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CANCELLATION RULE FOR INTERNAL DIRECT PRODUCT
DECOMPOSITIONS OF A CONNECTED PARTIALLY
ORDERED SET

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Abstract. In this note we deal with two-factor internal direct product decompositions of
a connected partially ordered set.

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Direct product decompositions of a connected partially ordered set have been
investigated by Hashimoto [1].

We apply the notion of internal direct product decomposition of a partially ordered
set in the same sense as in [2]; the definition is recalled in Section 1 below.

The following cancellation rule has been proved in [2]:

(A) Let \( L \) be a directed partially ordered set and \( x_0 \in L \). Let
\[
\phi^0 : L \to A^0 \times B^0,
\psi^0 : L \to A^1 \times B^1
\]
be internal direct product decompositions of \( L \) with the same central element
\( x^0 \). Suppose that \( A^0 = A^1 \). Then \( B^0 = B^1 \) and \( \phi^0(x) = \psi^0(x) \) for each
\( x \in L \).

The aim of the present paper is to generalize (A) to the case when \( L \) is a connected
partially ordered set.

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1. Preliminaries

We recall that a partially ordered set is called connected if for any \( x, y \in L \) there are elements \( x_0, x_1, x_2, \ldots, x_n \) in \( L \) such that

(i) \( x = x_0, y = x_n \);
(ii) if \( i \in \{1, 2, \ldots, n\} \), then the elements \( x_{i-1} \) and \( x_i \) are comparable.

Let \( L \) be a connected partially ordered set. Suppose that we have a direct product decomposition

\[
\phi: L \rightarrow \prod_{i \in I} L_i
\]

(i.e., \( \phi \) is an isomorphism of the partially ordered set \( L \) onto the direct product \( \prod L_i \)). For \( x \in L \) let \( \phi(x) = (\ldots, x_i, \ldots)_{i \in I} \). We denote \( x_i = x(L_i) \). Next we put

\[
L_i(x) = \{ z \in L : z(L_j) = x(L_j) \text{ for each } j \in I \setminus \{i\} \}.
\]

Let \( x^0 \) be a fixed element of \( L \). For each \( i \in I \) we denote \( L_i(x^0) = L_i^0 \).

For each \( x \in L \) and each \( i \in I \) there is a unique element \( y_i \) in \( L_i^0 \) such that

\[
\pi(L_i) = y_i(L_i).
\]

Then the relation

\[
\phi^0: L \rightarrow \prod_{i \in I} L_i^0
\]

is said to be an internal direct product decomposition of \( L \) with the central element \( x^0 \).

For each \( i \in I \), \( L_i^0 \) is isomorphic to \( L_i \).

2. Auxiliary results

In this section we suppose that \( L \) is a connected partially ordered set. Assume that we are given a direct product decomposition

\[
\psi: L \rightarrow A \times B.
\]

For \( x \in L \) we put \( \psi(x) = (x_A, x_B) \). Sometimes we write \( x(A) \) instead of \( x_A \), and similarly for \( x_B \).
Further, for each $x_0 \in L$ we put

\[ A(x_0) = \{ x \in L : x(B) = x_0(B) \}, \]
\[ B(x_0) = \{ x \in L : x(A) = x_0(A) \}. \]

Let $x_1 \in L$, $x_1 \notin A(x_0)$. We put $A(x_0) < A(x_1)$ if there are $x_0' \in A(x_0)$ and $x_1' \in A(x_1)$ such that $x_0' < x_1'$.

If $x, y, z \in L$ and $z = \sup \{ x, y \}$ in $L$, then we express this fact by writing $z = x \lor y$.

The meaning of $v = x \land y$ is analogous.

2.1. Lemma. Let $x_0, x_1 \in L, A(x_0) < A(x_1), x_2 \in A(x_1)$. Then there exists $x_3^0$ in $A(x_0)$ such that

(i) $x_3^0 < x_2$;

(ii) if $z \in A(x_0)$ and $z < x_2$, then $z \leq x_3^0$.

Proof. There exists $x_3^0 \in L(x_0)$ such that

\[ \phi(x_3^0) = (x_2(A), x_0(B)). \]

Then $x_3^0 \in A(x_0)$. We have

\[ x_0(B) = x_3^0(B) \leq x_1^0(B) = x_2(B), \]

where $x_3^0$ and $x_1^0$ are as in the definition of the relation $A(x_0) < A(x_1)$. Thus $x_3^0 \leq x_2$. Since $x_2 \notin A(x_0)$, we must have $x_3^0 < x_2$. Therefore (i) is valid.

