David Eric Edmunds; Jiří Rákosník
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ON A HIGHER-ORDER HARDY INEQUALITY

DAVID E. EDMUNDS, Sussex, JIŘÍ RÁKOSNÍK, Praha

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Dedicated to Professor A. Kufner on the occasion of his 65th birthday

Abstract. The Hardy inequality

\[ \int_\Omega |u(x)|^p d(x)^{-p} \, dx \leq c \int_\Omega |\nabla u(x)|^p \, dx \]

holds for \( u \in C^{0,0}(\Omega) \) if \( \Omega \subset \mathbb{R}^n \) is an open set with a sufficiently smooth boundary and if \( 1 < p < \infty \). P. Hajlasz proved the pointwise counterpart to this inequality involving a maximal function of Hardy-Littlewood type on the right hand side and, as a consequence, obtained the integral Hardy inequality. We extend these results for gradients of higher order and also for \( p = 1 \).

Keywords: Hardy inequality, capacity, \( p \)-thick set, maximal function, Sobolev space


1. INTRODUCTION

Let \( \Omega \) be a proper subdomain of \( \mathbb{R}^n \) and let \( d(x) = \text{dist}(x, \partial \Omega) \), \( x \in \Omega \), be the corresponding distance function.

It is well known that the Hardy inequality

\[ \int_\Omega |u(x)|^p d(x)^{-p} \, dx \leq c \int_\Omega |\nabla u(x)|^p \, dx, \]

holds for \( u \in C^{0,0}_b(\Omega) \) if \( 1 < p < \infty \) and the boundary of \( \Omega \) satisfies the Lipschitz condition or similar regularity conditions. For these results and further references we refer to [8], [10], [12].

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Different authors introduced the notions of capacity and of thick sets in various ways (see, e.g., [1], [4]-[9], etc.) in order to find weaker sufficient conditions for inequalities of Hardy, Poincaré and other types. We shall concentrate mainly on [4] and [6].

Let \( K \) be a compact subset of \( \Omega \) and let \( 1 \leq p < \infty \). The variational \((1,p)\)-capacity \( C_{1,p}(K,\Omega) \) of the condenser \((K,\Omega)\) is defined to be

\[
C_{1,p}(K,\Omega) = \inf \left\{ \int_{\Omega} |\nabla u(x)|^p \, dx : u \in C_0^\infty(\Omega), u(x) \geq 1 \text{ for } x \in K \right\}.
\]

By \( B(x,r) \) we denote the open ball in \( \mathbb{R}^n \) of radius \( r, 0 < r < \infty \), centered at \( x \in \mathbb{R}^n \).

**Definition 1.** A closed set \( K \subset \mathbb{R}^n \) is locally uniformly \((1,p)\)-thick, if there exist numbers \( b > 0 \) and \( r_0, 0 < r_0 \leq \infty \) such that

\[
C_{1,p}(B(x,r) \cap K, B(x,2r)) \geq b C_{1,p}(B(x,r), B(x,2r))
\]

for all \( x \in K \) and \( 0 < r < r_0 \). If \( r_0 = \infty \), then the set \( K \) is called uniformly \((1,p)\)-thick.

Note that a scaling argument yields

\[
C_{1,p}(B(x,r), B(x,2r)) = c(n,p)r^{n-p}.
\]

P. Hajłasz [4] used the Hardy-Littlewood maximal operator \( M \) and showed that for a domain \( \Omega \) with a locally uniformly \((1,p)\)-thick complement there exists \( q \in (1,p) \) such that every function \( u \in C_0^\infty(\Omega) \) satisfies the pointwise analogue of the Hardy inequality, which in a slightly simplified formulation reads

\[
|u(x)| \leq c d(x) \left[M(|u|)(x)\right]^{1/q}.
\]

As a corollary he obtained the integral Hardy inequality

\[
\int_\Omega |u(x)|^p d(x)^{-p} dx \leq c \int_\Omega |\nabla u(x)|^p d(x)^{p} dx,
\]

for small positive numbers \( a \). Similar results were obtained also by J. Kinnunen and O. Martio [6].

