

Vladimir G. Maz'ya; Tatjana Olegovna Shaposhnikova
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ON POINTWISE INTERPOLATION INEQUALITIES FOR
DERIVATIVES

VLADIMIR MAZ'YA, TATYANA SHAPOSHNIKOVA, Linköping

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. Pointwise interpolation inequalities, in particular,

$$|\nabla_k u(x)| \leq c (\mathcal{M}u(x))^{1-k/m} (\mathcal{M}\nabla_m u(x))^{k/m}, \quad k < m,$$

and

$$|I_z f(x)| \leq c (\mathcal{M}I_\zeta f(x))^{\operatorname{Re} z / \operatorname{Re} \zeta} (\mathcal{M}f(x))^{1-\operatorname{Re} z / \operatorname{Re} \zeta}, \quad 0 < \operatorname{Re} z < \operatorname{Re} \zeta < n,$$

where ∇_k is the gradient of order k , \mathcal{M} is the Hardy-Littlewood maximal operator, and I_z is the Riesz potential of order z , are proved. Applications to the theory of multipliers in pairs of Sobolev spaces are given. In particular, the maximal algebra in the multiplier space $M(W_2^m(\mathbb{R}^n) \rightarrow W_2^l(\mathbb{R}^n))$ is described.

Keywords: Landau inequality, interpolation inequalities, Hardy-Littlewood maximal operator, Gagliardo-Nirenberg inequality, Sobolev multipliers

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1. INTRODUCTION

It is well known that an arbitrary function on the real line \mathbb{R} with Lipschitz derivative u' satisfies

$$(1) \quad |u'(x)|^2 \leq 2 \sup |u| \sup |u''|,$$

where 2 is the best possible constant (Landau [1]). For the history of this estimate as well as of its analogs and generalizations one can consult, for example, Section 13.5 of our book [2].

In the present paper we are interested in purely pointwise modifications of (1), where both factors in the right-hand side are functions of x . It is trivial, of course, that the second sign of supremum cannot be removed. Moreover, (1) is no longer valid without the first supremum if f has a simple zero at a point.

However, under the additional assumption of non-negativity of u one obtains the following stronger variant of (1)

$$(2) \quad |u'(x)|^2 \leq 2u(x) \sup |u''|$$

for all $x \in \mathbb{R}$ (early applications and generalizations of this inequality can be found in [3] and [4]). Indeed, for any $t \in \mathbb{R}$ we have

$$0 \leq u(x+t) = u(x) + tu'(x) + \int_0^t (u'(x+\tau) - u'(x)) \, d\tau.$$

Therefore, the trinomial $u(x) + tu'(x) + \frac{1}{2}t^2 \sup |u''|$ is non-negative and the non-positivity of its discriminant is equivalent to (2).

Kufner and Maz'ya [5] gave generalizations of (2), and showed, in particular, that it can be easily improved as

$$(3) \quad |u'(x)|^2 \leq 2u(x) \mathcal{M}_{\pm} u''(x),$$

where the sign $+$ or $-$ is taken if $u'(x) \leq 0$ or $u'(x) \geq 0$, respectively, and \mathcal{M}_+ , \mathcal{M}_- are the Hardy-Littlewood right and left maximal operators:

$$\mathcal{M}_+ \varphi(x) = \sup_{\tau > 0} \frac{1}{\tau} \int_x^{x+\tau} |\varphi(y)| \, dy,$$

$$\mathcal{M}_- \varphi(x) = \sup_{\tau > 0} \frac{1}{\tau} \int_{x-\tau}^x |\varphi(y)| \, dy.$$

We claim that any complex valued function on \mathbb{R} with absolutely continuous first derivative is subject to the inequality

$$(4) \quad |u'(x)|^2 \leq 8\mathcal{M}u(x)\mathcal{M}u''(x),$$

where \mathcal{M} is the Hardy-Littlewood operator

$$\mathcal{M}\varphi(x) = \sup_{\tau > 0} \frac{1}{2\tau} \int_{x-\tau}^{x+\tau} |\varphi(y)| \, dy.$$

Moreover, (4) can be improved as

$$(5) \quad |u'(x)|^2 \leq 8\mathcal{M}^\circ u(x)\mathcal{M}u''(x),$$

where

$$(6) \quad \mathcal{M}^\circ u(x) = \sup_{\tau > 0} \frac{1}{2\tau} \left| \int_{x-\tau}^{x+\tau} \operatorname{sign}(x-y)u(y) \, dy \right|.$$

In fact, the following identity is readily checked by integration by parts

$$(7) \quad u'(0) = \frac{1}{t^2} \int_{-t}^t \operatorname{sign} y u(y) \, dy - \frac{1}{2} \int_{-t}^t \left(1 - \frac{|y|}{t}\right)^2 \operatorname{sign} y u''(y) \, dy$$

where $t > 0$. (We replace x by 0 to simplify the notation.) Hence

$$|u'(0)| \leq \frac{2}{t} \mathcal{M}^\circ u(0) + t \mathcal{M} u''(0),$$

which is equivalent to (5).

Another direct consequence of (7) is the inequality

$$|u'(x)|^2 \leq \frac{8}{3} \mathcal{M}^\circ u(x) \sup |u''|.$$

The constant $\frac{8}{3}$ in this inequality (and even in the weaker one with M° replaced by M) is best possible. In fact, one can easily check that the odd function u_0 given by

$$u_0(x) = \begin{cases} x(2-x) & \text{for } 0 \leq x < \frac{3}{2}, \\ \frac{(x-3)^2}{3} & \text{for } \frac{3}{2} \leq x < 3, \\ 0 & \text{for } x \geq 3 \end{cases}$$

satisfies

$$u'_0(0) = 2, \quad \mathcal{M}^\circ u_0(0) = \mathcal{M} u_0(0) = \frac{3}{4}, \quad \sup |u''_0| = 2.$$

As Kufner and Maz'ya noticed in [5], a simple change in the above argument leading to (2) results in the following generalization of (2) with the best possible constant

$$(8) \quad |u'(x)|^{\alpha+1} \leq \left(\frac{\alpha+1}{\alpha}\right)^\alpha u(x)^\alpha \sup_y \frac{|u'(y) - u'(x)|}{|y-x|^\alpha},$$

where $u(x) \geq 0$ and $0 < \alpha < 1$.

