{(0)} \) are differential operators in divergence form defined by (4.1) and (4.2), respectively. We assume that the coefficients functions \( a_{ij} \) satisfy the conditions \( (A_1) \) and \( (A_2) \) for all \( |a| < m \) and \( h \) is a given function in \( E_{d}^2(\Omega) \). We also assume that the conditions \( (A_3) \) holds implying the condition (iii) of Theorem 5.1 is true for \( \tilde{u} = \varphi \) and for any \( f \in W_m^m E_2(\Omega) \) (see [9]). Applying Theorem 5.1 and 5.2 we obtain following results for the existence of weak solution of (5.2).
(a) If \( A^{(1)} \) satisfies \( (A_3) \), then (5.2) is solvable for any \( h \in E_{d}^2(\Omega) \)
(b) If \( A^{(1)} \) satisfies \( (A_3)_1 \), then (5.2) is almost solvable for any \( h \in E_{d}^2(\Omega) \)
(c) If \( A^{(1)} \) satisfies \( (A_3)_1 \), and \( A^{(0)} \) has the form

\[
A^{(0)}u(x) = \sum_{|a| \\leq m} (-1)^{|a|} D^a a_{ij}(x, u, \nabla u, \ldots, \nabla^{m-1} u),
\]

then \( T_0 \) and \( T_1 + T_0 \) are pseudomonotone. Hence (5.2) is solvable for any \( h \in E_{d}^2(\Omega) \).

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References


Authors’ addresses: Vesa Mustonen, Matti Tienari, Department of Mathematical Sciences, University of Oulu, FIN 90570 Oulu, Finland, e-mail: vesa.mustonen@oulu.fi.