Aleksander Misiak; Alicja Ryż

$n$-inner product spaces and projections


Persistent URL: [http://dml.cz/dmlcz/126265](http://dml.cz/dmlcz/126265)

**Terms of use:**

© Institute of Mathematics AS CR, 2000

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*
n-INNER PRODUCT SPACES AND PROJECTIONS

ALEKSANDER MISIAK, ALICJA RYZ, Szczecin

(Received October 29, 1997)

Abstract. This paper is a continuation of investigations of n-inner product spaces given in [5, 6, 7] and an extension of results given in [3] to arbitrary natural n. It concerns families of projections of a given linear space \( L \) onto its n-dimensional subspaces and shows that between these families and n-inner products there exist interesting close relations.

Keywords: n-inner product space, n-normed space, ro-norm of projection

MSC 1991: 46C05, 46C50

1. n-INNER PRODUCTS AND n-NORMS

1.1. Let \( n \) be a natural number \( (n \neq 0) \), \( L \) a linear space with \( \dim L \geq n \) and let \( \langle \cdot, \cdot, \cdot \rangle \) be a real function on \( L^{n+1} = L \times \ldots \times L \) \( n+1 \) times

In the case \( n = 1 \), we also write \( \langle \cdot, \cdot \rangle \) instead of \( \langle \cdot, \cdot, \cdot \rangle \) and \( \langle a, a_2, \ldots, a_n \rangle \) is to be understood as the expression \( \langle a, b \rangle \). Let us assume the following conditions:

1. \( \langle a, b | a_2, \ldots, a_n \rangle \geq 0 \)
2. \( \langle a, b | a_2, \ldots, a_n \rangle = \langle b, a | a_2, \ldots, a_n \rangle \)
3. \( \langle a, b | a_2, \ldots, a_n \rangle = \langle a, b | a_1, \ldots, a_n \rangle \) for every permutation \( (i_2, \ldots, i_n) \) of \( (2, \ldots, n) \)
4. if \( n > 1 \), then \( \langle a, a | a_2, a_3, \ldots, a_n \rangle = \langle a_2, a_3, \ldots, a_n | a, a \rangle \)
5. \( \langle a, a, b | a_2, \ldots, a_n \rangle = a \langle a, b | a_2, \ldots, a_n \rangle \) for every real \( a \)
6. \( \langle a + b, c | a_2, \ldots, a_n \rangle = a \langle a, c | a_2, \ldots, a_n \rangle + b \langle c, a_2, \ldots, a_n \rangle \)

Then \( \langle \cdot, \cdot, \cdot \rangle \) is called an n-inner product on \( L \) (see [5]) and \( \langle L, \langle \cdot, \cdot, \cdot \rangle \rangle \) is called an n-inner product space. The concept of an n-inner product space is a generalization of the concepts of an inner product space \( (n = 1) \) and of a 2-inner product space (see [1]).
1.2. Let $n > 1$. An $n$-inner product space $L$ and its $n$-inner product $(\cdot | \cdot, \ldots, \cdot)$ are called simple if there exists an inner product $(\cdot, \cdot)$ on $L$ such that the relation

\[
(a, b | a_2, \ldots, a_n) =
\begin{pmatrix}
(a, b) & (a, a_2) & \cdots & (a, a_n) \\
(a_2, b) & (a_2, a_2) & \cdots & (a_2, a_n) \\
& \vdots & \ddots & \vdots \\
(a_n, b) & (a_n, a_2) & \cdots & (a_n, a_n)
\end{pmatrix}
\]

holds. The inner product $(\cdot, \cdot)$ is said to generate the $n$-inner product $(\cdot | \cdot, \ldots, \cdot)$. An element $a \in L$ is said to be orthogonal to a non-empty subset $S$ of $L$ if $(a, e_1 | e_2, \ldots, e_n) = 0$ for arbitrary $e_1, \ldots, e_n \in S$. A subset $S$ of $L$ is said to be orthogonal if it is linearly independent, contains at least $n$ elements and if every $e \in S$ is orthogonal to $S \setminus \{e\}$.

