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WEIGHTED MULTIDIMENSIONAL INEQUALITIES FOR MONOTONE FUNCTIONS

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(Received January 18, 1999)

Abstract. We discuss the characterization of the inequality

$$\left( \int_{\mathbb{R}^N_+} f^q u \right)^{1/q} \leq C \left( \int_{\mathbb{R}^N_+} f^p v \right)^{1/p}, \quad 0 < q, p < \infty,$$

for monotone functions $f \geq 0$ and nonnegative weights $u$ and $v$ and $N \geq 1$. We prove a new multidimensional integral modular inequality for monotone functions. This inequality generalizes and unifies some recent results in one and several dimensions.

Keywords: integral inequalities, monotone functions, several variables, weighted $L^p$ spaces, modular functions, convex functions, weakly convex functions

MSC 1991: 26D15, 26B99

1. INTRODUCTION

Let $\mathbb{R}^N_+ := \{(x_1, \ldots, x_N); x_i \geq 0, i = 1, 2, \ldots, N\}$ and $\mathbb{R}_+ := \mathbb{R}^1_+$. Assume that $f: \mathbb{R}^N_+ \to \mathbb{R}_+$ is monotone which means that it is monotone with respect to each variable. We denote $f \downarrow$, when $f$ is decreasing (= nonincreasing) and $f \uparrow$ when $f$ is increasing (= nondecreasing). Throughout this paper $\omega, u, v$ are positive measurable functions defined on $\mathbb{R}^N_+, N \geq 1$.

A function $P$ on $[0, \infty)$ is called a modular function if it is strictly increasing, with the values 0 at 0 and $\infty$ at $\infty$. For the definition of an N-function we refer to [7]. We say that a modular function $P$ is weakly convex if $2P(t) \leq P(Mt)$, for all $t > 0$ and some constant $M > 1$. All convex modular functions are obviously weakly convex. The function $P_1(t) = t^p, 0 < p < 1$ and the function $P_2(t) = \exp(\sqrt{t}) - 1$ are weakly convex, but not convex. See also [6].
In order to motivate this investigation and put it into a frame we use Section 2 to present the characterization of the inequality

\begin{equation}
\left( \int_{\mathbb{R}_+^N} f^q u \right)^{1/q} \leq C \left( \int_{\mathbb{R}_+^N} f^p v \right)^{1/p}, \quad 0 < p, q < \infty,
\end{equation}

for all \( f \downarrow \) or \( f \uparrow \).

In Section 3 we will characterize the weights \( \omega, u \) and \( v \) such that

\begin{equation}
Q^{-1} \left( \int_{\mathbb{R}_+^N} Q (\omega(x)f(x)) u(x) \, dx \right) \leq P^{-1} \left( \int_{\mathbb{R}_+^N} P (C f(x)) v(x) \, dx \right)
\end{equation}

holds for modular functions \( P \) and \( Q \), where \( P \) is weakly convex and \( 0 \leq f \downarrow \). Here and in the sequel \( C > 0 \) denotes a constant independent of \( f \).

Conventions and notation. Products and quotients of the form \( 0 \cdot \infty, \frac{\infty}{0}, 0 \) are taken to be \( 0 \). \( \mathbb{Z} \) stands for the set of all integers and \( \chi_E \) denotes the characteristic function of a set \( E \).

2. Weighted \( L^p \) inequalities for monotone functions

In the one-dimensional case the inequality (1) was characterized in [8, Proposition 1] for both alternative cases \( 0 < p \leq q < \infty \) and \( 0 < q < p < \infty \) as follows:

(a) If \( N = 1, 0 < p \leq q < \infty \), then (1) is valid for all \( f \downarrow \) if and only if

\[ A_0 := \sup_{t > 0} \left( \int_0^t u \right)^{1/q} \left( \int_0^t v \right)^{-1/p} < \infty \]

and the constant \( C = A_0 \) is sharp.