Let $z \in A(x_0)$ and $z < x_2$. Then $z(B) = x_0(B) = x_3^0(B)$ and $z(A) \leq x_2(A)$; hence $z < x_2$. Thus (ii) holds.

It is obvious that the element $x_3^0$ is uniquely determined if $x_2$ and $A(x_0)$ are given and if $A(x_2) > A(x_0)$.

2.2. Lemma. Let $x_0$ and $x_1$ be as in 2.1. Further, let $x_3 \in L, x_3 \geq x_1$. Then the following conditions are equivalent:

(i) $x_3 \in A(x_1)$;

(ii) $x_3 \lor x_1 = x_3$.

Proof. First we remark that from $x_3 \geq x_1$ we infer that $A(x_3) > A(x_0)$, whence in view of 2.1, the element $x_3^0$ does exist; moreover, we have

\[ \phi(x_3^0) = (x_3(A), x_0(B)). \]
Further, from the relation $A(x_0) < A(x_1)$ we conclude that whenever $t_1 \in A(x_0)$ and $t_2 \in A(x_1)$, then $t_1(B) < t_2(B)$. In particular, $x_0(B) < x_1(B)$. Thus $x_0(B) < x_3(B)$ and $x_0^3(B) < x_3(B)$.

Let (i) be valid. Hence $x_3(B) = x_1(B)$. From $x_3 \geq x_1$ we get $x_3(A) \geq x_1(A)$. Thus

$$(x_3(A), x_0(B)) \lor (x_1(A), x_1(B)) = (x_3(A), x_3(B)).$$

Therefore (ii) holds.

Conversely, let (ii) be valid. Then

$$x_0^3(B) \lor x_1(B) = x_3(B).$$

We already know that $x_0^3(B) \lor x_1(B) = x_1(B)$. Thus $x_1(B) = x_3(B)$. Hence (i) holds.  

2.3. Corollary. Let $x_0$ and $x_1$ be as in 2.1. Then the set $\{x \in A(x_1) : x \geq x_1\}$ is uniquely determined by $A(x_0)$ and $x_1$.

2.4. Lemma. Let $x_0$ and $x_1$ be as in 2.1. Further, let $x_4 \in L$, $x_4 \leq x_1$. Then $x_4$ belongs to $A(x_1)$ if and only if the following conditions are satisfied:

(i) $x_4 \lor x_1 = x_1$;
(ii) $x_4 \notin A(x_0)$;
(iii) there exists $t \in A(x_0)$ with $t < x_4$.

Proof. Suppose that $x_4$ belongs to $A(x_1)$. Then (ii) is obviously valid. In view of 2.1, the condition (iii) is satisfied.

For proving that (i) is valid we have to verify the validity of the relation

$$(x_4(A), x_4(B)) \lor (x_1^2(A), x_1^2(B)) = (x_1(A), x_1(B)).$$

We have

$$(x_1^2(A), x_1^2(B)) = (x_1(A), x_0(B)),$$

whence

$$(x_4(A), x_4(B)) = x_4(A) \lor x_1(A) = x_1(A).$$

Further, in view of (iii), $x_4(B) \geq t(B)$. Since $t \in A(x_0)$, we get $t(B) = x_0(B)$. Thus

$$(x_4(B), x_0(B)) = x_4(B) \lor x_0(B) = x_4(B) = x_1(B).$$

From (i) and (ii) we conclude that (i) is valid.
Conversely, suppose that the conditions (i), (ii) and (iii) are satisfied. From (i) we obtain
\[ x_4(B) \lor v_4(B) = x_3(B). \]
Further we have \( x_4(B) = t(B) \leq x_4(B) \), whence
\[ x_4(B) \lor x_4(B) = x_4(B) \lor t(B) = x_4(B). \]
Then \( x_4(B) = x_3(B) \), therefore \( x_4 \in A(x_3) \).

2.5. Corollary. Let \( x_0 \) and \( x_1 \) be as in 2.1. Then the set \( \{ x \in A(x_1) : x \leq x_1 \} \) is uniquely determined by \( A(x_0) \) and \( x_1 \).