1.3. An $n$-norm on $L$ is a real function $\|\cdot, \ldots, \cdot\|$ on $L^n$ which satisfies the following conditions:
1. $\|a_1, \ldots, a_n\| = 0$ if and only if $a_1, \ldots, a_n$ are linearly dependent,
2. $\|a_{i_1}, \ldots, a_{i_n}\| = \|a_{i_1}, \ldots, a_{i_n}\|$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$,
3. $\|\alpha a_1, a_2, \ldots, a_n\| = |\alpha| \|a_1, a_2, \ldots, a_n\|$ for every real number $\alpha$,
4. $\|a + b, a_2, \ldots, a_n\| \leq \|a, a_2, \ldots, a_n\| + \|b, a_2, \ldots, a_n\|$.

$L$ equipped with an $n$-norm $\|\cdot, \ldots, \cdot\|$ is called an $n$-normed space. The concept of an $n$-normed space is a generalization of the concepts of a normed ($n = 1$) and a 2-normed space (see [2]).

**Theorem 1.** (Theorem 7 of [5]) For every $n$-inner product $(\cdot | \cdot, \ldots, \cdot)$ on $L$,

1. $\|a_1, a_2, \ldots, a_n\| = \sqrt{(a_1, a_1 | a_2, \ldots, a_n)}$

defines an $n$-norm on $L$ for which

2. $(a, b | a_2, \ldots, a_n) = \frac{1}{2} \left( \|a + b, a_2, \ldots, a_n\|^2 - \|a - b, a_2, \ldots, a_n\|^2 \right)$

and

3. $\|a + b, a_2, \ldots, a_n\|^2 + \|a - b, a_2, \ldots, a_n\|^2 = 2 \|\|a, a_2, \ldots, a_n\|^2 + \|b, a_2, \ldots, a_n\|^2\|$ are true.

Conversely, for every $n$-norm $\|\cdot, \ldots, \cdot\|$ on $L$ with the property (3), (2) defines an $n$-inner product on $L$ for which (1) is true.

For every $n$-inner product $(\cdot | \cdot, \ldots, \cdot)$ on $L$ the $n$-norm given by (1) is said to be associated to $(\cdot | \cdot, \ldots, \cdot)$. If in connection with an $n$-inner product on $L$ an $n$-norm is used, then $\|\cdot, \ldots, \cdot\|$ always will be the $n$-norm associated to $(\cdot | \cdot, \ldots, \cdot)$.
2. PROJECTIONS IN n-INNER PRODUCT SPACES

2.1. Let $(L, (\cdot, \cdot, \ldots, \cdot))$ be an n-inner product space. For arbitrary linearly independent points $a_1, \ldots, a_n \in L$, let $pr_{a_1, \ldots, a_n}$ be the mapping of $L$ into $L$ given by

$$pr_{a_1, \ldots, a_n}(c) = \frac{(c, a_1 | a_2, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2} a_1 + \ldots + \frac{(c, a_n | a_1, \ldots, a_{n-1})}{\|a_1, \ldots, a_n\|^2} a_n$$

(see [3], where $n = 2$). We often use the notion

$$(c, a_k | a_1, \ldots, a_k, \ldots, a_n) = (c, a_k | a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n)$$

and

$$pr_{a_1, \ldots, a_k, \ldots, a_n}(c) = \frac{(c, a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2}$$

Then we have

$$pr_{a_1, \ldots, a_k}(c) = \sum_{k=1}^n \frac{(c, a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2} a_k$$

Theorem 2. $pr_{a_1, \ldots, a_n}$ is a projection of $L$ onto $L(\{a_1, \ldots, a_n\})$, the linear space generated by the set $\{a_1, \ldots, a_n\}$.

