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$n$-inner product spaces and projections


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Abstract. This paper is a continuation of investigations of $n$-inner product spaces given in [5, 6, 7] and an extension of results given in [3] to arbitrary natural $n$. It concerns families of projections of a given linear space $L$ onto its $n$-dimensional subspaces and shows that between these families and $n$-inner products there exist interesting close relations.

Keywords: $n$-inner product space, $n$-normed space, $r_0$-norm of projection

MSC 1991: 46C05, 46C50

1. $n$-INNER PRODUCTS AND $n$-NORMS

1.1. Let $n$ be a natural number ($n \neq 0$), $L$ a linear space with $\dim L \geq n$ and let $(\cdot | \cdot, \ldots, \cdot)$ be a real function on $L^{n+1} = L \times \ldots \times L$, $n+1$ times.

In the case $n = 1$, we also write $(\cdot, \cdot)$ instead of $(\cdot | \cdot, \ldots, \cdot)$ and $(a, b | a_2, \ldots, a_n)$ is to be understood as the expression $(a, b)$. Let us assume the following conditions:

1. $(a, b | a_2, \ldots, a_n) \geq 0$,
2. $(a, b | a_2, \ldots, a_n) = 0$ if and only if $a, a_2, \ldots, a_n$ are linearly dependent,
3. $(a, b | a_2, \ldots, a_n) = (a, b | a_{i_2}, \ldots, a_{i_n})$ for every permutation $(i_2, \ldots, i_n)$ of $(2, \ldots, n)$,
4. if $n > 1$, then $(a, b | a_2, a_3, \ldots, a_n) = (a_2, a_3 | a, a_3, \ldots, a_n),$
5. $(a, b | a_2, \ldots, a_n) = \alpha (a, b | a_2, \ldots, a_n)$ for every real $\alpha$, and
6. $(a + b, c | a_2, \ldots, a_n) = (a, c | a_2, \ldots, a_n) + (b, c | a_2, \ldots, a_n)$.

Then $(\cdot | \cdot, \ldots, \cdot)$ is called an $n$-inner product on $L$ (see [5]) and $(L, (\cdot | \cdot, \ldots, \cdot))$ is called an $n$-inner product space. The concept of an $n$-inner product space is a generalization of the concepts of an inner product space ($n = 1$) and of a 2-inner product space (see [1]).
1.2. Let \( n > 1 \). An \( n \)-inner product space \( L \) and its \( n \)-inner product \((\cdot, \cdot | \cdot, \cdot, \ldots)\) are called simple if there exists an inner product \((\cdot, \cdot)\) on \( L \) such that the relation
\[
(a, b | a_2, \ldots, a_n) = \begin{pmatrix}
(a, b) & (a, a_2) & \cdots & (a, a_n) \\
(a_2, b) & (a_2, a_2) & \cdots & (a_2, a_n) \\
\vdots & \vdots & \ddots & \vdots \\
(a_n, b) & (a_n, a_2) & \cdots & (a_n, a_n)
\end{pmatrix}
\]
holds. The inner product \((\cdot, \cdot)\) is said to generate the \( n \)-inner product \((\cdot, \cdot | \cdot, \cdot, \ldots)\).

An element \( a \in L \) is said to be orthogonal to a non-empty subset \( S \) of \( L \) if
\[
(a_1, e_1 | a_2, \ldots, e_n) = 0
\]
for arbitrary \( e_1, \ldots, e_n \in S \). A subset \( S \) of \( L \) is said to be orthogonal if it is linearly independent, contains at least \( n \) elements and if every \( e \in S \) is orthogonal to \( S \setminus \{e\} \).

1.3. An \( n \)-norm on \( L \) is a real function \(||\cdot, \ldots, \cdot||\) on \( L^n \) which satisfies the following conditions:
1. \(||a_1, \ldots, a_n|| = 0\) if and only if \( a_1, \ldots, a_n \) are linearly dependent,
2. \(||a_1, \ldots, a_n|| = ||a_{i_1}, \ldots, a_{i_n}||\) for every permutation \((i_1, \ldots, i_n)\) of \((1, \ldots, n)\),
3. \(||a_1, a_2, \ldots, a_n|| = ||a|| ||a_1, a_2, \ldots, a_n||\) for every real number \( a \),
4. \(||a+b, a_2, \ldots, a_n|| \leq ||a, a_2, \ldots, a_n|| + ||b, a_2, \ldots, a_n||\).