Our aim is to extend these results for derivatives of higher order.
If \( a = (a_1, \ldots, a_n) \) is an \( n \)-tuple of non-negative integers, \( |a| = \sum_{i=1}^{n} a_i \), and for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) we set \( x^a = x_1^{a_1} \cdots x_n^{a_n} \). The corresponding partial derivative operators will be denoted by

\[
D^a = D_1^{a_1} \cdots D_n^{a_n} = \frac{\partial^{|a|}}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}
\]

and the gradient of a real-valued function of order \( k \), \( k \in \mathbb{N} \), will be the vector \( \nabla^k u = \{D^a u\}_{|a|=k} \). For \( k = 1 \), \( \nabla^1 u = \nabla u \) is the usual gradient.

Given a measurable set \( E \subset \mathbb{R}^n \), we denote its Lebesgue \( n \)-measure by \( |E| \) and the characteristic function of \( E \) by \( \chi_E \). Constants \( c \) in estimates may vary during calculations but they always remain independent of all non-fixed entities.

2. The Pointwise Hardy Inequality

The fractional maximal function \( M_{\gamma,R} u \), \( 0 \leq \gamma \leq n \), \( 0 < R \leq \infty \), is defined for every \( u \in L_{loc}^1(\mathbb{R}^n) \) by

\[
M_{\gamma,R} u(x) = \sup_{0<r< R} |B(x,r)|^{\gamma/n-1} \int_{B(x,r)} |u(y)| \, dy, \quad x \in \mathbb{R}^n.
\]

Note that \( M_{0,\infty} u = M u \) is the classical Hardy-Littlewood maximal function.

**Theorem 1.** Let \( 1 \leq p < \infty \), let \( k \) be a positive integer and \( 0 \leq \gamma < k \). Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) such that \( \mathbb{R}^n \setminus \Omega \) is locally uniformly \((1,p)\)-thick and let \( b \) be the constant from Definition 1. Then there exists a constant \( c = c(k,p,n,b) > 0 \) such that every function \( u \in C^k(\Omega) \) satisfies the inequality

\[
|u(x)| \leq c d(x)^{k-\gamma/p} |M_{\gamma,R} u|^{1/p} \left[ |\nabla^k u| |\chi_{\Omega \cap \mathbb{R}^n} 2^{a(x)}(x)\right]^{1/p},
\]

where \( x \in \Omega \), \( d(x) < r_0 \), and \( x \in \partial \Omega \) is such that \( |x - \bar{x}| = d(x) \).

This is the main result of this section which extends Theorem 2 of [4]. To prove it we shall need several auxiliary assertions. The first one is a generalization of [3, Lemma 7.16]:

**Lemma 1.** Let \( k \) be a natural number. There exists a constant \( c = c(k,n) > 0 \) such that for every ball \( B \subset \mathbb{R}^n \) the inequality

\[
|u(x) - |B|^{-1} \int_B P(x,y) \, dy| \leq c \int_B \frac{|\nabla^k u(y)|}{|x-y|^{n-k}} \, dy, \quad x \in B,
\]

\( 115 \)
holds, where \( P \) is the polynomial of order \( \leq k - 1 \) given by

\[
P(x, y) = \sum_{|\alpha| \leq k - 1} \frac{(-1)^{|\alpha|}}{\alpha!} D^\alpha u(x) (y - x)^\alpha, \quad x, y \in B.
\]

Lemma 1 can be proved in a way similar to the proof of Lemma 7.16 in [3] using the Taylor expansion of the function \( v(\tau) = u(x + r\theta) \), where \( \tau = |x - y|, \theta = (y - x)/r \), \( x, y \in \Omega \). Note that assertions of this type can be found for instance in [1, §8.1] and [8, §1.1.10].