Proof. Obviously $pr_{a_1, \ldots, a_n}$ is linear. Since $pr_{a_1, \ldots, a_n}(a_k) = a_k$ for arbitrary $k \in \{1, \ldots, n\}$, $pr_{a_1, \ldots, a_n}$ maps $L$ onto $L(\{a_1, \ldots, a_n\})$. Moreover,

$$pr^2_{a_1, \ldots, a_n}(c) = \sum_{k=1}^n \frac{(pr_{a_1, \ldots, a_n}(c), a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2} a_k$$

from which by virtue of

$$\frac{(pr_{a_1, \ldots, a_n}(c), a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2}$$

$$= \sum_{k=1}^n \frac{(c, a_{k-1} | a_1, \ldots, a_{k-1}, a_k) (a_{k-1}, a_{k-1} | a_1, \ldots, a_{k-1}, a_k)}{\|a_1, \ldots, a_n\|^4}$$

$$= \frac{(c, a_k | a_1, \ldots, a_k, \ldots, a_n)}{\|a_1, \ldots, a_n\|^2}$$
we get
\[ pr_{a_1, \ldots, a_n}(c) = pr_{a_1, \ldots, a_n}(c). \]

**Theorem 3.** \( pr_{a_1, \ldots, a_n} \) is independent of the special choice of \( a_1, \ldots, a_n \) in \( L(\{a_1, \ldots, a_n\}) \); this means, for arbitrary linearly independent points \( a^*_i = \sum_{k=1}^n a_{i,k} a_k \), \( i = 1, \ldots, n \), we have
\[ pr_{a_1, \ldots, a_n} = pr_{a^*_1, \ldots, a^*_n}. \]

**Proof.** Let linearly independent points \( a^*_i = \sum_{k=1}^n a_{i,k} a_k \), \( i = 1, \ldots, n \) be given. Then
\[ \begin{bmatrix} a_{1,1} & \ldots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \ldots & a_{n,n} \end{bmatrix} \neq 0. \]
For arbitrary \( c \in L \),
\[ pr_{a_1, a_2, \ldots, a_n}(c) = \sum_{i, j=1}^n a_{i,j} \left( c \sum_{k=1}^n a_{i,k} a_k + \sum_{k=1}^n a_{i,k} a_k + \sum_{k=1}^n a_{i,k} a_k + \sum_{k=1}^n a_{i,k} a_k \right) \]
Using the notion \( \sum' \), which means that summation is taken only with respect to different indices, formula (8) in Theorem 6 of [6] implies that
\[ \sum_{i=1}^n a_{i,1} \left( c, \sum_{k=1}^n a_{i,k} a_k \right) = \sum_{j, k_1, < k_2 \ldots < k_n} \left( c, a_j \mid a_{k_1}, \ldots, a_{k_n} \right) \]
\[ \sum_{j, k_1, < k_2 \ldots < k_n} \left( c, a_j \mid a_{k_1}, \ldots, a_{k_n} \right) = \left\{ \begin{array}{c} a_{1,1} \ldots a_{1,n} \\ \vdots \\ a_{n,1} \ldots a_{n,n} \end{array} \right\} \]
This yields that
\[
\sum_{k=1}^{n} \sum_{\ell=1}^{n} a_{\ell,k} a_{k,\ell} = \left( \begin{array}{ccc} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{array} \right) \left( \begin{array}{c} a_{1,1} \\ \vdots \\ a_{n,1} \end{array} \right)^2.
\]

This yields that
\[
p_{\mathbf{a}_1, \ldots, \mathbf{a}_n}(c) = \sum_{k=1}^{n} \frac{c \cdot a_k}{\|a_1, \ldots, a_n\|^2} a_k = p_{\mathbf{a}_1, \ldots, \mathbf{a}_n}(c)
\]
which proves the theorem. \(\square\)

**Theorem 4.** For arbitrary \(c \in L\), \(c - p_{\mathbf{a}_1, \ldots, \mathbf{a}_n}(c)\) is orthogonal to \(L(\{a_1, \ldots, a_n\})\).