\( L \) equipped with an \( n \)-norm \(||\cdot, \ldots, \cdot||\) is called an \( n \)-normed space. The concept of an \( n \)-normed space is a generalization of the concepts of a normed \((n = 1)\) and a 2-normed space (see [2]).

**Theorem 1.** (Theorem 7 of [5]) For every \( n \)-inner product \((\cdot, \cdot | \cdot, \cdot, \ldots)\) on \( L \),
\[
||a_1, a_2, \ldots, a_n|| = \sqrt{(a_1, a_1 | a_2, \ldots, a_n)}
\]
defines an \( n \)-norm on \( L \) for which
\[
(a, b | a_2, \ldots, a_n) = \frac{1}{2} \left(||a + b, a_2, \ldots, a_n||^2 - ||a - b, a_2, \ldots, a_n||^2\right)
\]
and
\[
||a+b, a_2, \ldots, a_n||^2 + ||a-b, a_2, \ldots, a_n||^2 = 2(||a, a_2, \ldots, a_n||^2 + ||b, a_2, \ldots, a_n||^2)
\]
are true.

Conversely, for every \( n \)-norm \(||\cdot, \ldots, \cdot||\) on \( L \) with the property (3), (2) defines an \( n \)-inner product on \( L \) for which (1) is true.

For every \( n \)-inner product \((\cdot, \cdot | \cdot, \cdot, \ldots)\) on \( L \) the \( n \)-norm given by (1) is said to be associated to \((\cdot, \cdot | \cdot, \cdot, \ldots)\). If in connection with an \( n \)-inner product on \( L \) an \( n \)-norm is used, then \(||\cdot, \ldots, \cdot||\) always will be the \( n \)-norm associated to \((\cdot, \cdot | \cdot, \cdot, \ldots)\).
2. PROJECTIONS IN $n$-INNER PRODUCT SPACES

2.1. Let $(L, (\cdot, \cdot, \cdot, \ldots, \cdot))$ be an $n$-inner product space. For arbitrary linearly independent points $a_1, \ldots, a_n \in L$, let $\text{pr}_{a_1, \ldots, a_n}$ be the mapping of $L$ into $L$ given by

$$\text{pr}_{a_1, \ldots, a_n}(c) = \frac{(c, a_1 | a_2, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2} a_1 + \cdots + \frac{(c, a_n | a_1, \ldots, a_{n-1})}{\|a_1, \ldots, a_n\|^2} a_n$$

(see [3], where $n = 2$). We often use the notion

$$(c, a_k | a_1, \ldots, a_n) = (c, a_k | a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$$

and

$$\text{pr}_{a_1, \ldots, a_k, \ldots, a_n}(c) = \frac{(c, a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2}$$

Then we have

$$\text{pr}_{a_1, \ldots, a_n}(c) = \sum_{k=1}^n \frac{(c, a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2} a_k$$

Theorem 2. $\text{pr}_{a_1, \ldots, a_n}$ is a projection of $L$ onto $L((a_1, \ldots, a_n))$, the linear space generated by the set $(a_1, \ldots, a_n)$.

Proof. Obviously $\text{pr}_{a_1, \ldots, a_n}$ is linear. Since $\text{pr}_{a_1, \ldots, a_n}(a_k) = a_k$ for arbitrary $k \in \{1, \ldots, n\}$, $\text{pr}_{a_1, \ldots, a_n}$ maps $L$ onto $L((a_1, \ldots, a_n))$. Moreover,

$$\text{pr}^2_{a_1, \ldots, a_n}(c) = \sum_{k=1}^n \frac{(\text{pr}_{a_1, \ldots, a_n}(c), a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2} a_k$$

from which by virtue of

$$\frac{(\text{pr}_{a_1, \ldots, a_n}(c), a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2}$$

$$= \sum_{i=1}^n \frac{(c, a_i | a_1, \ldots, a_i, \ldots, a_n, a_i, a_i | a_1, \ldots, a_i, \ldots, a_n)}{\|a_1, \ldots, a_n\|^4}$$

$$= \frac{(c, a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2}$$
we get
\[ p_{a_1, \ldots, a_n}(c) = p_{a_1, \ldots, a_n}(c). \]

\[ \text{Theorem 3.} \ p_{a_1, \ldots, a_n} \text{ is independent of the special choice of } a_1, \ldots, a_n \text{ in } L(a_1, \ldots, a_n); \text{ this means, for arbitrary linearly independent points } a'_i = \sum_{k=1}^n a_{i,k} a_k, i = 1, \ldots, n, \text{ we have } p_{a_1, \ldots, a_n} = p_{a'_1, \ldots, a'_n} . \]

\[ \text{Proof.} \text{ Let linearly independent points } a'_i = \sum_{k=1}^n a_{i,k} a_k, i = 1, \ldots, n \text{ be given.} \]

