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A GENERALIZED MAXIMUM PRINCIPLE FOR BOUNDARY VALUE PROBLEMS FOR DEGENERATE PARABOLIC OPERATORS WITH DISCONTINUOUS COEFFICIENTS

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Dedicated to Professor Alois Kufner on the occasion of his 65th birthday

Abstract. We prove a generalized maximum principle for subsolutions of boundary value problems, with mixed type unilateral conditions, associated to a degenerate parabolic second-order operator in divergence form.

Keywords: weak subsolution, generalized maximum principle, comparison theorem, degenerate equation

MSC 1991: 35B50, 35K10, 35K65, 35K85

1. INTRODUCTION

In [14] M. G. Platone Garroni has extended the classical generalized maximum principle (see, for instance, [15]), when the coefficients of the operator are discontinuous, to subsolutions of elliptic linear second order equations with mixed type boundary unilateral conditions, that is, on a portion of the boundary $\partial\Omega$ of $\Omega$, the values of the solution are assigned, while on the other part a unilateral condition on the solution and its conormal derivative is given. In the present paper we will establish a similar result (see Theorem 5.1) for degenerate parabolic equations, using a technique different from that of [14]. As a corollary, we obtain a comparison theorem (see Theorem 6.1). Our procedure, rather similar to that followed in [12] and in [13] allows us to obtain more general results. Other sufficient conditions for the boundedness of weak subsolutions of Cauchy-Dirichlet problem, in the non degenerate case, may be obtained from [6] and [17], while in the degenerate case some results are announced in [3] and in [4].
2. FUNCTIONAL SPACES

Let $\mathbb{R}^m$ be the Euclidean space $(m > 2)$ with a generic point $x = (x_1, x_2, \ldots, x_m)$, $\Omega$ a bounded open subset of $\mathbb{R}^m$ whose boundary satisfies locally a Lipschitz condition, $T$ a real positive number. Let us denote by $Q(\tau_1, \tau_2)$ $(0 \leq \tau_1 < \tau_2 \leq T)$ the cylinder $\Omega \times [\tau_1, \tau_2]$ and let $Q = Q(0, T)$; $\Gamma$ is the parabolic boundary of $Q$, that is $\Gamma = (\Omega \times \{t = 0\}) \cup (\partial \Omega \times [0, T])$.

Let $\partial \Omega_1$ be a closed subset of $\partial \Omega$, $\Gamma_2 = \partial \Omega_1 \times [0, T]$, and let us set $\partial \Omega_2 = \partial \Omega \setminus \partial \Omega_1$, $\Gamma_1 = \partial \Omega_2 \times [0, T]$.

The symbol $\text{meas}_m$ will henceforth denote the $m$-dimensional measure.

If $u(x)$ is a measurable function defined in $\Omega$, we will denote by $|u|_p$ $(1 \leq p \leq \infty)$ the usual norm in the space $L^p(\Omega)$.

**Hypothesis 2.1.** Let $\nu(x)$ be a positive function defined in $\Omega$ such that

$$\nu(x) \in L^{\frac{m-n}{n-1}}(\Omega), \quad \nu^{-1}(x) \in L^1(\Omega), \quad g > m.$$

The symbol $\tilde{H}^1(\nu, \Omega)$ stands for the completion of $C^1(\Omega)$ with respect to the norm

$$\|u\| = \left(\|u\|_2^2 + \sum_{i=1}^m \nu \frac{\partial u}{\partial x_i}^2\right)^{\frac{1}{2}}.$$

$C^*(\Omega)$ denotes the following linear subspace of $C^\infty(\overline{\Omega})$:

$$C^*(\Omega) = \{u \in C^\infty(\overline{\Omega}) : u = 0 \text{ on } \partial \Omega_2\}.$$

$H^*(\nu, \Omega)$ denotes the closure of $C^*(\Omega)$ in $\tilde{H}^1(\nu, \Omega)$.

If $u(x, t)$ is a measurable real function in $Q$, we will denote by $|u|_{p,q}$ $(1 \leq p, q \leq +\infty)$ the usual norm in the space $L^{p,q}(Q)$, with the obvious modification if $p$ or $q$ are $+\infty$.

**Hypothesis 2.2.** Let $\psi(t)$ be a positive monotone nondecreasing function defined in $[0, T]$ such that

$$\psi(t) \in L^1(0, T).$$

The symbol $\tilde{H}^{1,q}(\psi, Q(\tau_1, \tau_2))$ $(0 \leq \tau_1 < \tau_2 \leq T)$ stands for the completion of $C^1(Q(\tau_1, \tau_2))$ with respect to the norm

$$\|u\|_{1,q,(\tau_1, \tau_2)} = \left(\|u\|^2 + \sum_{i=1}^m \nu \psi \frac{\partial u}{\partial x_i}^2 dx \, dt\right)^{\frac{1}{2}},
\|u\|_{1,q} = \|u\|_{1,q,(0,T)}.$$
- $H^{1,0}(\nu|Q(T_1, T_2))$ is a Hilbert space with respect to the norm $||u||_{H^{1,0}(T_1, T_2)}$.
- $C^*(Q(T_1, T_2))$ (0 ≤ $t_1 < t_2$ ≤ $T$) denotes the following linear subspace of $C^0(Q(T_1, T_2)) \cap C^0(Q(T_1, T_2))$: $C^*(Q(T_1, T_2)) = \{ u \in C^0(Q(T_1, T_2)) \cap C^0(Q(T_1, T_2)) : u = 0$ on $\partial \Omega_i \times [t_1, t_2] \}$.
- $H^{1,0}(\nu|Q(T_1, T_2))$ (0 ≤ $t_1 < t_2$ ≤ $T$) is the closure of $C^*(Q(T_1, T_2))$ in $H^{1,0}(\nu|Q(T_1, T_2))$.