**Proof.** For arbitrary \(c = \sum_{k=1}^{n} a_k, i = 1, \ldots, n\), by means of (8) in Theorem 6 (see [6]) we get
\[
\left( c - p_{\mathbf{a}_1, \ldots, \mathbf{a}_n}(c), \sum_{k=1}^{n} a_{1,k} a_k, \sum_{k=1}^{n} a_{2,k} a_k, \ldots, \sum_{k=1}^{n} a_{n,k} a_k \right)
= \left( \sum_{k=1}^{n} \left( \frac{c \cdot a_k}{\|a_1, \ldots, a_n\|^2} a_k \right) \right) \sum_{k=1}^{n} a_{1,k} a_k, \sum_{k=1}^{n} a_{2,k} a_k, \ldots, \sum_{k=1}^{n} a_{n,k} a_k \right)
= \sum_{j,k=1}^{n} \left( \begin{array}{c} 1 \\ 0 \\ \vdots \\ 0 \\ a_{1,j} \\ a_{1,k} \\ a_{2,j} \\ a_{2,k} \\ \vdots \\ \vdots \\ \vdots \\ a_{n,j} \\ a_{n,k} \end{array} \right) \left( \begin{array}{c} a_{1,1} \\ \vdots \\ \vdots \\ a_{n,1} \\ a_{1,1} \\ \vdots \\ \vdots \\ \vdots \\ a_{n,1} \end{array} \right)
= 0.
\]
This was to be proved. \(\square\)

2.2. From Theorem 2 of [7] we know the following: if \((\cdot, \cdot | \cdots, \cdot)\) is a simple \(n\)-inner product on \(L\) and \((\cdot, \cdot)\) generates \((\cdot, \cdot | \cdots, \cdot)\), then for arbitrary \(a \in L\) and
arbitrary $S \subset L$ which generates a linear subspace of $L$ of dimension $\geq n$, $a$ is orthogonal to $S$ relative to $(\cdot, \cdot | \cdot , \cdot $) if and only if $a$ is orthogonal to $S$ relative to $(\cdot , \cdot )$. From this and Theorem 4 it follows that if $(\cdot , \cdot | \cdot , \cdot $) is simple and $(\cdot , \cdot )$ is an inner product on $L$ generating $(\cdot , \cdot | \cdot , \cdot $), then for arbitrary $c \in L$, $c - pr_{o_1, \ldots , o_n}(c)$ is orthogonal to $L((o_1, \ldots , o_n))$ relative to $(\cdot , \cdot $).

2.3. From Theorem 3 of [6] we know that if $S$ is an orthogonal set in $L$, for every $e \in S$, distinct $e_2, \ldots , e_n \in S \setminus \{e\}$, distinct $e'_2, \ldots , e'_n \in S \setminus \{e\}$ and every $c$ from the linear space generated by $S$, we have

$$
\frac{(c, e | e_2, \ldots , e_n)}{\|e, e_2, \ldots , e_n\|^2} = \frac{(c, e | e'_2, \ldots , e'_n)}{\|e, e'_2, \ldots , e'_n\|^2},
$$

which implies $pr_{E_{e_2, \ldots , e_n}}(c) = pr_{E_{e'_2, \ldots , e'_n}}(c)$. This means that under the above conditions the coordinate $pr_{E_{e_2, \ldots , e_n}}(c)$ of $pr_{E_{e'_2, \ldots , e'_n}}(c)$ is independent of $e_2, \ldots , e_n$.

For every $n$-dimensional linear subspace $U$ of $L$ let $S_U$ be the set of all subsets $\{o_1, \ldots , o_n\}$ of $L'$ such that $\|o_1, \ldots , o_n\| = 1$. Then for arbitrary $\{o_1, \ldots , o_n\}$, $\{o'_1, \ldots , o'_n\} \in S_U$, we have $d'_i = \sum_{k=1}^{n} \alpha_i k a_k, i = 1, \ldots , n$ with

\[
\begin{bmatrix}
\alpha_{1,1} & \ldots & \alpha_{1,n} \\
\vdots & \ddots & \vdots \\
\alpha_{n,1} & \ldots & \alpha_{n,n}
\end{bmatrix}
= \pm 1.
\]

$S$ is maximal in the sense that if $\{o_1, \ldots , o_n\} \in S_U$, then for arbitrary points $d'_i = \sum_{k=1}^{n} \alpha_i k a_k, i = 1, \ldots , n$ with

\[
\begin{bmatrix}
\alpha_{1,1} & \ldots & \alpha_{1,n} \\
\vdots & \ddots & \vdots \\
\alpha_{n,1} & \ldots & \alpha_{n,n}
\end{bmatrix}
= \pm 1.
\]

we have $\{o'_1, \ldots , o'_n\} \in S_U$.