Then
\[ \begin{bmatrix}
  a_{1,1} & \ldots & a_{1,n} \\
  \vdots & \ddots & \vdots \\
  a_{n,1} & \ldots & a_{n,n}
\end{bmatrix} \neq 0. \]

For arbitrary \( c \in L \),
\[ p_{a'_1, \ldots, a'_n}(c) = \sum_{i=1}^n a_{i,j} \left( \sum_{k=1}^n a_{i,k} a_k \right) \begin{bmatrix}
  c, a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c, a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix} \]
\[ = \begin{bmatrix}
  \sum_{j,k=1}^n a_{j,k} a_k a_{i,j} & \cdots & \sum_{j,k=1}^n a_{j,k} a_k a_{i,n} \\
  \vdots & \ddots & \vdots \\
  \sum_{j,k=1}^n a_{j,k} a_k a_{n,j} & \cdots & \sum_{j,k=1}^n a_{j,k} a_k a_{n,n}
\end{bmatrix} \begin{bmatrix}
  c, a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c, a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix} \]
\[ = \begin{bmatrix}
  a_{1,1} & \ldots & a_{1,n} \\
  \vdots & \ddots & \vdots \\
  a_{n,1} & \ldots & a_{n,n}
\end{bmatrix} \begin{bmatrix}
  c, a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  \vdots & \ddots & \vdots & \vdots \\
  c, a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix} \]
\[ = \begin{bmatrix}
  a_{1,1} & \ldots & a_{1,n} \\
  \vdots & \ddots & \vdots \\
  a_{n,1} & \ldots & a_{n,n}
\end{bmatrix} \begin{bmatrix}
  c, a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  \vdots & \ddots & \vdots & \vdots \\
  c, a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix} \]
\[ = \begin{bmatrix}
  \sum_{j,k=1}^n a_{j,k} a_k a_{i,j} & \cdots & \sum_{j,k=1}^n a_{j,k} a_k a_{i,n} \\
  \vdots & \ddots & \vdots \\
  \sum_{j,k=1}^n a_{j,k} a_k a_{n,j} & \cdots & \sum_{j,k=1}^n a_{j,k} a_k a_{n,n}
\end{bmatrix} \begin{bmatrix}
  c, a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  \vdots & \ddots & \vdots & \vdots \\
  c, a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix} \]
\[ = \begin{bmatrix}
  \sum_{j,k=1}^n a_{j,k} a_k a_{i,j} & \cdots & \sum_{j,k=1}^n a_{j,k} a_k a_{i,n} \\
  \vdots & \ddots & \vdots \\
  \sum_{j,k=1}^n a_{j,k} a_k a_{n,j} & \cdots & \sum_{j,k=1}^n a_{j,k} a_k a_{n,n}
\end{bmatrix} \begin{bmatrix}
  c, a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
  \vdots & \ddots & \vdots & \vdots \\
  c, a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix} \]
and
\[ \left\| \sum_{k=1}^{n} \alpha_{1,k} a_{k} \ldots \sum_{k=1}^{n} \alpha_{n,k} a_{k} \right\|^2 = \begin{bmatrix} \alpha_{1,1} & \ldots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \ldots & \alpha_{n,n} \end{bmatrix} \begin{bmatrix} \|a_{1}\| & \ldots & \|a_{n}\| \end{bmatrix}^2. \]

This yields that
\[ p_{[a_{1},...,a_{n}]}(c) = \sum_{k=1}^{n} (c, a_{k} | a_{k}) \frac{a_{k}}{\|a_{k}\|^2} = p_{[a_{1},...,a_{n}]}(c) \]

which proves the theorem. \( \square \)

**Theorem 4.** For arbitrary \( c \in L \), \( c - p_{[a_{1},...,a_{n}]}(c) \) is orthogonal to \( L \{ [a_{1},...,a_{n}] \} \).