2.6. Definition. The interval \([u, v]\) of \( L \) is said to have the property (a) if
(i) there exist \( u^0, v^0 \in A(x_0) \) such that the relations
\[ u^0 = \max\{ x \in A(x_0) : x \leq u \}, \quad v^0 = \max\{ x \in A(x_0) : x \geq v \} \]
are valid;
(ii) \( u^0 \lor v = u \).

2.7. Lemma. Let \( x_0 \) and \( x_1 \) be as in 2.1. Let \( z \in L \). The following conditions (a) and (b) are equivalent:
(a) There are elements \( z_0, z_1, \ldots, z_n \) in \( L \) such that \( z_0 = x_1, z_n = z \) and for each \( i \in \{1, 2, \ldots, n\} \) we have
(i) the elements \( z_{i-1}, z_i \) are comparable;
(ii) if \( z_{i-1} \leq z_i \), then the interval \([z_{i-1}, z_i]\) satisfies the condition (a);
(iii) if \( z_{i-1} \geq z_i \), then the interval \([z_i, z_{i-1}]\) satisfies the condition (a).
(b) \( z \in A(x_1) \).

Proof. Assume that (a) is valid. Then in view of 2.2 and 2.4 we obtain \( z_1 \in A(x_1) \). Now it suffices to apply induction with respect to \( n \).

Conversely, assume that (b) is valid. Since \( L \) is connected, the partially ordered set \( A \) is connected as well. It is obvious that the partially ordered sets \( A \) and \( A(x_1) \) are isomorphic; hence \( A(x_1) \) is connected as well. Thus there are elements \( z_0, z_1, \ldots, z_n \) in \( A(x_1) \) such that \( z_0 = x_1, z_n = z \) and for each \( i \in \{1, 2, \ldots, n\} \) the elements \( z_{i-1}, z_i \) are comparable. Then by using 2.1, 2.2 and 2.4 we conclude that (a) is valid.

2.8. Corollary. Let \( x_0 \) and \( x_1 \) be as in 2.1. Then the set \( A(x_1) \) is uniquely determined by \( A(x_0) \) and \( x_1 \).
By a dual argument we obtain

2.9. Corollary. Let \( x_0, x_1 \in L \) be such that \( A(x_0) > A(x_1) \). Then the set \( A(x_1) \) is uniquely determined by \( A(x_0) \) and \( x_1 \).

From 2.8, 2.9 and from the fact that \( L \) is connected we conclude

2.10. Lemma. Let \( x_0, x_1 \in L \). Then the set \( A(x_1) \) is uniquely determined by \( A(x_0) \) and \( x_1 \).

Let \( x_0, x_1 \in L, x_0 \leq x_1 \). In view of 2.1 there exists \( a(x_0, x_1) \in L \) such that

\[ a(x_0, x_1) = \max\{x \in A(x_0) : x \leq x_1\}. \]

Dually, if \( x_0, x_1 \in L, x_0 \geq x_1 \), then there is \( b(x_0, x_1) \in L \) with

\[ b(x_0, x_1) = \min\{x \in A(x_0) : x \geq x_1\}. \]

2.11. Lemma. Let \( x_0, x_1 \in L \), \( x_0 \leq x_1 \). Then

\[ x_1 \in B(x_0) \iff a(x_0, x_1) = x_0. \]

Proof. Suppose that \( a(x_0, x_1) = x_0 \). Hence \( x_0(A) = x_1(A) \) and therefore \( x_1 \in B(x_0) \).
Conversely, suppose that \( x_1 \in B(x_0) \). Then \( x_1(A) = x_0(A) \). From \( x_0 \leq x_1 \) we conclude that \( x_0(B) \leq x_1(B) \).
Let \( x \in A(x_0) \), \( x \leq x_1 \). We get \( x(A) \leq x_1(A) \), whence \( x(A) \leq x_0(A) \). Further, \( x(B) = x_0(B) \). Therefore \( x \leq x_0 \). This yields that \( a(x_0, x_1) = x_0 \). \qed
By a dual argument we obtain

2.12. Lemma. Let \( x_0, x_1 \in L \), \( x_0 \geq x_1 \). Then

\[ x_1 \in B(x_0) \iff b(x_0, x_1) = x_0. \]

2.13. Lemma. Let \( x_0, x \in L \). The following conditions are equivalent:

(a) There exist elements \( z_0, z_1, z_2, \ldots, z_n \) in \( L \) such that \( z_0 = z_n = x \), for each \( i \in \{1, 2, \ldots, n\} \) the elements \( z_{i-1}, z_i \) are comparable and \( z_i \in B(z_{i-1}) \);
(b) \( x \in B(x_0) \).