From the proof of Theorem 4 we know that

\[
\begin{bmatrix}
(c, \sum_{k=1}^{n} \alpha_{1,k} a_k | \sum_{k=1}^{m} \alpha_{2,k} a_k, \ldots , \sum_{k=1}^{m} \alpha_{n,k} a_k)
\end{bmatrix}
= \begin{bmatrix}
pr_{E_{\alpha_{2,1}, \ldots , \alpha_{2,m}}}(c) & \ldots & pr_{E_{\alpha_{n,1}, \ldots , \alpha_{n,m}}}(c)
\end{bmatrix}
\begin{bmatrix}
\alpha_{1,1} & \ldots & \alpha_{1,n} \\
\alpha_{2,1} & \ldots & \alpha_{2,n} \\
\vdots & \ddots & \vdots \\
\alpha_{n,1} & \ldots & \alpha_{n,n}
\end{bmatrix}
\]
Theorem 5. Let $L'$ and $L^+$ be $n$-dimensional linear subspaces of $L$ such that 
$\dim (L' \cap L^+) = n - 1$ and let $\{o'_1, o'_2, \ldots, o'_n\} \in S_L$ and $\{o^+, o_2, \ldots, o_n\} \in S_{L^+}$. Then
$$p_{o'_1, o'_2, \ldots, o'_n}(a') = p_{o^+, o_2, \ldots, o_n}(a^+).$$

Proof. Evident. \qed

3. Generation of $n$-inner products by means of families of projections

3.1. Let $L$ be an arbitrary linear space of dimension $\geq n$. For every $n$-dimensional linear subspace $L'$ of $L$ let $S_L'$ be a maximal set of subsets $\{a_1, \ldots, a_n\}$ of linearly independent points of $L'$ such that for arbitrary $\{a_1, \ldots, a_n\}$, $\{a'_1, \ldots, a'_n\} \in S_L'$ we have
$$a'_i = \sum_{k=1}^n \alpha_{i,k} a_k, \quad i = 1, \ldots, n$$
with
$$\begin{vmatrix}
\alpha_{1,1} & \cdots & \alpha_{1,n} \\
\vdots & \ddots & \vdots \\
\alpha_{n,1} & \cdots & \alpha_{n,n}
\end{vmatrix} = \pm 1.$$ 
Moreover, let us assume the following:

1. For every $n$-dimensional linear subspace $L'$ of $L$ there is a projection $p_L$ of $L$ onto $L'$ for which for every $\{a_1, \ldots, a_n\} \in S_L$ we also will use the notation
$$p_{a_1, \ldots, a_n} = \sum_{k=1}^n p_{a_1, \ldots, a_{k-1}, a_k} a_k.$$

2. If $L'$, $L^+$ are $n$-dimensional linear subspaces of $L$ such that $\dim (L' \cap L^+) = n - 1$ and if $\{a'_1, a'_2, \ldots, a'_n\} \in S_L$ and $\{a^+, a_2, \ldots, a_n\} \in S_{L^+}$ then
$$p_{a'_1, a'_2, \ldots, a'_n}(a') = p_{a^+, a_2, \ldots, a_n}(a^+).$$