**Proof.** For arbitrary \( a' = \sum_{k=1}^{n} \alpha_{k,i} a_{k}, i = 1, \ldots, n \), by means of (8) in Theorem 6 (see [6]) we get
\[
\left( c - p_{[a_{1},...,a_{n}]}(c), \sum_{k=1}^{n} \alpha_{1,k} a_{k} \sum_{k=1}^{n} \alpha_{2,k} a_{k} \ldots \sum_{k=1}^{n} \alpha_{n,k} a_{k} \right) = \left( c, \sum_{k=1}^{n} \alpha_{1,k} a_{k} \sum_{k=1}^{n} \alpha_{2,k} a_{k} \ldots \sum_{k=1}^{n} \alpha_{n,k} a_{k} \right) 
\]
\[ = \left( \sum_{k=1}^{n} (c, a_{k} | a_{k}) \frac{a_{k}}{\|a_{k}\|^2} \sum_{k=1}^{n} \alpha_{1,k} a_{k} \sum_{k=1}^{n} \alpha_{2,k} a_{k} \ldots \sum_{k=1}^{n} \alpha_{n,k} a_{k} \right) 
\]
\[ = \sum_{j,k<...<nk} \begin{vmatrix} 1 & 0 & \ldots & 0 & \alpha_{1,j} & \alpha_{1,k} & \ldots & \alpha_{1,n} \\ 0 & \alpha_{2,j} & \ldots & \alpha_{2,k} & \alpha_{2,j} & \ldots & \alpha_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \alpha_{n,j} & \ldots & \alpha_{n,k} & \alpha_{n,j} & \ldots & \alpha_{n,n} \end{vmatrix} \begin{vmatrix} (c, a_{1} | a_{1}) & \ldots & (c, a_{1} | a_{n}) \\ \cdots & \cdots & \cdots \end{vmatrix} 
\]
\[ = 0. \]

This was to be proved. \( \square \)

2.2. From Theorem 2 of [7] we know the following: if \( (\cdot, \cdot, \ldots, \cdot) \) is a simple \( n \)-inner product on \( L \) and \( (\cdot, \cdot) \) generates \( (\cdot, \cdot, \ldots, \cdot) \), then for arbitrary \( a \in L \) and
arbitrary $S \subseteq L$ which generates a linear subspace of $L$ of dimension $\geq n$, $a$ is orthogonal to $S$ relative to $(\cdot, \cdot | \cdot, \cdots , \cdot)$ if and only if $a$ is orthogonal to $S$ relative to $(\cdot, \cdot)$. From this and Theorem 4 it follows that if $(\cdot, \cdot | \cdot, \cdots , \cdot)$ is simple and $(\cdot, \cdot)$ is an inner product on $L$ generating $(\cdot, \cdot | \cdot, \cdots , \cdot)$, then for arbitrary $c \in L$, $c - pr_{a_1, \ldots , a_n}(c)$ is orthogonal to $L((a_1, \ldots , a_n))$ relative to $(\cdot, \cdot)$.

2.3. From Theorem 3 of [6] we know that if $S$ is an orthogonal set in $L$, for every $e \in S$, distinct $e_2, \ldots , e_n \in S \setminus \{e\}$, distinct $e'_2, \ldots , e'_n \in S \setminus \{e\}$ and every $c$ from the linear space generated by $S$, we have

$$\langle c, c | e_2, \ldots , e_n \rangle \|e, e_2, \ldots , e_n\|^2 = \|e, e_2, \ldots , e_n\|^2,$$

which implies $pr_{L, e_2, \ldots , e_n}(c) = pr_{L, e'_2, \ldots , e'_n}(c)$. This means that under the above conditions the coordinate $pr_{L, e_2, \ldots , e_n}(c)$ of $pr_{L, e'_2, \ldots , e'_n}(c)$ is independent of $e_2, \ldots , e_n$.

For every $n$-dimensional linear subspace $L'$ of $L$ let $S_{L'}$ be the set of all subsets $\{a_1, \ldots , a_n\}$ of $L'$ such that $\|a_1, \ldots , a_n\| = 1$. Then for arbitrary $\{a_1, \ldots , a_n\}$, $\{a'_1, \ldots , a'_n\} \in S_{L'}$ we have $d'_i = \sum k=1^n \alpha_{k,i} a_k$, $i = 1, \ldots , n$ with

$$\begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{bmatrix} = \pm 1.$$

$S$ is maximal in the sense that if $\{a_1, \ldots , a_n\} \in S_{L'}$, then for arbitrary points $d'_i = \sum k=1^n \alpha_{k,i} a_k$, $i = 1, \ldots , n$ with

$$\begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{bmatrix} = \pm 1$$

we have $\{a'_1, \ldots , a'_n\} \in S_{L'}$.