Proof. The implication (a) \( \Rightarrow \) (b) is obvious. Suppose that (b) is valid. The partially ordered set \( B \) is connected, hence so is \( B(x_0) \). Thus there exist \( z_0, z_1, \ldots, z_n \in B(x_0) \) with the properties as in (a). \qed
From 2.10-2.13 we obtain

2.14. Lemma. Let \( x_0 \in L \). Then the set \( B(x_0) \) is uniquely determined by \( A(x_0) \) and \( x_0 \).

In 2.10, \( A \) can be replaced by \( B \). Hence 2.14 yields

2.15. Corollary. Let \( x_0, x \in L \). Then the set \( B(x) \) is uniquely determined by \( A(x_0) \) and \( x \).

3. Cancellation Rule

Suppose that \( L \) is a connected partially ordered set and consider direct product decompositions

1. \( \varphi: L \to A \times B \),
2. \( \varphi_1: L \to A_1 \times B_1 \).

Let \( x_0 \in L \). Then from (1) and (2) we can construct internal direct product decompositions

1'. \( \varphi^0: L \to A^0 \times B^0 \),
2'. \( \varphi_1^0: L \to A_1^0 \times B_1^0 \),

with the central element \( x_0 \).

In view of the definition of the internal direct product decomposition we have

3. \( A^0 = A(x_0) \), \( B^0 = B(x_0) \),
4. \( A_1^0 = A_1(x_0) \), \( B_1^0 = B_1(x_0) \);

further, if \( x \in L \) and \( \varphi^0(x) = (x_1, x_2), \varphi_1^0(x) = (x'_1, x'_2) \), then

5. \( \{x_1\} = A^0 \cap B(x), \{x_2\} = B^0 \cap A(x) \),
6. \( \{x'_1\} = A_1^0 \cap B_1(x), \{x'_2\} = B_1^0 \cap A_1(x) \).

3.1. Theorem. Let (1') and (2') be an internal direct product of a connected partially ordered set \( L \) with the central element \( x_0 \). Suppose that \( A^0 = A_1^0 \). Then \( B^0 = B_1^0 \). Moreover, for each \( x \in L \) we have \( \varphi^0(x) = \varphi_1^0(x) \).

Proof. The first assertion is a consequence of 2.10, 2.15 and of the relations (3), (4). Then in view of (5) and (6) we infer that \( \varphi^0(x) = \varphi_1^0(x) \) for each \( x \in L \). \( \square \)

Let us remark that if \( \varphi: L \to A \times B \) and \( \psi: L \to A_1 \times B_1 \) are direct product decompositions of a connected partially ordered set \( L \) and if \( A \) is isomorphic to \( A_1 \), then \( B \) need not be isomorphic to \( B_1 \).
Example. Let $N$ be the set of all positive integers and let $X$ be a linearly ordered set having more than one element. Put

$$L = \prod_{n \in N} X_n,$$

where $X_n = X$ for each $n \in N$. We denote

$$A = \prod_{n \in N} X_n, \quad B = X_1,$$

$$A_1 = \prod_{n > 1} X_n, \quad B_1 = X_1 \times X_2.$$

Then we have direct product decompositions

$$\varphi: L \to A \times B, \quad \psi \to A_1 \times B_1,$$

$A$ is isomorphic to $A_1$, but $B$ fails to be isomorphic to $B_1$.

Further, the notion of the internal direct product decomposition can be used in group theory (where the central element coincides with the neutral element of the corresponding group); cf., e.g. Kurosh [3], p. 104. The result analogous to 3.1 does not hold, in general, for internal direct product decompositions of a group.

Example. Let $X$ be the additive group of all reals, $Y = X$, $G = X \times Y$. We put

$$X^0 = \{(x, 0) : x \in X\},$$

$$Y^0 = \{(0, y) : y \in Y\},$$

$$Z^0 = \{(x, y) : y \in G : x = y\}.$$

Then $Y^0 \neq Z^0$. The group $G$ is the internal direct product of $X^0$ and $Y^0$; at the same time, $G$ is the internal direct product of $X^0$ and $Z_0$.

We conclude by remarking that the assumption of connectedness of $L$ cannot be omitted in 3.1.

References


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