One can arrive at an analogue of (8) for arbitrary complex valued functions, where the role of Hölder's seminorm is played by the function

$$\mathcal{D}_{p,m} u(x) = \left(\int_{\mathbb{R}} \frac{|u'(x) - u'(y)|^p}{|x-y|^{p(m-1)+1}} \, dy \right)^{1/p}, \quad m \in (1, 2), \quad p \in [1, \infty].$$

This function is important because its L_p -norm is a seminorm in the fractional Sobolev space $W_p^m(\mathbb{R})$.

We note that

$$(9) \quad u'(0) = \frac{1}{t^2} \int_{-t}^t \text{sign } y u(y) dy + \frac{1}{t^2} \int_{-t}^t (t - |y|) (u'(0) - u'(y)) dy.$$

By Hölder's inequality, the absolute value of the second term on the right-hand side is dominated by

$$t^{m-1} (2\mathbf{B}(qm, q+1))^{1/q} \mathcal{D}_{p,m} u(0),$$

where $m \in (1, 2)$, $p^{-1} + q^{-1} = 1$ and \mathbf{B} is Euler's Beta-function. This along with (9) implies

$$|u'(0)| \leq 2t^{-1} \mathcal{M}^\circ u(0) + t^{m-1} (2\mathbf{B}(qm, q+1))^{1/q} \mathcal{D}_{p,m} u(0).$$

Minimizing the right-hand side we conclude that the inequality

$$(10) \quad |u'(x)|^m \leq m \left(\frac{2m}{m-1} \right)^{m-1} (2\mathbf{B}(qm, q+1))^{1/q} (\mathcal{M}^\circ u(x))^{m-1} \mathcal{D}_{p,m} u(x)$$

is valid for almost all $x \in \mathbb{R}$. In particular, for $p = \infty$ and $m = \alpha + 1$, $0 < \alpha < 1$, we have the following analogue of (8):

$$(11) \quad |u'(x)|^{\alpha+1} \leq \frac{2^{\alpha+1}}{\alpha+2} \left(\frac{\alpha+1}{\alpha} \right)^\alpha (\mathcal{M}^\circ u(x))^\alpha \sup_y \frac{|u'(y) - u'(x)|}{|y-x|^\alpha}.$$

The constant factor in this inequality (and even in the weaker one with \mathcal{M}° replaced by \mathcal{M}) is best possible as can be checked by the odd function u_α given for $x \geq 0$ by

$$u_\alpha(x) = \begin{cases} (\alpha+1)x - x^{\alpha+1} & \text{for } 0 \leq x < (\frac{\alpha+2}{2})^{1/\alpha}, \\ \frac{\alpha}{\alpha+2} (2(\frac{\alpha+2}{2})^{1/\alpha} - x)^{\alpha+1} & \text{for } (\frac{\alpha+2}{2})^{1/\alpha} \leq x < 2(\frac{\alpha+2}{2})^{1/\alpha}, \\ 0 & \text{for } x \geq 2(\frac{\alpha+2}{2})^{1/\alpha}. \end{cases}$$

In the sequel we prove n -dimensional generalizations of the above interpolation inequalities and give applications to the theory of pointwise multipliers in pairs of Sobolev spaces.

2. MULTIDIMENSIONAL VARIANTS OF INEQUALITY (4)

Let u be a function on \mathbb{R}^n such that its distributional derivatives of order m are locally summable. By $\nabla_m u$ we mean the gradient of u of order m , i.e.

$$\nabla_m u = \{\partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u\}, \quad \alpha_1 + \dots + \alpha_n = m,$$

where ∂_{x_j} is a partial derivative. The Euclidean length of $\nabla_m u$ will be denoted by $|\nabla_m u|$ and we write ∂ or ∇ instead of ∇_1 .

Let \mathcal{M} be the Hardy-Littlewood maximal operator over centered balls defined by

$$\mathcal{M}\varphi(x) = \sup_{r>0} (\text{meas}_n B_r)^{-1} \int_{B_r(x)} |\varphi(y)| \, dy,$$

where φ is a scalar or vector-valued function in \mathbb{R}^n , $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$ and $B_r = B_r(0)$.

Our goal is to prove the following generalization of inequality (4).

Theorem 1. *Let k, l and m be integers, $0 \leq l \leq k \leq m$. Then there exists a positive constant $c = c(k, l, m, n)$ such that*

$$(12) \quad |\nabla_k u(x)| \leq c (\mathcal{M}\nabla_l u(x))^{\frac{m-l}{m-k}} (\mathcal{M}\nabla_m u(x))^{\frac{k-l}{m-k}}$$

for almost all $x \in \mathbb{R}^n$.

Proof. Clearly, it suffices to prove (12) for $l = 0$ when it becomes

$$(13) \quad |\nabla_k u(x)| \leq c (\mathcal{M}u(x))^{\frac{m-k}{m}} (\mathcal{M}\nabla_m u(x))^{\frac{k}{m}}.$$

Let η be a function in the ball B_1 with Lipschitz derivatives of order $m - 2$ which vanishes on ∂B_1 together with all these derivatives. Also let

$$\int_{B_1} \eta(y) \, dy = 1.$$

We shall use the Sobolev integral representation:

$$\begin{aligned} v(0) &= \sum_{|\beta| < m-k} t^{-n} \int_{B_t} \frac{(-y)^\beta}{\beta!} \partial^\beta v(y) \eta\left(\frac{y}{t}\right) \, dy \\ &+ (-1)^{m-k} (m-k) \sum_{|\alpha|=m-k} \int_{B_t} \frac{y^\alpha}{\alpha!} \partial^\alpha v(y) \int_{|y|/t}^\infty \eta\left(\frac{y}{|y|}\right) e^{n-1} \, d\varrho \frac{dy}{|y|^n} \end{aligned}$$

(see [7], Section 1.5.1).