From the proof of Theorem 4 we know that

$$\left\langle c, \sum_{k=1}^n \alpha_{k,i} a_k \mid \sum_{k=1}^n \alpha_{2,k} a_k, \ldots , \sum_{k=1}^n \alpha_{n,k} a_k \right\rangle
= \begin{bmatrix} pr_{a_1, \ldots , a_n}(c) & \cdots & pr_{a_1, \ldots , a_n}(c) \\ \alpha_{2,1} & \cdots & \alpha_{2,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{bmatrix} \begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{bmatrix}$$
whenever $c \in L$ and \{o$_1$, $\ldots$, o$_n$\} $\in S_L$.

**Theorem 5.** Let $L'$ and $L^+$ be $n$-dimensional linear subspaces of $L$ such that $\dim (L' \cap L^+) = n - 1$ and let \{a$'$, a$_2$, $\ldots$, a$_n$\} $\in S_L$ and \{a$^+$, a$_2$, $\ldots$, a$_n$\} $\in S_L$. Then

$$\Pr_{\alpha_1, \ldots, \alpha_n} (a') = \Pr_{\alpha_2, \ldots, \alpha_n} (a^+).$$

**Proof.** Evident. $\square$

3. **Generation of $n$-inner products by means of families of projections**

3.1. Let $L$ be an arbitrary linear space of dimension $\geq n$. For every $n$-dimensional linear subspace $L'$ of $L$ let $S_L$ be a maximal set of subsets \{o$_1$, $\ldots$, o$_n$\} of linearly independent points of $L'$ such that for arbitrary \{a$_1$, $\ldots$, a$_n$\}, \{a$'$, a$_2$, $\ldots$, a$_n$\} $\in S_L$ we have $a'_i = \sum_{k=1}^n \alpha_{i,k} a_k$, $i = 1, \ldots, n$ with

$$\begin{vmatrix}
\alpha_{1,1} & \cdots & \alpha_{1,n} \\
\vdots & \ddots & \vdots \\
\alpha_{n,1} & \cdots & \alpha_{n,n}
\end{vmatrix} = \pm 1.$$

Moreover, let us assume the following:

1. For every $n$-dimensional linear subspace $L'$ of $L$ there is a projection $\text{pr}_{L'}$ of $L$ onto $L'$ for which for every \{o$_1$, $\ldots$, o$_n$\} $\in S_L$ we also will use the notation

$$\text{pr}_{\alpha_1, \ldots, \alpha_n} = \sum_{k=1}^n \text{pr}_{\alpha_1, \ldots, \alpha_k, \ldots, \alpha_n} a_k.$$

2. If $L', L^+$ are $n$-dimensional linear subspaces of $L$ such that $\dim (L' \cap L^+) = n - 1$ and if \{a$'$, a$_2$, $\ldots$, a$_n$\} $\in S_L$ and \{a$^+$, a$_2$, $\ldots$, a$_n$\} $\in S_L$ then

$$\Pr_{\alpha_1, \ldots, \alpha_n} (a') = \Pr_{\alpha_2, \ldots, \alpha_n} (a^+).$$

Every $n$ points a$'_1$, $\ldots$, a$'_n$ of $L$ can be written in the form $a'_i = \sum_{k=1}^n \alpha_{i,k} a_k$, $i = 1, \ldots, n$, by means of \{a$_1$, $\ldots$, a$_n$\} $\in S_L$ with a suitable $L'$. Let us define

$$\begin{vmatrix}
\alpha_{2,1} & \cdots & \alpha_{2,n} \\
\vdots & \ddots & \vdots \\
\alpha_{n,1} & \cdots & \alpha_{n,n}
\end{vmatrix} = \begin{vmatrix}
\Pr_{\alpha_1, \ldots, \alpha_n} (c) & \cdots & \Pr_{\alpha_1, \ldots, \alpha_n} (c) \\
\vdots & \ddots & \vdots \\
\alpha_{1,1} & \cdots & \alpha_{1,n} \\
\vdots & \ddots & \vdots \\
\alpha_{n,1} & \cdots & \alpha_{n,n}
\end{vmatrix}$$