The next assertion is a variation of a well-known result of L. I. Hedberg.

**Lemma 2.** Let \( 0 \leq \gamma < \kappa \) and let \( B \subset \mathbb{R}^n \) be a ball of radius \( R \). Then there exists a constant \( c = c(n, \gamma, \kappa) > 0 \) such that every function \( g \in L^1_{loc}(B) \) satisfies the inequality

\[
\int_B \frac{|g(y)|}{|x - y|^{n-\kappa}} \, dy \leq c R^{n-\gamma} M_1,2R(g)(x), \quad x \in B.
\]

**Proof.** Fix \( x \in B \) and for \( i \in \mathbb{N} \) set \( A_i = (B(x, 2^{-i+1}R) \setminus B(x, 2^{-i}R)) \cap B \). Then

\[
\int_B \frac{|g(y)|}{|x - y|^{n-\kappa}} \, dy = \sum_{i=0}^\infty \int_{A_i} \frac{|g(y)|}{|x - y|^{n-\kappa}} \, dy
\]

\[
\leq \max(1, 2^{-\gamma}) \sum_{i=0}^\infty (2^{-i})^{n-\kappa} \int_{B(x, 2^{i+1}R)} |g(y)| \, dy
\]

\[
\leq |B(0,1)|^{-1} \max(1, 2^{-\gamma}) 2^{n-\gamma} R^{n-\gamma} \sum_{i=0}^\infty 2^{-i(n-\kappa)} M_1,2R(g)(x).
\]

We shall also need the following inequality of Poincaré type which follows from the considerations in [8, Sections 9.3 and 10.1.2].

**Lemma 3.** Let \( 1 \leq p \leq \infty \). Let \( B = B(x, R) \) be a ball in \( \mathbb{R}^n \) and let \( K \) be a closed subset of \( B \). Then every function \( u \in C^m(B) \) such that \( \text{dist}(\text{supp} u, K) > 0 \) satisfies the inequality

\[
\int_B |u(x)|^p \, dx \leq c \frac{R^n}{C_{1,\delta}(K, B(x, 2R))} \int_B |\nabla u(x)|^p \, dx,
\]

where \( c \) is a positive constant independent of \( B, K \) and \( u \).
Proof of Theorem 1. Let \( z \in \Omega \) be such that \( d(z) < r_0 \), where \( r_0 \) is the number from Definition 1. Let \( x \) satisfy \( |x - z| = d(x) = R \) and let \( u \in C^p_0(\Omega) \). Set \( B = B(z, 2R) \). Then \( x \in B \) and

\[
|u(x)| \leq |u(x) - P_B(x)| + |P_B(x)|,
\]

where \( P_B(z) = |B|^{-1} \int_B P(x, y) \, dy \) and \( P \) is the polynomial from Lemma 1. Using Lemma 1, Lemma 2 and the Hölder inequality we obtain

\[
|u(x) - P_B(x)| \leq c \int_B \frac{\|\nabla^k u(y)\|}{|x - y|^{k-1}} \, dy \leq c R^{k-1} M_{k, \alpha, \beta} \langle \|\nabla^k u \| \chi_B \rangle(x)
\]

\[
\leq c R^{k-1/p} M_{k, \alpha, \beta} \langle \|\nabla^k u \| \chi_B \rangle(x) \rangle^{1/p}.
\]

From (2.2) we have

\[
|P_B(x)| \leq |B|^{-1} \int_B |P(x, y)| \, dy \leq c \sum_{i=0}^{k-1} R^i |B|^{-1} \int_B \|\nabla^i u(y)\| \, dy
\]

\[
\leq c \sum_{i=0}^{k-1} R^i \left( |B|^{-1} \int_B \|\nabla^i u\| \, dy \right)^{1/p}.
\]