(b) If \( N = 1, 0 < q < p < \infty \), then (1) is true for all \( f \downarrow \) if and only if

\[ B_0 := \left( \int_0^\infty \left( \int_0^t u \right)^{\tau/p} \left( \int_0^t v \right)^{-\tau/p} u(t) \, dt \right)^{1/r} < \infty. \]

Moreover,

\[ \left( \frac{q^2}{pr} \right)^{1/p} B_0 \leq C \leq \left( \frac{r}{q} \right)^{1/r} B_0 \]

and

\[ B_0^* = q \left( \int_0^\infty u \right)^{r/q} + q \left( \int_0^\infty v \right)^{-r/q} \int_0^\infty \left( \int_0^t u \right)^{\tau/p} \left( \int_0^t v \right)^{-\tau/p} v(t) \, dt. \]
Similar characterizations are valid when $f \uparrow$, with the only change that the integrals over $[0, t]$ are replaced by integrals over $[t, \infty]$.

Since the one-dimensional inequality (1) expresses the embedding of classical Lorentz spaces, further generalizations and references in this directions can be found in [3].

The multidimensional case was recently treated in [1, Theorem 2.2], for the case $0 < p \leq q < \infty$ and in [2, Theorem 4.1], for the case $0 < q < p < \infty$ as follows:

(a) If $0 < p \leq q < \infty$, then (1) is valid for all $f \downarrow$ if and only if

$$A_N := \sup_{D \in \mathcal{D}_d} \frac{\left( \int_D u \right)^{1/q}}{\left( \int_D v \right)^{1/p}} < \infty$$

and the constant $C = A_N$ is sharp. Here the supremum is taken over the set $\mathcal{D}_d$ of all "decreasing" domains, i.e., for which the characteristic function is a decreasing function in each variable.

(b) If $0 < q < p < \infty$, then (1) is valid for all $f \downarrow$ if and only if

$$B_N^r := \sup_{0 \leq t \leq \epsilon} \int_0^\infty \left( \int_{D_{h,t}} u \right)^{-r/p} d \left( \left( \int_{D_{h,t}} u \right)^{r/q} \right) < \infty,$$

where

$$D_{h,t} = \{ x \in \mathbb{R}^N_+ ; h(x) > t \}.$$ 

Moreover,

$$\frac{1}{2^{1/q}(2r/q + 2r/p)^1/r} B_N \leq C \leq 4^{1/q} B_N.$$ 

If $N = 1$, $P$ and $Q$ are N-functions and $Q \circ P^{-1}$ is convex, then some weight characterizations of the inequality (2) have been obtained in [4] and [5].

For $N > 1$, $P$ and $Q$ N-functions and $Q \circ P^{-1}$ convex, (2) holds for all $0 \leq f \downarrow$ if and only if there exists a constant $A = A(\Phi_1, \Phi_2, u, v, \omega)$ such that, for all $\epsilon > 0$ and $D \in \mathcal{D}_d$,

$$Q^{-1} \left( \int_D Q(\epsilon \omega(x)) u(x) \, dx \right) \leq P^{-1} \left( P(A\epsilon) \int_D v(x) \, dx \right).$$

This characterization can be found in [2, Theorem 2.1].

However, if $Q$ and $P$ are not N-functions (hence not convex) and $Q \circ P^{-1}$ is not convex, then the problem of characterizing weights for which (2) holds seems to be to a large extent open. For $N = 1$ the first characterization of this type was given in [6].

In the next section we characterize the weights for which (2) holds when $P$ is weakly convex. This result generalizes both the corresponding one-dimensional result.
obtained in [6] and the multidimensional case obtained in [2]. Some particular cases of (2) will also be pointed out.