**Theorem 6.** (c, a$'_1$ | a$'_2$, $\ldots$, a$'_n$) given by (5) is independent of the special choice of \{a$_1$, $\ldots$, a$_n$\).
Proof. Let \( \{a_1, \ldots, a_n\}, \{\bar{a}_1, \ldots, \bar{a}_n\} \in S_L \) and \( a_k = \sum_{i=1}^{n} \alpha_{i,k} \bar{a}_i, k = 1, \ldots, n \). Then
\[
\begin{bmatrix}
\bar{a}_{1,1} & \cdots & \bar{a}_{1,n} \\
\vdots & \ddots & \vdots \\
\bar{a}_{n,1} & \cdots & \bar{a}_{n,n}
\end{bmatrix} = \pm 1
\]
and \( a'_k = \sum_{i=1}^{n} \alpha_{i,k} a_k = \sum_{i=1}^{n} \alpha_{i,k} \bar{a}_i, i = 1, \ldots, n \). From
\[
\sum_{i=1}^{n} \text{pr}_{\cdots, a_i, \cdots, a'_n}(c) \bar{a}_i = \sum_{i=1}^{n} \text{pr}_{\cdots, a_i, \cdots, a'_n}(c) a_k
\]
we get \( \text{pr}_{\cdots, a_i, \cdots, a'_n}(c) = \sum_{i=1}^{n} \text{pr}_{\cdots, a_i, \cdots, a'_n}(c) a_k, i = 1, \ldots, n \), and consequently
\[
\begin{bmatrix}
\text{pr}_{\cdots, a_i, \cdots, a'_n}(c) & \cdots & \text{pr}_{\cdots, a_i, \cdots, a'_n}(c) \\
\sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{i,k} \bar{a}_k & \cdots & \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{i,k} a_k \\
\vdots & \ddots & \vdots \\
\sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{i,k} \bar{a}_k & \cdots & \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{i,k} a_k
\end{bmatrix}
\begin{bmatrix}
\sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{i,k} \bar{a}_k \\
\vdots \\
\sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{i,k} a_k
\end{bmatrix} = \begin{bmatrix}
\sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{i,k} \bar{a}_k \\
\vdots \\
\sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{i,k} a_k
\end{bmatrix}
\begin{bmatrix}
\sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{i,k} \bar{a}_k & \cdots & \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{i,k} a_k \\
\vdots & \ddots & \vdots \\
\sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{i,k} \bar{a}_k & \cdots & \sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{i,k} a_k
\end{bmatrix}
\begin{bmatrix}
\sum_{k=1}^{n} \sum_{i=1}^{n} \alpha_{i,k} \bar{a}_k \\
\vdots \\
\sum_{i=1}^{n} \sum_{k=1}^{n} \alpha_{i,k} a_k
\end{bmatrix}
\]
By virtue of (5) the last equation proves the theorem. \( \square \)

Theorem 7. (\( \cdot, \cdot, \ldots, \cdot \)) given by (5) is an n-inner product on \( L \) where for every n-dimensional linear subspace \( L' \) of \( L \) and arbitrary \( \{a_1, \ldots, a_n\} \in S_L \) we have \( \|a_1, \ldots, a_n\| = 1 \).

Proof. Let \( a_1, \ldots, a_n \) be arbitrary in \( L \), let \( L' \) be an n-dimensional linear subspace of \( L \) containing \( a_1, \ldots, a_n \) and let \( \{a'_1, \ldots, a'_n\} \in S_{L'} \). Then \( a_i = \sum_{k=1}^{n} \alpha_{i,k} a'_k \),
Hence we get
\[
(\alpha_1, \alpha_1 | \alpha_2, \ldots, \alpha_n) = \left( \sum_{k=1}^{n} \alpha_1 k \alpha_1 k, \ldots, \sum_{k=1}^{n} \alpha_1 k \alpha_1 k \right) = \left( \sum_{k=1}^{n} \alpha_2 k \alpha_2 k, \ldots, \sum_{k=1}^{n} \alpha_2 k \alpha_2 k \right) = \left( \sum_{k=1}^{n} \alpha_3 k \alpha_3 k, \ldots, \sum_{k=1}^{n} \alpha_3 k \alpha_3 k \right) = \cdots = \left( \sum_{k=1}^{n} \alpha_n k \alpha_n k, \ldots, \sum_{k=1}^{n} \alpha_n k \alpha_n k \right)
\]

which implies that \((\alpha_1, \alpha_1 | \alpha_2, \ldots, \alpha_n) \neq 0\) and moreover that \((\alpha_1, \alpha_1 | \alpha_2, \ldots, \alpha_n) = 0\) if and only if \(\alpha_1, \ldots, \alpha_n\) are linearly dependent.