Every $n$ points $a'_1, \ldots, a'_n$ of $L$ can be written in the form $a'_i = \sum_{k=1}^n \alpha_{i,k} a_k, \quad i = 1, \ldots, n$, by means of $\{a_1, \ldots, a_n\} \in S_L$ with a suitable $L'$. Let us define
$$\begin{pmatrix}
\langle c, a'_1 \rangle & \langle c, a'_2 \rangle & \cdots & \langle c, a'_n \rangle \\
\langle a_2, a'_1 \rangle & \langle a_2, a'_2 \rangle & \cdots & \langle a_2, a'_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle a_n, a'_1 \rangle & \langle a_n, a'_2 \rangle & \cdots & \langle a_n, a'_n \rangle
\end{pmatrix} = 
\begin{pmatrix}
\langle c, a_1 \rangle & \langle c, a_2 \rangle & \cdots & \langle c, a_n \rangle \\
\langle a_2, a_1 \rangle & \langle a_2, a_2 \rangle & \cdots & \langle a_2, a_n \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle a_n, a_1 \rangle & \langle a_n, a_2 \rangle & \cdots & \langle a_n, a_n \rangle
\end{pmatrix}$$

Theorem 6. $\langle c, a'_1 \mid a'_2, \ldots, a'_n \rangle$ given by (5) is independent of the special choice of $\{a_1, \ldots, a_n\}$. 

93
Proof. Let \( \{a_1, \ldots, a_n\}, \{\tilde{a}_1, \ldots, \tilde{a}_n\} \in S_L \) and \( a_k = \sum_{i=1}^n \alpha_{k,i} \tilde{a}_i, k = 1, \ldots, n. \) Then

\[
\begin{pmatrix}
\tilde{a}_{1,1} & \cdots & \tilde{a}_{1,n} \\
\vdots & \ddots & \vdots \\
\tilde{a}_{n,1} & \cdots & \tilde{a}_{n,n}
\end{pmatrix} = \pm 1
\]

and \( a'_k = \sum_{k=1}^n \alpha_{k,i} a_k = \sum_{k=1}^n \alpha_{k,i} \tilde{a}_k, i = 1, \ldots, n. \) From

\[
\sum_{i=1}^n \text{pr}_{a_1, \ldots, a_n}(c) \tilde{a}_i = \sum_{k=1}^n \text{pr}_{a_1, \ldots, a_n}(c) \tilde{a}_k
\]

we get \( \text{pr}_{a_1, \ldots, a_n}(c) = \sum_{k=1}^n \text{pr}_{a_1, \ldots, a_n}(c) \tilde{a}_k, l = 1, \ldots, n, \) and consequently

\[
\text{pr}_{a_1, \ldots, a_n}(c) = \begin{pmatrix}
\text{pr}_{a_1, \ldots, a_n}(c) \\
\vdots \\
\text{pr}_{a_1, \ldots, a_n}(c)
\end{pmatrix} = \begin{pmatrix}
\alpha_{1,1} & \cdots & \alpha_{1,n} \\
\vdots & \ddots & \vdots \\
\alpha_{n,1} & \cdots & \alpha_{n,n}
\end{pmatrix}
\]

By virtue of (5) the last equation proves the theorem.

Theorem 7. \((\cdot, \cdot, \ldots, \cdot)\) given by (5) is an \( n \)-inner product on \( L \) where for every \( n \)-dimensional linear subspace \( L' \) of \( L \) and arbitrary \( \{a_1, \ldots, a_n\} \in S_L \) we have \( \|a_1, \ldots, a_n\| = 1. \)

Proof. Let \( a_1, \ldots, a_n \) be arbitrary in \( L \), let \( L' \) be an \( n \)-dimensional linear subspace of \( L \) containing \( a_1, \ldots, a_n \) and let \( \{a'_1, \ldots, a'_n\} \in S_L \). Then \( a_i = \sum_{k=1}^n \alpha_{i,k} a'_k \).
\[ i = 1, \ldots, n. \] Hence we get

\[ (6) \quad (a_1, a_1 | a_2, \ldots, a_n) = \left( \sum_{k=1}^{n} a_{1,k} a'_{1,k}, \sum_{k=1}^{n} a_{1,k} a'_k \right) \]

which implies that \((a_1, a_1 | a_2, \ldots, a_n) \geq 0\) and moreover that \((a_1, a_1 | a_2, \ldots, a_n) = 0\) if and only if \(a_1, \ldots, a_n\) are linearly dependent.