Repeated application of Lemma 3 and of (1.2) and (1.3) yields

\[
\int_B \|\nabla^j u(x)\|^p \, dx \leq c \int_{\mathbb{R}^n} \|\nabla^j u(x)\|^p \, dx
\]

\[
\leq c R^j \int_B \|\nabla^j u(x)\|^p \, dx
\]

\[
\leq c R^{j(k-1)p} \int_B \|\nabla^j u(x)\|^p \, dx, \quad i = 0, \ldots, k-1.
\]

Hence,

\[
|P_B(x)| \leq c R^k \left( |B|^{-1} \int_B \|\nabla^k u \| \, dx \right)^{1/p}
\]

\[
\leq c R^{k-1/p} M_{k, \alpha, \beta} \langle \|\nabla^k u \| \chi_B \rangle(x) \rangle^{1/p}.
\]

The inequality (2.1) follows from (2.3)–(2.5). \( \square \)
3. INTEGRAL INEQUALITIES

In this section we shall use Theorem 1 to obtain higher-order analogues of the classical Hardy inequality. As in [4] and [6], in further considerations we shall essentially use the openness of the \((1,p)\)-thickness with respect to \(p\). This deep property was originally proved by J.L. Lewis [7, Theorem 1] and later on in another way by P. Mikkonen [9, Theorem 8.2]. The following lemma can be obtained as a particular case of Lewis’ and Mikkonen’s results. It is not important for our purpose that Lewis dealt with another type of capacity.

**Lemma 4.** Let \(1 < p < \infty\) and let \(K \subset \mathbb{R}^n\) be a closed locally uniformly \((k,p)\)-thick set. Then there exists \(q, 1 < q < p\), depending only on \(n, k, p\) and \(b\), such that \(K\) is locally uniformly \((k,q)\)-thick with the same value of \(r_0\) as for \(p\).

For \(r > 0\) we set

\[\Omega_r = \{x \in \Omega: d(x) < r\}.

**Theorem 2.** Let \(1 < p < \infty\) and let \(k\) be a positive integer. Let \(\Omega\) be an open subset of \(\mathbb{R}^n\) such that \(\mathbb{R}^n \setminus \Omega\) is locally uniformly \((1,p)\)-thick. Then there exists a positive constant \(c = c(k,p,n,b)\) such that the inequality

\[
\frac{1}{\Omega_r} \left\{ \frac{|u(x)|}{d(x)^k} \right\}^p dx \leq c \int_{\Omega_r} |\nabla^k u(x)|^p dx
\]

holds for every function \(u \in C_0^\infty(\Omega)\) and for every \(r \in (0, r_0)\), where \(r_0\) is the parameter given in Definition 1.

**Proof.** Let \(p > 1\) and let \(q \in (1,p)\) be from Lemma 4, and suppose that \(r \in (0, r_0)\). It follows from (2.1) that for all \(u \in C_0^\infty(\Omega),\)

\[
\frac{|u(x)|}{d(x)^k} \leq c \left[ M(\|
abla^k u\|^p_{\mathcal{X}_{\Omega}})(x) \right]^{1/p}, \quad x \in \Omega_r.
\]

We use the boundedness of \(M: L^{q/p} \to L^{p/q}\) and the Hölder inequality to obtain

\[
\int_{\Omega_r} \left( \frac{|u(x)|}{d(x)^k} \right)^p dx \leq c \int_{\Omega_r} [M(\|
abla^k u\|^p_{\mathcal{X}_{\Omega}})(x)]^{p/q} dx \leq c \int_{\Omega_r} |\nabla^k u(x)|^p dx.
\]

Note that the norm of the maximal operator \(M\) and, consequently, also the constant \(c\) depend on the value of \(p/q\). \(\square\)
If $p = 1$, we cannot use Lemma 4. Instead we use the fact that for $n$ with $|\Omega| < \infty$ the maximal operator $M$ is a bounded mapping of $L \log L(\Omega)$ in $L^1(\Omega)$ (see [2], p. 74). Recall that $L \log L(\Omega)$ is the Zygmund space which consists of all measurable functions $u$ with $\int_\Omega |u(x)| \log^+ |u(x)| \, dx < \infty$, endowed with the norm

$$
\|u\|_{L \log L(\Omega)} = \int_0^{||u||} u^*(t) \log \frac{|u|}{t} \, dt,
$$

where $u^*$ is the non-increasing rearrangement of $u$.