Now we shall show that for arbitrary \(\alpha', \alpha^+, \alpha_2, \ldots, \alpha_n\) we have
\[
(\alpha', \alpha^+ | \alpha_2, \ldots, \alpha_n) = (\alpha^+, \alpha' | \alpha_2, \ldots, \alpha_n).
\]

If \(\alpha', \alpha_2, \ldots, \alpha_n\) or \(\alpha^+, \alpha_2, \ldots, \alpha_n\) are linearly dependent, then \((\alpha', \alpha^+ | \alpha_2, \ldots, \alpha_n)\) and \((\alpha^+, \alpha' | \alpha_2, \ldots, \alpha_n)\) both are 0. Hence we may restrict our considerations to the case that \(\alpha', \alpha_2, \ldots, \alpha_n\) and \(\alpha^+, \alpha_2, \ldots, \alpha_n\) are linearly independent. Let \(L', L^+\) denote the linear subspaces of \(L\) generated by \(\alpha', \alpha_2, \ldots, \alpha_n\) or \(\alpha^+, \alpha_2, \ldots, \alpha_n\), respectively. There exist reals \(\alpha', \alpha^+\) different from 0 such that \(\{\alpha' \alpha', \alpha_2, \ldots, \alpha_n\} \in S_{L'}\) and \(\{\alpha^+ \alpha^+, \alpha_2, \ldots, \alpha_n\} \in S_{L^+}\). This together with (4) and (5) yields
\[
(\alpha', \alpha^+ | \alpha_2, \ldots, \alpha_n) = \frac{1}{\alpha^+} (\alpha', \alpha^+ | \alpha_2, \ldots, \alpha_n) = \frac{1}{\alpha^+} \text{pr}_{\alpha^+ \alpha^+} (\alpha' \alpha^+)
\]
which shows that \((\alpha, \alpha | \alpha_2, \ldots, \alpha_n) = (\alpha, \alpha | \alpha_2, \ldots, \alpha_n)\) for every permutation \((i_2, \ldots, i_n)\) of \((2, \ldots, n)\). Also the linearity of \((\alpha, \alpha | \alpha_2, \ldots, \alpha_n)\) with respect to \(\alpha\) is evident. From (5) we immediately see that, moreover, for every \(\{\alpha_1, \ldots, \alpha_n\} \in S_L\) we have \(||\alpha_1, \ldots, \alpha_n|| = 1\). \(\square\)

3.2. If dim \(L = n\), then in Assumption 2 of 3.1 we necessarily have \(L' = L^+\), hence \(\alpha^+ = \alpha^+ + \sum_{k=1}^{n} \alpha_k \alpha_k\) and \(\text{pr}_{\alpha^+ \alpha^+} = \text{pr}_{\alpha^+ \alpha^+} \) is the identical mapping.

From this we see that in this case, equation (4) becomes trivial. We can choose \(S_L\) arbitrarily and the corresponding \(n\)-inner products differ only by a factor.

Let now \(\text{dim} L > n\). Then obviously (4) contains restrictions to the projections \(\text{pr}_{L'}\) if the sets \(S_{L'}\) are fixed, and conversely for fixed projections \(\text{pr}_{L'}\) it contains restrictions to the sets \(S_{L'}\).
4. n-NORM OF PROJECTIONS

4.1. Concerning the problem of the relations between norms \(\|b_1, \ldots, b_n\|\) and 
\(\|pr_{a_1, \ldots, a_n}(b_1), \ldots, pr_{a_1, \ldots, a_n}(b_n)\|\) we have the following results.

**Theorem 8.** Let \((L, \langle \cdot, \cdot | \cdot, \cdot\rangle)\) be an n-inner product space which in the case 
\(n > 1\) is simple. Then

\[
\|b_1, \ldots, b_n\| \geq \|pr_{a_1, \ldots, a_n}(b_1), \ldots, pr_{a_1, \ldots, a_n}(b_n)\|.
\]