By setting here $v = \partial^\gamma u$ with an arbitrary multiindex γ of order k and integrating by parts in the first integral, we arrive at the identity

$$(14) \quad \begin{aligned} \partial^\gamma u(0) &= (-1)^k t^{-n} \int_{B_t} u(y) \sum_{|\beta| < m-k} \frac{1}{\beta!} \partial^{\beta+\gamma} \left(y^\beta \eta\left(\frac{y}{t}\right) \right) dy \\ &+ \sum_{|\alpha|=m-k} (-1)^{m-k} (m-k) \int_{B_t} \frac{y^\alpha}{\alpha!} \partial^{\alpha+\gamma} u(y) \int_{|y|/t}^\infty \eta\left(\frac{y}{|y|}\right) e^{n-1} d e \frac{dy}{|y|^n}. \end{aligned}$$

Hence

$$(15) \quad |\nabla_k u(0)| \leq c_1 t^{-n-k} \int_{B_t} |u(y)| dy + c_2 \int_{B_t} |\nabla_m u(y)| \frac{dy}{|y|^{n-m+k}}.$$

If $m-k \geq n$, the second integral does not exceed

$$t^{m-k-n} \int_{B_t} |\nabla_m u(y)| dy.$$

In the case $m-k < n$ the second integral in (15) equals

$$t^{m-k-n} \int_{B_t} |\nabla_m u(y)| dy + (n-m+k) \int_0^t \frac{d\tau}{\tau^{n-m+k+1}} \int_{B_\tau} |\nabla_m u(y)| dy.$$

Therefore

$$(16) \quad \int_{B_t} |\nabla_m u(y)| \frac{dy}{|y|^{n-m+k}} \leq \frac{n}{m-k} t^{m-k} \sup_{\tau \leq t} \tau^{-n} \int_{B_\tau} |\nabla_m u(y)| dy.$$

Thus, for any $t > 0$,

$$(17) \quad |\nabla_k u(0)| \leq c_3 t^{-k} \mathcal{M}u(0) + c_4 t^{m-k} \mathcal{M}\nabla_m u(0)$$

which implies (13). The result follows. \square

Remark 1. The above proof enables one to improve (13) replacing \mathcal{M} by the maximal operator

$$\sup_{t>0} \frac{1}{\text{meas}_n B_t} \sum_{|\gamma|=k} \left| \int_{B_t(x)} u(y) H_\gamma(t^{-1}(y-x)) dy \right|.$$

Here $\{H_\gamma\}_{|\gamma|=k}$ is a collection of bounded measurable functions such that, for all multiindices α of order $|\alpha| \leq k$,

$$\int_{B_t} y^\alpha H_\gamma(y) dy = \delta_\alpha^\gamma,$$

where δ_α^γ is Kronecker's symbol.

Remark 2. One can specify the constants c_3 and c_4 in (17) but we do not dwell upon this being unaware if the values obtained are best possible. In the special case $k = 1$, $m = 2$ a direct generalization of (5) is proved as follows. First, (14) becomes

$$\begin{aligned} \frac{\partial u(0)}{\partial x_i} &= -t^{-n} \int_{B_t} u(y) \frac{\partial}{\partial y_i} \left(\eta \left(\frac{y}{t} \right) \right) dy \\ &\quad - \sum_{j=1}^n \int_{B_t} y_j \frac{\partial^2 u(y)}{\partial y_i \partial y_j} \int_{|y|/t}^{\infty} \eta \left(\frac{y}{|y|} \right) e^{n-1} d\rho \frac{dy}{|y|^n}. \end{aligned}$$

We choose

$$(18) \quad \eta(y) = \frac{n+1}{\text{meas}_n B_1} (1 - |y|)_+.$$

Then

$$\begin{aligned} |\nabla u(0)| &\leq t^{-1} \frac{n+1}{\text{meas}_n B_t} \left| \int_{B_t} u(y) \frac{y}{|y|} dy \right| \\ &\quad + \frac{1}{\text{meas}_{n-1} \partial B_1} \int_{B_t} |\nabla_2 u(y)| \frac{dy}{|y|^{n-1}}. \end{aligned}$$

By (16) the second term on the right is not greater than $t\mathcal{M}\nabla_2 u(0)$. Therefore

$$|\nabla u(0)| \leq \frac{n+1}{t} \mathcal{M}^\circ u(0) + t\mathcal{M}\nabla_2 u(0),$$

where \mathcal{M}° is a multidimensional generalization of (6):

$$(19) \quad \mathcal{M}^\circ u(x) = \sup_{t>0} \frac{1}{\text{meas}_n B_t} \left| \int_{B_t(x)} u(y) \frac{y-x}{|y-x|} dy \right|.$$

Finally,

$$|\nabla u(x)|^2 \leq 4(n+1)\mathcal{M}^\circ u(x) \mathcal{M}\nabla_2 u(x)$$

for almost all $x \in \mathbb{R}^n$.

Remark 3. Suppose that instead of \mathcal{M} we use the modified maximal operator \mathcal{M}_δ given by

$$\mathcal{M}_\delta \varphi(x) = \sup_{0 < r < \delta} (\text{meas}_n B_r)^{-1} \int_{B_r(x)} |\varphi(y)| dy.$$

Then the proof of Theorem 1 (with small changes) leads to the following alternatives: either

$$\mathcal{M}_\delta u(x) < \frac{m-k}{m} \delta^m \mathcal{M}_\delta \nabla_m u(x)$$

and

$$|\nabla_k u(x)| \leq c (\mathcal{M}_\delta u(x))^{1-k/m} (\mathcal{M}_\delta \nabla_m u(x))^{k/m}$$

or

$$\mathcal{M}_\delta u(x) \geq \frac{m-k}{m} \delta^m \mathcal{M}_\delta \nabla_m u(x)$$

and

$$|\nabla_k u(x)| \leq c \delta^{-k} \mathcal{M}_\delta u(x).$$

As a consequence, the local variant of (13)

$$|\nabla_k u(x)| \leq c (\delta^{-m} \mathcal{M}_\delta u(x) + \mathcal{M}_\delta \nabla_m u(x))^{k/m} (\mathcal{M}_\delta u(x))^{1-k/m}$$

is valid with c independent of δ . A similar remark can be made concerning Theorems 2–4 in the next sections.