3. A MULTIDIMENSIONAL MODULAR INEQUALITY

Let $0 \leq h(x) \downarrow$ and $t > 0$. Denote

$$D_{h,t} := \{x \in \mathbb{R}^N_+; h(x) > t\},$$

and

$$D_{d} := \bigcup_{0 \leq h \downarrow} \bigcup_{t > 0} D_{h,t}.$$ 

The set $D_{d}$ consists of all “decreasing” domains $D_{h,t}$. In particular, $\chi_{D_{h,t}}$ is decreasing in each variable. For a strictly decreasing, positive sequence $\{t_k\}$, such that $t_k \to 0$ as $k \to \infty$ we put

$$D_k = D_{h,t_k} := \{x \in \mathbb{R}^N_+; h(x) > t_k\}, k \in \mathbb{Z}.$$ 

Obviously, $D_{k+1} \supset D_k$ and we define

$$\Delta_k = \Delta_{h,t_k} := D_{k+1} \setminus D_k.$$ 

Hence, $\Delta_k \cap \Delta_n = \emptyset$, $k \neq n$ and $\mathbb{R}^N_+ = \bigcup_k \Delta_k$. For simplicity we also assume in the sequel that

$$\int_{\mathbb{R}^N_+} v(x) \, dx = \infty.$$ 

**Theorem 3.1.** Let $Q$ and $P$ be modular functions and $P$ weakly convex. Then (2) holds for all $0 \leq f \downarrow$ if and only if there exists a constant $B > 0$ such that

$$Q^{-1}\left(\sum_{k \in \mathbb{Z}} \int_{\Delta_k} Q\left(\frac{\varepsilon_k}{B} \omega(x)\right) u(x) \, dx\right) \leq P^{-1}\left(\sum_{k \in \mathbb{Z}} P(\varepsilon_k) \int_{\Delta_k} v(x) \, dx\right)$$

is satisfied for all positive decreasing sequences $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ and all increasing sequences of decreasing sets $\{D_k\}_{k \in \mathbb{Z}}$ such that $\int_{D_k} v(x) \, dx = 2^k$.

**Proof.** The necessity follows, if we replace $f$ in (2) by the decreasing function

$$f = \sum_{k \in \mathbb{Z}} \varepsilon_k \chi_{\Delta_k}, \{\varepsilon_k\}_k \text{ being a decreasing sequence.}$$
Next we consider the sufficiency. Fix \( f \downarrow \) and set \( \epsilon_k = Bt_k, D_k = D_{f,t_k} \) and \( \Delta_k = \Delta_{f,t_k} \). Because \( \mathbb{R}^N_+ = \bigcup_k \Delta_k \) we obtain, using also (4) and the facts that \( Q, P, Q^{-1}, P^{-1} \) are increasing and \( f \) is decreasing,

\[
Q^{-1} \left( \int_{\mathbb{R}^N_+} Q(\omega(x)f(x))u(x)\,dx \right) = Q^{-1} \left( \sum_{k \in \mathbb{Z}} \int_{\Delta_k} Q(\omega(x)f(x))u(x)\,dx \right) \\
\leq Q^{-1} \left( \sum_{k \in \mathbb{Z}} \int_{\Delta_k} Q(\omega(x)t_k)u(x)\,dx \right) \\
\leq P^{-1} \left( \sum_{k \in \mathbb{Z}} P(Bt_k) \int_{\Delta_k} v(x)\,dx \right) \\
= P^{-1} \left( \sum_{k \in \mathbb{Z}} 2P(Bt_k) \int_{\Delta_{k-1}} v(x)\,dx \right) \\
\leq P^{-1} \left( \sum_{k \in \mathbb{Z}} \int_{\Delta_{k-1}} 2P(Bf(x))v(x)\,dx \right).
\]

Therefore, by using the assumption that \( P \) is weakly convex, we find that

\[
Q^{-1} \left( \int_{\mathbb{R}^N_+} Q(\omega(x)f(x))u(x)\,dx \right) \leq P^{-1} \left( \sum_{k \in \mathbb{Z}} \int_{\Delta_{k-1}} P(MBf(x))v(x)\,dx \right) \\
= P^{-1} \left( \int_{\mathbb{R}^N_+} P(MBf(x))v(x)\,dx \right),
\]

i.e., (2) holds with \( C = MB \). The proof is complete. \( \square \)

We will give now two important corollaries of Theorem 3.1.