**Proof.** In the case \(n = 1\) the assertion of the theorem is well known. For further considerations let \(n > 1\). Let \(\langle \cdot, \cdot | \cdot, \cdot\rangle\) be an inner product generating \(\langle \cdot, \cdot | \cdot, \cdot\rangle, \ldots, \langle \cdot, \cdot | \cdot, \cdot\rangle\). Because of Theorem 3 we may restrict our considerations to the case that 
\(\langle a_k, a_i \rangle = \delta_{k,i}\) for \(k, i \in \{1, \ldots, n\}\). If 
\(pr_{a_1, \ldots, a_n}(b_1), \ldots, pr_{a_1, \ldots, a_n}(b_n)\) are linearly dependent, then obviously (7) is true. Therefore, in what follows we may assume that 
\(pr_{a_1, \ldots, a_n}(b_1), \ldots, pr_{a_1, \ldots, a_n}(b_n)\) are linearly independent. Since for arbitrary points 
\(c_1, \ldots, c_n \in L\) and arbitrary reals \(\gamma_{1,k}, \ldots, \gamma_{n,k}\), we have

\[
\left\| \sum_{k=1}^{n} \gamma_{1,k} c_k, \ldots, \sum_{k=1}^{n} \gamma_{n,k} c_k \right\|^2 = \begin{vmatrix} 
\gamma_{1,1} & \cdots & \gamma_{1,n} \\
\vdots & \ddots & \vdots \\
\gamma_{n,1} & \cdots & \gamma_{n,n} 
\end{vmatrix} \|c_1, \ldots, c_n\|^2,
\]

we can see that, moreover, the restriction to the case 
\(pr_{a_1, \ldots, a_n}(b_k) = a_k, \ k = 1, \ldots, n\) is possible. Then we have 
\(\langle b_k, a_i | a_1, \ldots, \hat{a}_i, \ldots, a_n \rangle = \delta_{k,i}\) for \(k, i \in \{1, \ldots, n\}\) and because of

\[
\begin{pmatrix}
(b_k, a_1) & (b_k, a_1) & \cdots & (b_k, a_{i-1}) & (b_k, a_i) & \cdots & (b_k, a_n) \\
(a_1, a_1) & (a_1, a_1) & \cdots & (a_1, a_{i-1}) & (a_1, a_i) & \cdots & (a_1, a_n) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
(a_{i-1}, a_{i-1}) & (a_{i-1}, a_{i-1}) & \cdots & (a_{i-1}, a_{i-1}) & (a_{i-1}, a_i) & \cdots & (a_{i-1}, a_n) \\
(a_{i-1}, a_{i-1}) & (a_{i-1}, a_{i-1}) & \cdots & (a_{i-1}, a_{i-1}) & (a_{i-1}, a_i) & \cdots & (a_{i-1}, a_n) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
(b_n, a_1) & (b_n, a_1) & \cdots & (b_n, a_{i-1}) & (b_n, a_i) & \cdots & (b_n, a_n) \\
\end{pmatrix} = (b_k, a_i)
\]
we get \((b_k, a_l) = \delta_{kl}\) for \(k, l \in \{1, \ldots, n\}.\) In view of this we see that for arbitrary \(k \in \{1, \ldots, n\},\)
\[
\begin{align*}
(a_k, b_k - a_k | a_1, \ldots, a_{k-1}, b_{k+1}, \ldots, b_n) &= \begin{pmatrix}
(a_k, a_k - a_k) & (a_k, a_1) & \cdots & (a_k, a_{k-1}) & (a_k, b_{k+1}) & \cdots & (a_k, b_n) \\
(a_1, a_k) & (a_1, a_1) & \cdots & (a_1, a_{k-1}) & (a_1, b_{k+1}) & \cdots & (a_1, b_n) \\
& \vdots & & \vdots & \vdots & & \vdots \\
& \vdots & & \vdots & \vdots & & \vdots \\
(b_n, a_k - a_k) & (b_n, a_1) & \cdots & (b_n, a_{k-1}) & (b_n, b_{k+1}) & \cdots & (b_n, b_n)
\end{pmatrix} \\
&= 0.
\end{align*}
\]
This yields
\[
\begin{align*}
\|b_1, \ldots, b_n\|^2 &= \|a_1, b_2, \ldots, b_n\|^2 + \|b_1 - a_1, b_2, \ldots, b_n\|^2 + 2(a_1, b_1 - a_1 | b_2, \ldots, b_n) \\
&\geq \|a_1, b_2, \ldots, b_n\|^2 \\
&\geq \cdots \\
&\geq \|a_1, \ldots, a_n\|^2 \\
&= \|\text{pr}_{a_1, \ldots, a_n}(b_1), \ldots, \text{pr}_{a_1, \ldots, a_n}(b_n)\|^2,
\end{align*}
\]
hence the theorem is proved.

In the case \(n > 1,\) (7) need not always be true as is shown by an example (with \(n = 2\)) given in [3].

References


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