**Theorem 3.** Let $p = 1$ and let $k$ be a positive integer. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,1)$-thick. Then there exists a positive constant $c = c(k, n, b)$ such that the inequality

$$
(2.9) \quad \int_\Omega \frac{|u(x)|}{d(x)^k} \, dx \leq c \|\nabla^b u\|_{L \log L(\Omega)}
$$

holds for every function $u \in C^0_0(\Omega)$ and for every $r \in (0, r_0)$, where $r_0$ is the parameter given in Definition 1.

**Proof.** From the estimate (2.1) we have

$$
|u(x)| d(x)^{-k} \leq c M(\nabla^b u)(x), \quad x \in \Omega.
$$

Integrating both sides of the inequality over $\Omega$, and using the boundedness of $M : L \log L(\Omega) \rightarrow L^1(\Omega)$ we arrive at the inequality (2.9).

**Corollary 1.** Let $1 < p < \infty$ and let $k$ be a positive integer. Let $\Omega$ be an open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,p)$-thick. Then there exists a number $c_0 > 0$ such that the inequality

$$
(2.10) \quad \int_\Omega \left(\frac{|u(x)|}{d(x)^k}\right)^p d(x)^p \, dx \leq c \int_\Omega |\nabla^b u(x)|^p d(x)^p \, dx
$$

holds for all $u \in C^0_0(\Omega)$, $r \in (0, r_0)$ and $0 \leq \varepsilon < c_0$. The constant $c > 0$ depends on $n$, $p$, $k$, $b$ and on the number $q$ from Lemma 4.

**Proof.** Fix $\varepsilon > 0$ and let $u \in C^0_0(\Omega)$ be such that the integral on the right hand side of (2.10) is finite.

If $k = 1$, we set $v(x) = |u(x)| d(x)$. Then

$$
(2.11) \quad |\nabla v(x)| \leq |\nabla u(x)| d(x)^k + c |u(x)| d(x)^{k-1} \quad \text{for a.e.} \quad x \in \Omega,
$$

119
and (2.10) implies that $v$ belongs to the Sobolev space $W^{1,p}_0(\Omega)$. Applying Theorem 2 to functions from $C_0^{\infty}(\Omega)$ which approximate $v$ in $W^{1,p}_0(\Omega)$ and passing to the limit we obtain

$$
\int_{\Omega} \left( \frac{\|v(x)\|}{d(x)} \right)^p d(x)^p dx \leq c \int_{\Omega} \left( \frac{\|\nabla v(x)\|}{d(x)} \right)^p d(x)^p dx
$$

for $0 < \varepsilon < \varepsilon_0$. By (2.11), we have

$$
\int_{\Omega} \left( \frac{\|u(x)\|}{d(x)} \right)^p d(x)^p dx \leq c \left( \int_{\Omega} \|\nabla u(x)\| d(x)^p dx + c^p \int_{\Omega} \left( \frac{\|u(x)\|}{d(x)} \right)^p d(x)^p dx \right).
$$

Thus, the inequality (2.10) holds for $0 < \varepsilon < \varepsilon_0 = c^{-1/p}$.