Remark 4. Estimate (13) leads directly to the Gagliardo-Nirenberg ([8], [9]) inequality

$$(20) \quad \|\nabla_k u; \mathbb{R}^n\|_{L^q} \leq c \|u; \mathbb{R}^n\|_{L^q}^{1-k/m} \|\nabla_m u; \mathbb{R}^n\|_{L^p}^{k/m},$$

where $1 < q \leq \infty, 1 < p \leq \infty$ and

$$\frac{1}{q} = \frac{k}{m} \frac{1}{p} + \left(1 - \frac{k}{m}\right) \frac{1}{q}.$$

Indeed, by (13) the left-hand side of (20) does not exceed

$$c \left(\int_{\mathbb{R}^n} (\mathcal{M}u(x))^{s(1-k/m)} (\mathcal{M}\nabla_m u(x))^{sk/m} dx \right)^{1/s}$$

which by Hölder's inequality is majorized by

$$c \|\mathcal{M}u; \mathbb{R}^n\|_{L^q}^{1-k/m} \|\mathcal{M}\nabla_m u; \mathbb{R}^n\|_{L^p}^{k/m},$$

and it remains to refer to the boundedness of the operator \mathcal{M} in $L_\sigma(\mathbb{R}^n)$, $1 < \sigma \leq \infty$.

3. INTERPOLATION INEQUALITY FOR THE RIESZ POTENTIALS

If m is even and u is the Riesz potential of order m with non-negative density, then the estimate

$$(21) \quad |\nabla_k u(x)| \leq c \mathcal{M}u(x)^{1-k/m} \left(\mathcal{M}\Delta^{m/2}u(x) \right)^{k/m},$$

which is stronger than (2), follows directly from Hedberg's inequality

$$(22) \quad I_t f(x) \leq c (I_\tau f(x))^{t/\tau} (\mathcal{M}f(x))^{1-t/\tau},$$

where $0 < t < \tau < n$, f is a non-negative locally summable function, and the Riesz potential is defined by

$$(23) \quad I_t f(x) = c \int_{\mathbb{R}^n} \frac{f(y)}{|y-x|^{n-t}} dy$$

(see [10] or [6], Proposition 3.1.2(b)). The constant c in (23) is chosen in such a way that $I_t = (-\Delta)^{-t/2}$, i.e.

$$I_t f(x) = F_{\xi \rightarrow x}^{-1} |\xi|^{-t} F_{x \rightarrow \xi} f(x),$$

where F is the Fourier transform in \mathbb{R}^n .

However, in our case u is not a potential with non-negative density and we cannot refer to (22). Nevertheless, (21) is a direct corollary of the following assertion which seems to be of independent interest.

Theorem 2. *Let z and ζ be complex numbers subject to*

$$0 < \operatorname{Re} z < \operatorname{Re} \zeta < n$$

and let $I_z f$ denote the Riesz potential of order z defined by (23) where $t = z$ and f is a complex valued function in $L_1(\mathbb{R}^n)$ with compact support. Then there exists a constant c independent of f such that

$$(24) \quad |I_z f(x)| \leq c (\mathcal{M}I_\zeta f(x))^{\operatorname{Re} z / \operatorname{Re} \zeta} (\mathcal{M}f(x))^{1-\operatorname{Re} z / \operatorname{Re} \zeta}$$

for almost all $x \in \mathbb{R}^n$.

Proof. Denote by χ a function in the Schwartz space \mathcal{S} such that $F\chi = 1$ in a neighbourhood of the origin. From the identity

$$|\xi|^{-z} = |\xi|^{\zeta-z} F\chi(\xi) |\xi|^{-\zeta} + |\xi|^{-z} (1 - F\chi(\xi)), \quad \xi \in \mathbb{R}^n,$$

it follows that

$$(25) \quad I_z f(0) = P * I_\zeta f(0) + Q * f(0),$$

where $*$ stands for the convolution and

$$\begin{aligned} P(x) &= c_1 F_{\xi \rightarrow x}^{-1}(|\xi|^{\zeta-z} F\chi(\xi)), \\ Q(x) &= c_2 F_{\xi \rightarrow x}^{-1}(|\xi|^{-z}(1 - F\chi(\xi))). \end{aligned}$$

Let m be a positive integer such that

$$0 \leq 2m - \operatorname{Re} \zeta + \operatorname{Re} z < 2.$$

In the case $n \geq 2$ we have

$$|P(x)| = c \left| \Delta_x^n \int_{\mathbb{R}^n} \frac{\chi(y) dy}{|x-y|^{n-2m+\zeta-z}} \right| \leq c(|x|+1)^{-n-\operatorname{Re} \zeta + \operatorname{Re} z}.$$

Analogously, for $n = 1$ we obtain

$$|P(x)| = c \left| \partial_x \int_{\mathbb{R}} \frac{(x-y)\chi(y) dy}{|x-y|^{1+\zeta-z}} \right| \leq c(|x|+1)^{-1-\operatorname{Re} \zeta + \operatorname{Re} z}.$$