Now we shall show that for arbitrary \(a', a^+, a_2, \ldots, a_n\) we have

\[ (a', a^+ | a_2, \ldots, a_n) = (a^+, a' | a_2, \ldots, a_n). \]

If \(a', a_2, \ldots, a_n\) or \(a^+, a_2, \ldots, a_n\) are linearly dependent, then \((a', a^+ | a_2, \ldots, a_n)\) and \((a^+, a' | a_2, \ldots, a_n)\) both are 0. Hence we may restrict our considerations to the case that \(a', a_2, \ldots, a_n\) and \(a^+, a_2, \ldots, a_n\) are linearly independent. Let \(L', L^+\) denote the linear subspaces of \(L\) generated by \(a', a_2, \ldots, a_n\) or \(a^+, a_2, \ldots, a_n\), respectively. There exist reals \(a', a^+\) different from 0 such that \(\{a'a', a_2, \ldots, a_n\} \in S_{L'}\) and \(\{a^+a^+, a_2, \ldots, a_n\} \in S_{L^+}\). This together with (4) and (5) yields

\[ (a', a^+ | a_2, \ldots, a_n) = \frac{1}{a'^+} (a', a^+ a^+ | a_2, \ldots, a_n) = \frac{1}{a'^+} pr_{a'a'^+ a_2, \ldots, a_n} (a' a^+) \]

Using (5) we see that \((a, b | a_2, \ldots, a_n) = (a, b | a_1, \ldots, a_n)\) for every permutation \((i_2, \ldots, i_n)\) of \((2, \ldots, n)\). And (6) shows that if \(n > 1\), then \((a, a | a_2, a_3, \ldots, a_n) = (a_2, a_2 | a, a_3, \ldots, a_n)\). Also the linearity of \((a, b | a_2, \ldots, a_n)\) with respect to \(a\) is evident. From (5) we immediately see that, moreover, for every \(\{a_1, \ldots, a_n\} \in S_L\), we have \(|a_1, \ldots, a_n| = 1\).

3.2. If \(\dim L = n\), then in Assumption 2 of 3.1 we necessarily have \(L' = L^+\), hence \(a^+ = a + \sum b_k a_k\), and \(pr_{a_2, \ldots, a_n} = pr_{a_2, \ldots, a_n}\) is the identical mapping. From this we see that in this case, equation (4) becomes trivial. We can choose \(S_L\) arbitrarily and the corresponding \(n\)-inner products differ only by a factor.

Let now \(\dim L > n\). Then obviously (4) contains restrictions to the projections \(pr_L\) if the sets \(S_L\) are fixed, and conversely for fixed projections \(pr_L\) it contains restrictions to the sets \(S_L\).
4. N-NORM OF PROJECTIONS

4.1. Concerning the problem of the relations between norms \( \| b_1, \ldots, b_n \| \) and \( \| \text{pr}_{a_1, \ldots, a_n}(b_1), \ldots, \text{pr}_{a_1, \ldots, a_n}(b_n) \| \) we have the following results.

Theorem 8. Let \( (L, (\cdot, \cdot), \ldots, \cdot) ) \) be an \( n \)-inner product space which in the case \( n > 1 \) is simple. Then

\[
\| b_1, \ldots, b_n \| \geq \| \text{pr}_{a_1, \ldots, a_n}(b_1), \ldots, \text{pr}_{a_1, \ldots, a_n}(b_n) \|.
\]

Proof. In the case \( n = 1 \) the assertion of the theorem is well known. For further considerations let \( n > 1 \). Let \( (\cdot, \cdot) \) be an inner product generating \( (\cdot, \cdot), \ldots, \cdot \).