Let $k > 1$ and suppose that the inequality (2.10) holds for $j = 1, 2, \ldots, k-1$ and $0 < \varepsilon < \varepsilon_0$. Let $\phi$ be the regularized distance function equivalent to $d$ and satisfying the estimate:

$$
\|\nabla^j \phi(x)\| \leq c j d(x)^{1-j}, \quad x \in \Omega, \quad j = 1, 2, \ldots,
$$

(see, e.g., [11, p. 171]). Set $u(x) = |u(x)| \phi(x)^k$. Then

$$
\|\nabla^k u(x)\| \leq \|\nabla^k u(x)\| \phi(x)^k + c \sum_{j=1}^k Q_j(\varepsilon) \|\nabla^{k-j} u(x)\| \phi(x)^{k-j},
$$

where $Q_j$ are polynomials of degree $j$. Thus, we have

$$
\int_{\Omega} \left( \frac{|u(x)|}{d(x)} \right)^p d(x)^p dx \leq c \int_{\Omega} \left( \frac{|u(x)|}{\phi(x)^k} \right)^p d(x)^p dx
$$

$$
\leq c \int_{\Omega} \left\{ \|\nabla^k u(x)| \phi(x)^k \| d(x)^p dx + c^p \sum_{j=1}^k Q_j(\varepsilon) \int_{\Omega} \left( \frac{|u(x)|}{\phi(x)^k} \right)^p \phi(x)^{k-j} d(x)^p dx
$$

$$
\leq c \int_{\Omega} \left\{ \|\nabla^k u(x)| \phi(x)^k \| d(x)^p dx + c^p \sum_{j=1}^k Q_j(\varepsilon) \int_{\Omega} \left( \frac{|u(x)|}{\phi(x)^k} \right)^p \phi(x)^{k-j} d(x)^p dx
$$

and the inequality (2.10) holds for $0 < \varepsilon < c^{-1/p}$.

**Corollary 2.** Let $\Omega$ be such that $\mathbb{R}^n \setminus \Omega$ is locally uniformly $(1,p)$-thick with $r_0 > \frac{1}{2} \text{diam}(\Omega)$. Then the inequality (2.1) holds for every $x \in \Omega$ and the assertions of Theorem 2, Theorem 3 and Corollary 1 hold with $\Omega$ in place of $\Omega_r$ and for all functions $u$ from the corresponding Sobolev spaces $W^{1,p}_0(\Omega)$ on $\Omega$.

**Proof.** It suffices to observe that $\Omega_r = \Omega$ for $r > \frac{1}{2} \text{diam}(\Omega)$ and that the constant $c$ does not depend on the parameter $r_0$. \qed
Note that the assumption of Corollary 2 holds, in particular, if $\mathbb{R}^n \setminus \Omega$ is uniformly $(1,p)$-thick (i.e., $r_0 = \infty$).

An open problem. Additional weights could be introduced into the inequality (2.6) by applying a weighted inequality for the maximal function. Following the proof of Theorem 2 we can multiply both sides of inequality (2.7) (or, more precisely, of inequality (2.11)) by $d(x)^p$ and integrate over $\Omega$. However, to make the final step in (2.8) we have to know that the maximal function satisfies the weighted inequality

$$\int_{\Omega} \left( M(\nabla^2 u(x_0, \cdot)(x) \right)^{\frac{p}{2}} d(x)^p \, dx \leq c \int_{\Omega} |\nabla^2 u(x)|^p d(x)^p \, dx.$$ 

Note that we are dealing with the global maximal function (the balls in the construction of $M_{x_0, d(x)}$ from inequality (2.1) cross the complement of $\Omega$) and so to use the known weighted inequalities for $M$ we would have to consider $d(x)$ extended properly outside $\Omega$. The question is, if the sufficient conditions for such weighted estimate would not override the condition of $(1,p)$-thickness of $\mathbb{R}^n \setminus \Omega$.

References

Authors’ addresses: D. E. Edmunds, Centre for Mathematical Analysis and Its Applications, School of Mathematical and Physical Sciences, University of Sussex, Falmer, Brighton, BN1 9QH, United Kingdom, e-mail: D.E.Edmunds@sussex.ac.uk; J. Rákosník, Mathematical Institute, Academy of Sciences of the Czech Republic, Zitná 25, 115 67 Praha 1, Czech Republic, e-mail: rakosnak@math.cas.cz.