Hence

$$(26) \quad \begin{aligned} |P * I_\zeta f(0)| &\leq c \int_{\mathbb{R}^n} \frac{|I_\zeta f(y)| dy}{(|y|+1)^{n+\operatorname{Re} \zeta - \operatorname{Re} z}} \\ &= c \int_0^\infty \int_{B_\varrho} |I_\zeta f(y)| dy \frac{d\varrho}{(\varrho+1)^{n+1+\operatorname{Re} \zeta - \operatorname{Re} z}} \\ &\leq c \mathcal{M} I_\zeta f(0). \end{aligned}$$

The function $|\xi|^{-z}(1 - F\eta(\xi))$ is smooth which implies

$$(27) \quad |Q(y)| \leq c(N)|y|^{-N}$$

for $|y| \geq 1$ and for sufficiently large N . If $|y| < 1$ we have

$$|Q(y)| \leq c|y|^{-n+\operatorname{Re} z} + |I_z \chi(y)|$$

and since the second term on the right is bounded,

$$(28) \quad |Q(y)| \leq c|y|^{-n+\operatorname{Re} z}$$

for $|y| < 1$. Using (27) and (28) we arrive at

$$|Q * f(0)| \leq c \left(\int_{|y|>1} \frac{|f(y)| dy}{|y|^{n+1}} + \int_{|y|<1} \frac{|f(y)| dy}{|y|^{n-\operatorname{Re} z}} \right) \leq c \mathcal{M}f(0).$$

Combining this inequality with (26) we obtain from (25) that

$$|I_z f(0)| \leq c(\mathcal{M}I_\zeta f(0) + \mathcal{M}f(0)).$$

Now the dilation $y \rightarrow y/r$ with an arbitrary positive r implies

$$|I_z f(0)| \leq c(r^{\operatorname{Re} z - \operatorname{Re} \zeta} \mathcal{M}I_\zeta f(0) + r^{\operatorname{Re} z} \mathcal{M}f(0))$$

and it remains to minimize the right-hand side with respect to r . \square

The following analogue of the inequality (20) can be easily obtained from (24).

Corollary 1. *Let $1 < q \leq \infty$, $1 < p \leq \infty$, and*

$$\frac{1}{s} = \left(1 - \frac{\operatorname{Re} z}{\operatorname{Re} \zeta}\right) \frac{1}{p} + \frac{\operatorname{Re} z}{\operatorname{Re} \zeta} \frac{1}{q}.$$

Then

$$(29) \quad \|I_z f; \mathbb{R}^n\|_{L_s} \leq c \|I_\zeta f; \mathbb{R}^n\|_{L_q}^{\operatorname{Re} z / \operatorname{Re} \zeta} \|f; \mathbb{R}^n\|_{L_p}^{1 - \operatorname{Re} z / \operatorname{Re} \zeta}.$$

Proof. By (24), the left-hand side in (29) does not exceed

$$\int_{\mathbb{R}^n} (\mathcal{M}I_\zeta f(x))^{s \operatorname{Re} z / \operatorname{Re} \zeta} (\mathcal{M}f(x))^{s(1 - \operatorname{Re} z / \operatorname{Re} \zeta)} dx$$

which by Hölder's inequality is majorized by

$$c \| \mathcal{M}I_\zeta f; \mathbb{R}^n \|_{L_q}^{s \operatorname{Re} z / \operatorname{Re} \zeta} \| \mathcal{M}f; \mathbb{R}^n \|_{L_p}^{s(1 - \operatorname{Re} z / \operatorname{Re} \zeta)}.$$

It remains to refer to the boundedness of the operator \mathcal{M} in $L_\sigma(\mathbb{R}^n)$ for $1 < \sigma \leq \infty$. \square

Remark 5. Note that Hedberg's inequality (22) with $f \geq 0$ follows from (24) since, obviously,

$$\int_{B_r} I_r f(y) dy \leq c \int_{\mathbb{R}^n} \frac{r^n f(y) dy}{r^{n-\tau} + |y|^{n-\tau}}$$

and hence

$$(30) \quad \mathcal{M}I_r f(x) \leq c I_r f(x) \quad \text{a.e.}$$

if f is non-negative. Moreover, the proof of Corollary 1, along with (30), gives inequality (29) with real z, ζ and with non-negative f also for $q \in (0, 1]$. (This is an alternative proof of the corresponding inequality in Theorem 3.1.6 of [6].)

4. MULTIDIMENSIONAL VARIANTS OF INEQUALITY (10)

Let m be a fractional number with $[m]$ and $\{m\}$ denoting its integer and fractional parts. We introduce the function

$$\left(\mathcal{D}_{p,m}u\right)(x) = \left(\int_{\mathbb{R}^n} |\nabla_{[m]}u(x) - \nabla_{[m]}u(y)|^p |x-y|^{-n-p\{m\}} dy\right)^{1/p}.$$

Theorem 3. *Let k, l be integers and let m be fractional, $0 \leq l \leq k < m$. Then there exists a positive constant $c = c(k, l, m, n)$ such that*

$$(31) \quad |\nabla_k u(x)| \leq c [\mathcal{M}\nabla_l u(x)]^{\frac{m-l}{n-l}} [\mathcal{D}_{p,m}u(x)]^{\frac{l-l}{m-l}}$$

for almost all $x \in \mathbb{R}^n$.