**Corollary 3.2.** If \( P \) and \( Q \) are as in Theorem 3.1 and \( Q \circ P^{-1} \) is convex, then (2) holds if and only if, for all \( \epsilon > 0 \) and decreasing sets \( D \), there exists \( C > 0 \) such that

(5) \[
Q^{-1} \left( \int_D Q(\frac{\omega(x)}{C}P^{-1}(\frac{\epsilon}{\int_D v}))u(x)\,dx \right) \leq P^{-1}(\epsilon).
\]

**Proof.** For the necessity we just have to substitute \( f \) in (2) with the function

\[
f_0(x) = \frac{P^{-1}(\frac{\epsilon}{\int_D v})}{C} \chi_D(x).
\]

Next we prove the sufficiency, i.e., that (5) implies (2). According to Theorem 3.1 it is sufficient to prove that (5) implies (4). By applying (5) with \( \epsilon = P(C\epsilon_k) \int_{D_{k+1}} v \) for
each decreasing set $D_{k+1}$ and using the convexity of $Q \circ P^{-1}$ and the weak convexity of $P$ we find that

$$
\left( \sum_{k \in \mathbb{Z}} \int_{\Delta_k} Q(\epsilon_k \omega(x)) u(x) \, dx \right) \leq \left( \sum_{k \in \mathbb{Z}} \int_{D_{k+1}} Q(\epsilon_k \omega(x)) u(x) \, dx \right) \\
\leq \sum_{k \in \mathbb{Z}} Q \circ P^{-1} \left( P(C \epsilon_k) \int_{D_{k+1}} v \right) \\
\leq Q \circ P^{-1} \left( \sum_{k \in \mathbb{Z}} 2P(C \epsilon_k) \int_{D_k} v \right) \\
\leq Q \circ P^{-1} \left( \sum_{k \in \mathbb{Z}} P(MC \epsilon_k) 2^k \right) \\
= Q \circ P^{-1} \left( \sum_{k \in \mathbb{Z}} P(MC \epsilon_k) \int_{\Delta_k} v \right)
$$

Hence (4) follows with $B = MC$ and the corollary is proved. \hfill \Box

Remark. If $Q(x) = x^q$ and $P(x) = x^p$, $0 < p \leq q < \infty$, then $Q \circ P^{-1}$ is convex and the condition (5) coincides with condition (3). Hence, Corollary 3.2 generalizes Theorem 2.2(d) in [1].

Remark. For $N = 1$ the condition (5) reads

$$
Q^{-1} \left( \int_0^r Q \left( \frac{\omega(x)}{B} P^{-1} \left( \frac{x}{\int_0^r v} \right) \right) u(x) \, dx \right) \leq P^{-1} (\epsilon), \quad \forall r > 0.
$$

Thus, if $N = 1$, then Corollary 3.2 coincides with Corollary 1 in [6].

Finally we apply Theorem 3.1 with $P(x) = x^p$ and $Q(x) = x^q$, $0 < p, q < \infty$, and obtain the following result:

**Corollary 3.3.** The inequality (1) holds for all $0 < f \downarrow$ if and only if there exists a constant $K = K(p, q)$ such that

$$
\left( \sum_{k \in \mathbb{Z}} \epsilon_k^q \int_{\Delta_k} u(x) \, dx \right)^{1/q} \leq K \left( \sum_{k \in \mathbb{Z}} \epsilon_k^p \int_{\Delta_k} v(x) \, dx \right)^{1/p}
$$

for all positive decreasing sequences $\{\epsilon_k\}_{k \in \mathbb{Z}}$ and such that $\int_{D_k} v(x) \, dx = 2^k$.

Remark. For $N = 1$ a similar characterization is given in [6]. For other multidimensional characterizations of (1) in the case $0 < p \leq q < \infty$ see [1] and in the case $0 < q < p < \infty$ see [2] (cf. Section 2).
Final remarks. (i) The results in this paper can also be formulated when we remove the technical assumption (3) (cf. [2], [8]).

(ii) Similar results to all results in this paper can be formulated also for increasing functions of several variables.

References


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