Because of Theorem 3 we may restrict our considerations to the case that \( (a_k, a_i) = \delta_{ki} \) for \( k, i \in \{1, \ldots, n\} \). If \( \text{pr}_{a_1, \ldots, a_n}(b_1), \ldots, \text{pr}_{a_1, \ldots, a_n}(b_n) \) are linearly dependent, then obviously (7) is true. Therefore, in what follows we may assume that \( \text{pr}_{a_1, \ldots, a_n}(b_1), \ldots, \text{pr}_{a_1, \ldots, a_n}(b_n) \) are linearly independent. Since for arbitrary points \( c_1, \ldots, c_n \in L \) and arbitrary reals \( \gamma_1, \ldots, \gamma_n \), we have

\[
\| \sum_{k=1}^{n} \gamma_k c_k, \ldots, \sum_{k=1}^{n} \gamma_k c_k \|^2 = \begin{bmatrix} \gamma_1,1 & \cdots & \gamma_1,n \\ \vdots & \ddots & \vdots \\ \gamma_n,1 & \cdots & \gamma_n,n \end{bmatrix} \begin{bmatrix} \| c_1,1, c_2,1, \ldots, c_n,1 \| \\ \| c_1,2, c_2,2, \ldots, c_n,2 \| \\ \vdots \\ \| c_1,n, c_2,n, \ldots, c_n,n \| \end{bmatrix},
\]

we can see that, moreover, the restriction to the case \( \text{pr}_{a_1, \ldots, a_n}(b_k) = a_k, k = 1, \ldots, n \) is possible. Then we have \( (b_k, a_l | a_{l+1}, \ldots, a_n) = \delta_{kl} \) for \( k, l \in \{1, \ldots, n\} \) and because of

\[
(b_k, a_l | a_{l+1}, \ldots, a_n) = \begin{bmatrix} (b_k, a_l) & (b_k, a_{l+1}) & \cdots & (b_k, a_n) \\ (a_{l+1}, a_l) & (a_{l+1}, a_{l+1}) & \cdots & (a_{l+1}, a_n) \\ \vdots & \vdots & \ddots & \vdots \\ (a_n, a_l) & (a_n, a_{l+1}) & \cdots & (a_n, a_n) \end{bmatrix},
\]

we have

\[
= (b_k, a_l).
\]
we get \((b_k, a_l) = \delta_{kl}\) for \(k, l \in \{1, \ldots, n\}\). In view of this we see that for arbitrary \(k \in \{1, \ldots, n\}\),

\[
\begin{align*}
\begin{pmatrix}
(a_k, b_k - a_k | a_1, \ldots, a_{k-1}, b_{k+1}, \ldots, b_n) \\
(a_{k-1}, b_{k-1} - a_{k-1} | a_1, a_2, \ldots, a_{k-1}) \\
\vdots \\
(b_{n+1}, b_n - a_n | a_1, a_2, \ldots, a_{n-1}, b_{n+1}) \\
\end{pmatrix}
\end{align*}
\]

\[
= \begin{pmatrix}
(a_k, b_k - a_k) \\
(a_{k-1}, b_{k-1} - a_{k-1}) \\
\vdots \\
(b_{n+1}, b_n - a_n) \\
\end{pmatrix}
\]

\[
= 0.
\]

This yields

\[
\|b_1, \ldots, b_n\|^2 = \|a_1, b_2, \ldots, b_n\|^2 + \|b_1 - a_1, a_2, \ldots, b_n\|^2 + 2(a_1, b_1 - a_1 | b_2, \ldots, b_n)
\]

\[
\geq \|a_1, b_2, \ldots, b_n\|^2
\]

\[
\geq \ldots
\]

\[
= \|p_{a_1, \ldots, a_n}(b_1), \ldots, p_{a_1, \ldots, a_n}(b_n)\|^2,
\]

hence the theorem is proved.

In the case \(n > 1\), (7) need not always be true as is shown by an example (with \(n = 2\)) given in [3].

References


Authors' address: Aleksander Misiak, Alicja Ryż, Instytut Matematyki, Politechnika Szczecińska, Al. Piastów 17, 70-310 Szczecin, Poland.