Proof. It suffices to prove inequality (31) for $l = 0$ and $x = 0$. By (14) we have

$$(32) \quad |\nabla_k u(0)| \leq c \left(t^{-k} \mathcal{M}u(0) + t^{[m]-k} |\nabla_{[m]}u(0)| + \int_{B_t} \frac{|\nabla_{[m]}u(y) - \nabla_{[m]}u(0)|}{|y|^{n-[m]+k}} dy \right).$$

Hölder's inequality implies

$$(33) \quad \int_{B_t} \frac{|\nabla_{[m]}u(y) - \nabla_{[m]}u(0)|}{|y|^{n-[m]+k}} dy \leq c t^{m-k} \mathcal{D}_{p,m}u(0).$$

Using the function η , introduced in the proof of Theorem 1, we obtain for any multiindex γ of order $[m]$

$$(34) \quad \partial^\gamma u(0) = t^{-n} \int_{B_t} \eta\left(\frac{y}{t}\right) \partial^\gamma u(y) dy + t^{-n} \int_{B_t} \eta\left(\frac{y}{t}\right) [\partial^\gamma u(0) - \partial^\gamma(y)] dy.$$

Hence

$$(35) \quad |\nabla_{[m]}u(0)| \leq t^{-n-[m]} \left| \int_{B_t} u(y) (\nabla_{[m]}\eta)\left(\frac{y}{t}\right) dy \right| + t^{\{m\}} \left(\int_{B_t} |\eta(y)|^q |y|^{\{m\}q} dy \right)^{1/q} \mathcal{D}_{p,m}u(0),$$

where $p^{-1} + q^{-1} = 1$. Combining (32), (33) and (35) we arrive at

$$|\nabla_k u(0)| \leq c (t^{-k} \mathcal{M}u(0) + t^{m-k} \mathcal{D}_{p,m}u(0)).$$

The minimization of the right-hand side in t completes the proof. \square

Remark 6. The same argument as in Remark 4 applied to (31) gives the inequality

$$\|\nabla_k u; \mathbb{R}^n\|_{L^s} \leq c \|u; \mathbb{R}^n\|_{L^q}^{1-k/m} \|\mathcal{D}_{p,m} u; \mathbb{R}^n\|_{L^p}^{k/m}$$

where m is fractional, $k < m$, $1 \leq p \leq \infty$, and q, s are the same as in Remark 4.

Remark 7. By (35), inequality (10) can be easily extended to the n -dimensional case. Indeed, let $m \in (1, 2)$. Then, inserting the function (18) into (35) we arrive at

$$\begin{aligned} |\nabla u(0)| &\leq t^{-1} \frac{n+1}{\text{meas}_n B_t} \left| \int_{B_t} u(y) \frac{y}{|y|} dy \right| \\ &\quad + t^{m-1} \frac{n+1}{\text{meas}_n B_1} \left(\int_{B_1} (1-|y|)^q |y|^{\left(\frac{n}{p}+m-1\right)q} dy \right)^{1/q} \mathcal{D}_{p,m} u(0) \end{aligned}$$

which implies

$$\begin{aligned} |\nabla u(0)| &\leq t^{-1} (n+1) \mathcal{M}^\circ u(0) \\ &\quad + \frac{t^{m-1} (n+1) n^{1/q}}{(\text{meas}_n B_1)^{1/p}} (\mathbf{B}(q(m+n-1), q+1))^{1/q} \mathcal{D}_{p,m} u(0) \end{aligned}$$

with \mathcal{M}° given by (19). The minimization of the right-hand side results in the inequality

$$|\nabla u(x)|^m \leq \frac{(n+1)^m m^m (n\mathbf{B}(q(m+n-1), q+1))^{1/q}}{(m-1)^{m-1} (\text{meas}_n B_1)^{1/p}} (\mathcal{M}^\circ u(x))^{m-1} \mathcal{D}_{p,m} u(x)$$

containing (10) as a special case. In particular for $p = \infty$, $m = \alpha + 1$, $0 < \alpha < 1$, we have

$$|\nabla u(x)|^{\alpha+1} \leq \frac{n(n+1)^{\alpha+1} (\alpha+1)^{\alpha+1}}{(n+\alpha)(n+\alpha+1)\alpha^\alpha} (\mathcal{M}^\circ u(x))^\alpha \sup_y \frac{|\nabla u(y) - \nabla u(x)|}{|y-x|^\alpha}$$

which is a multidimensional generalization of (11).

We conclude this section with two inequalities of the same nature as (31).

Theorem 4.

(i) Let k, m be integers, and let l be noninteger, $0 < l < k \leq m$. Then there exists a positive constant $c = c(k, l, m, n)$ such that

$$(36) \quad |\nabla_k u(x)| \leq c (\mathcal{D}_{p,l} u(x))^{\frac{m-l}{m-l}} (\mathcal{M} \nabla_m u(x))^{\frac{l-l}{m-l}}$$

for almost all $x \in \mathbb{R}^n$.

(ii) Let k be integer and let l, m be noninteger, $0 < l < k < m$. Then there exists a positive constant $c = c(k, l, m, n)$ such that

$$(37) \quad |\nabla_k u(x)| \leq c (\mathcal{D}_{p,l} u(x))^{\frac{m-k}{m-l}} (\mathcal{D}_{p,m} u(x))^{\frac{k-l}{m-l}}$$

for almost all $x \in \mathbb{R}^n$.

Proof. It is sufficient to take $l \in (0, 1)$ and $x = 0$.

(i) Since the function $\partial^{\theta+\gamma} (y^\beta \eta(y))$ in (14) is orthogonal to 1 in $L_2(B_1)$, it follows from (14) that

$$(38) \quad |\nabla_k u(0)| \leq c \left(t^{-n-k} \int_{B_t} |u(y) - u(0)| \, dy + \int_{B_t} \frac{|\nabla_m u(y)|}{|y|^{n-m+k}} \, dy \right).$$

By Hölder's inequality, applied to the first integral and by (16) we have

$$|\nabla_k u(0)| \leq c (t^{-k} \mathcal{D}_{p,l} u(0) + t^{m-k} \mathcal{M} \nabla_m(x)).$$

The result follows.

(ii) By (38) with m replaced by $[m]$

$$|\nabla_k u(0)| \leq c \left(t^{l-k} \mathcal{D}_{p,l} u(0) + t^{[m]-k} |\nabla_{[m]} u(0)| + \int_{B_t} \frac{|\nabla_{[m]} u(y) - \nabla_{[m]} u(0)|}{|y|^{n-[m]+k}} \, dy \right).$$

By (33) the third term in the right-hand side does not exceed

$$t^{m-k} \mathcal{D}_{p,m} u(0).$$

Now we note that (35) implies

$$|\nabla_{[m]} u(0)| \leq c (t^{l-[m]} \mathcal{D}_{p,l} u(0) + t^{[m]} \mathcal{D}_{p,m} u(0)).$$

Hence

$$|\nabla_k u(0)| \leq c (t^{-k} \mathcal{D}_{p,l} u(0) + t^{m-k} \mathcal{D}_{p,m} u(0)).$$

The result follows. \square

5. APPLICATIONS TO THE THEORY OF MULTIPLIERS IN SOBOLEV SPACES

5.1. The maximal algebra in $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$.

Let m and l be integers, $m \geq l$, and let $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$ denote the space of pointwise multipliers acting from $W_p^m(\mathbb{R}^n)$ to $W_p^l(\mathbb{R}^n)$ (see [11]). Analytical descriptions of $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$ as well as separate necessary and sufficient conditions for the membership in this multiplier space can be found in [11]. We characterize the maximal algebra in $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$ by using inequality (13).

Theorem 5. *The maximal Banach algebra in $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$, $m \geq l$, $1 < p < \infty$, is isomorphic to the space*

$$(39) \quad M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n)) \cap L_\infty(\mathbb{R}^n).$$

Remark 8. In the case $m = l$ the statement of Theorem 5 is trivial since the multiplier space $M(W_p^l(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$ is an algebra and is embedded into $L_\infty(\mathbb{R}^n)$.

Proof of Theorem 5. Let A be a subalgebra of $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$. Then, for any $N = 1, 2, \dots$ and for any $\gamma \in A$, $u \in W_p^m(\mathbb{R}^n)$,

$$\|\gamma^N u\|_{L_p}^{1/N} \leq \|\gamma^N u\|_{W_p^l}^{1/N} \leq c^{1/N} \|\gamma\|_A \|u\|_{W_p^m}^{1/N}.$$

(Here and elsewhere in the present section we omit \mathbb{R}^n in the notations of norms.) Passing to the limit as $N \rightarrow \infty$ we obtain $\gamma \in L_\infty(\mathbb{R}^n)$. Hence A is a part of the intersection (39).

Let γ_1, γ_2 belong to (39). Then, for any $u \in W_p^m(\mathbb{R}^n)$,

$$(40) \quad \begin{aligned} \|\nabla_l(\gamma_1 \gamma_2 u)\|_{L_p} &\leq c \left(\|\gamma_1\|_{L_\infty} \|\nabla_l(\gamma_2 u)\|_{L_p} \right. \\ &\quad + \|\gamma_2\|_{L_\infty} \sum_{h=1}^l \|\nabla_h \gamma_1\| \|\nabla_{l-h} u\|_{L_p} \\ &\quad \left. + \sum_{h=1}^{l-1} \sum_{k=1}^{l-h} \|\nabla_h \gamma_1\| \|\nabla_k \gamma_2\| \|\nabla_{l-h-k} u\|_{L_p} \right). \end{aligned}$$

The first term in the right-hand side is majorized by

$$c \|\gamma_1\|_{L_\infty} \|\gamma_2\|_{M(W_p^m \rightarrow W_p^l)} \|u\|_{W_p^m}.$$

Before estimating the second term we note that if $\Gamma \in M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$ then, for any $h = 0, \dots, l$,

$$\nabla_h \Gamma \in M(W_p^m(\mathbb{R}^n) \rightarrow W_p^{l-h}(\mathbb{R}^n)) \subset M(W_p^{m-l+h}(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n))$$

and the estimate

$$(41) \quad \|\nabla_h \Gamma\|_{M(W_p^{m-l+h} \rightarrow L_p)} \leq c \|\Gamma\|_{M(W_p^m \rightarrow W_p^l)}$$

holds (see [11], Section 1.3). Therefore, the second term in the right-hand side of (30) is not greater than

$$c \|\gamma_2\|_{L_\infty} \|\gamma_1\|_{M(W_p^m \rightarrow W_p^l)} \|u\|_{W_p^m}.$$

To estimate the remaining terms in the right-hand side of (40) we need the inequality

$$|\nabla_h \Gamma(x)| \leq c \|\Gamma\|_{L_\infty}^{\frac{k}{k+l}} (\mathcal{M} \nabla_{h+k} \Gamma(x))^{\frac{l}{k+l}}$$

stemming from (13). Hence

$$\begin{aligned} & \|\|\nabla_h \gamma_1\| \|\nabla_k \gamma_2\| \|\nabla_{l-h-k} u\|\|_{L_p} \\ & \leq c \|\gamma_1\|_{L_\infty}^{\frac{k}{k+l}} \|\gamma_2\|_{L_\infty}^{\frac{l}{k+l}} \|(\mathcal{M} \nabla_{h+k} \gamma_1)^{\frac{k}{k+l}} (\mathcal{M} \nabla_{h+k} \gamma_2)^{\frac{l}{k+l}} \|\nabla_{l-h-k} u\|\|_{L_p} \\ & \leq c \|\gamma_1\|_{L_\infty}^{\frac{k}{k+l}} \|\gamma_2\|_{L_\infty}^{\frac{l}{k+l}} \|(\mathcal{M} \nabla_{h+k} \gamma_1)\|_{L_p}^{\frac{k}{k+l}} \|(\mathcal{M} \nabla_{h+k} \gamma_2)\|_{L_p}^{\frac{l}{k+l}} \|\nabla_{l-h-k} u\|\|_{L_p}^{\frac{k}{k+l}}. \end{aligned}$$

By Verbitsky's theorem (see [12], Lemma 3.1)

$$(42) \quad \|\mathcal{M} \Gamma\|_{M(W_p^s \rightarrow L_p)} \leq c \|\Gamma\|_{M(W_p^s \rightarrow L_p)}$$

which along with (41) implies

$$\begin{aligned} & \|\|\nabla_h \gamma_1\| \|\nabla_k \gamma_2\| \|\nabla_{l-h-k} u\|\|_{L_p} \\ & \leq c \|\gamma_1\|_{L_\infty}^{\frac{k}{k+l}} \|\gamma_2\|_{L_\infty}^{\frac{l}{k+l}} \|\|\nabla_{h+k} \gamma_1\| \|\nabla_{l-h-k} u\|\|_{L_p}^{\frac{k}{k+l}} \|\|\nabla_{h+k} \gamma_2\| \|\nabla_{l-h-k} u\|\|_{L_p}^{\frac{l}{k+l}} \\ & \leq c \|\gamma_1\|_{L_\infty}^{\frac{k}{k+l}} \|\gamma_2\|_{L_\infty}^{\frac{l}{k+l}} \|\gamma_1\|_{M(W_p^m \rightarrow W_p^l)}^{\frac{k}{k+l}} \|\gamma_2\|_{M(W_p^m \rightarrow W_p^l)}^{\frac{l}{k+l}} \|u\|_{W_p^m}. \end{aligned}$$

The proof is complete. \square

From Theorem 5 and the description of $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$ obtained in [11, Chapter 1] we arrive at

Corollary 2. *The maximal algebra in $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$, $m \geq l$, consists of $\gamma \in W_{p,\text{loc}}^l(\mathbb{R}^n)$ with the finite norm*

$$(43) \quad \sup_{e \subset \mathbb{R}^n, \text{diam}(e) \leq 1} \frac{\|\nabla_l \gamma; e\|_{L_p}}{(\text{cap}(e, W_p^m))^{1/p}} + \|\gamma\|_{L_\infty},$$

where e is a compact set and $\text{cap}(e, W_p^m)$ is the capacity of e generated by the norm in $W_p^m(\mathbb{R}^n)$, i.e.

$$\text{cap}(e, W_p^m) = \inf\{\|u\|_{W_p^m}^p : u \in C_0^\infty(\mathbb{R}^n), u \geq 1 \text{ on } e\}.$$

In the case $mp > n$ the norm (43) can be simplified as

$$\sup_{x \in \mathbb{R}^n} \|\nabla_l \gamma; B_1(x)\|_{L_p} + \|\gamma\|_{L_\infty},$$

which also follows from the fact that the norm in $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$ is equivalent to the norm

$$\sup_{x \in \mathbb{R}^n} \|\gamma; B_1(x)\|_{W_p^m}$$

in the case $mp > n$ ([11, Chapter 1]).

5.2. Estimate for the norm in $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$.

According to Theorem 1.3.2/1 [11], the equivalence relation

$$\|\gamma\|_{M(W_p^m \rightarrow W_p^l)} \sim \|\nabla_l \gamma\|_{M(W_p^m \rightarrow L_p)} + \|\gamma\|_{M(W_p^{m-l} \rightarrow L_p)}$$

holds with $mp \leq n$ and $m \geq l$. The proof of the upper estimate for the norm in $M(W_p^m(\mathbb{R}^n) \rightarrow W_p^l(\mathbb{R}^n))$ given in [11] is based on the complex interpolation. The inequality (13) enables one to arrive at the same result in a different way. The argument is as follows.

Let $\gamma \in M(W_p^{m-l}(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n))$ and $\nabla_l \gamma \in M(W_p^m(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n))$. For any $u \in C_0^\infty(\mathbb{R}^n)$

$$\|\gamma u\|_{W_p^l} \leq c \left(\sum_{k=0}^l \|\nabla_k \gamma \nabla_{l-k} u\|_{L_p} + \|\gamma u\|_{L_p} \right).$$

By (13) and the Hölder inequality we have for $k = 1, \dots, l-1$

$$(44) \quad \begin{aligned} \|\nabla_k \gamma \nabla_{l-k} u\|_{L_p} &\leq c \|(\mathcal{M}\gamma)^{1-k/l} (\mathcal{M}\nabla_l \gamma)^{k/l} (\mathcal{M}u)^{k/l} (\mathcal{M}\nabla_l u)^{1-k/l}\|_{L_r} \\ &\leq c \|(\mathcal{M}\gamma) (\mathcal{M}\nabla_l u)\|_{L_r}^{1-k/l} \|(\mathcal{M}\nabla_l \gamma) (\mathcal{M}u)\|_{L_r}^{k/l}. \end{aligned}$$

Clearly,

$$|\nabla_t u| \leq c I_{m-t} |\nabla_m u|, \quad |u| \leq c I_m |\nabla_m u|.$$

Hence,

$$\mathcal{M}\nabla_t u \leq c I_{m-t} \mathcal{M}\nabla_m u, \quad \mathcal{M}u \leq c I_m \mathcal{M}\nabla_m u.$$

This along with (42) leads to

$$\begin{aligned} \|(\mathcal{M}\gamma)(\mathcal{M}\nabla_t u)\|_{L_p} &\leq c \|\gamma\|_{M(W_p^m \rightarrow W_p^t)} \|I_{m-t} \mathcal{M}\nabla_m u\|_{W_p^{m-t}} \\ &\leq c \|\gamma\|_{M(W_p^{m-t} \rightarrow L_p)} \|\mathcal{M}\nabla_m u\|_{L_p} \end{aligned}$$

and similarly,

$$\begin{aligned} \|(\mathcal{M}\nabla_t \gamma)(\mathcal{M}u)\|_{L_p} &\leq c \|\nabla_t \gamma\|_{M(W_p^m \rightarrow W_p^t)} \|I_m \mathcal{M}\nabla_m u\|_{W_p^m} \\ &\leq c \|\nabla_t \gamma\|_{M(W_p^m \rightarrow L_p)} \|\mathcal{M}\nabla_m u\|_{L_p}. \end{aligned}$$

The result follows from (44) and the boundedness of the operator \mathcal{M} in $L_p(\mathbb{R}^n)$.

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Authors' addresses: Vladimir Maz'ya, Tatyana Shaposhnikova, Department of Mathematics, Linköping University, S-581 83 Linköping, Sweden, e-mails: vlmaz@mai.liu.se; tasha@mai.liu.se.