Finally, we will denote by $V^{1,0}(\nu|Q)$ the space of functions $u(x,t)$ belonging to $H^{1,0}(\nu|Q)$, continuous in $[0,T]$ with values in $L^2(\Omega)$.

**Definition 1.** Given a real number $h$, if $u \in H^{1,0}(\nu|Q(T_1, T_2))$ (0 ≤ $t_1 < t_2$ ≤ $T$), we will say that $u(x,t) \leq h (\geq h)$ on $\partial \Omega_i \times [t_1, t_2]$ if there exists a sequence $(u_n)$ of functions from $C^1(Q(T_1, T_2))$ such that

$$u_n(x,t) \leq h (\geq h) \text{ on } \partial \Omega_i \times [t_1, t_2]$$

and

$$\lim_{n \to \infty} ||u_n - u||_{H^{1,0}(T_1, T_2)} = 0.$$ 

If $k$ is such that $u(x,t) \leq k$ on $\partial \Omega_i \times [t_1, t_2]$, we will say that $u(x,t)$ is bounded from above on $\partial \Omega_i \times [t_1, t_2]$.

**Definition 2.** If $u(x,t), w(x,t)$ belong to $H^{1,0}(\nu|Q(T_1, T_2))$ (0 ≤ $t_1 < t_2$ ≤ $T$) and $w(x,t) \geq 0$ on $\partial \Omega_i \times [t_1, t_2]$ (i = 1, 2), let us denote

$$\sup u = \inf \{ h \in \mathbb{R} : u(x,t) - hw(x,t) \geq 0 \text{ on } \partial \Omega_i \times [t_1, t_2] \}.$$ 

We will consider the following generalized problem:

$$\begin{cases}
- \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \right) + d_i u + \left( \sum_{i=1}^{m} b_i \frac{\partial u}{\partial x_i} + c \right) + \frac{\partial u}{\partial T} = 0 & \text{in } Q \\
\frac{\partial u}{\partial T} + \alpha u + \sum_{i=1}^{m} d_i \cos nx_i \geq 0 & \text{on } \Gamma_1,
\end{cases}$$

where

$$\frac{\partial u}{\partial T} = \sum_{k,j=1}^{m} a_{kj} \cos nx_j \frac{\partial u}{\partial x_i}$$

and $\cos nx_j$ is the j-th directional cosine of $x$, normal to $\Gamma_1$ and external to $Q$.

---

1 For more details concerning hypotheses (2.1), (2.2) see also [5], [7], [8] and [9].
2 We suppose that $\inf B = +\infty$.
By an $L^{r}-$subsolution ($L^{r}-$supersolution) of problem (2.1) we mean any function $u \in V^{r,0}(\nu\psi, Q)$ satisfying the following conditions:

$$
\mathcal{L}(u, \varphi) = \int_{Q} \left( \sum_{i,j=1}^{m} a_{ij} \frac{\partial u}{\partial x_{ij}} + \frac{\partial u}{\partial t} + \sum_{i=1}^{m} b_{i} \frac{\partial u}{\partial x_{i}} + cu \varphi \right) dx \, dt + \int_{\Gamma_{1}} a u \varphi \, d\sigma \leq 0 \quad (\geq 0)
$$

for any $\varphi \in C^{r}(\Omega)$ such that $\varphi(x,t) \geq 0$ a.e. in $Q$, $\varphi(x,0) = \varphi(x,T) = 0$ for a.e. $x \in \Omega$.

Of particular interest are $L^{r}$-subsolutions ($L^{r}$-supersolutions) such that

$$
\int_{Q} \left( \sum_{i,j=1}^{m} a_{ij} \frac{\partial u}{\partial x_{ij}} + \frac{\partial u}{\partial t} + \sum_{i=1}^{m} b_{i} \frac{\partial u}{\partial x_{i}} + cu \varphi \right) dx \, dt + \int_{\Gamma_{1}} a u \varphi \, d\sigma \leq 0 \quad (\geq 0)
$$

for any $\varphi \in C^{r}(\Omega)$, $\varphi(x,t) \geq 0$ on $\Gamma_{1}$, $\varphi(x,0) = \varphi(x,T) = 0$ for a.e. $x \in \Omega$.

In fact, problem (2.3) is equivalent, at least “formally,” to the problem

$$
\begin{array}{c}
\left( \sum_{i=1}^{m} \frac{\partial u}{\partial x_{i}} + d_{i} u \right) + \left( \sum_{i=1}^{m} b_{i} \frac{\partial u}{\partial x_{i}} + cu \right) + \frac{\partial u}{\partial t} = 0 \quad \text{in} \quad Q \\
\frac{\partial u}{\partial \nu} + a u + \sum_{i=1}^{m} d_{i} u \cos n_{x_{i}} \leq 0 \quad (\geq 0) \quad \text{on} \quad \Gamma_{1},
\end{array}
$$

Let us consider the problem

$$
\begin{array}{c}
\left\{ \begin{array}{l}
\left( \sum_{i,j=1}^{m} a_{ij} \frac{\partial u}{\partial x_{ij}} + d_{i} u \right) + \left( \sum_{i=1}^{m} b_{i} \frac{\partial u}{\partial x_{i}} + cu \right) + \frac{\partial u}{\partial t} = f \quad \text{in} \quad Q \\
\int_{\Gamma_{1}} a u \varphi \, d\sigma = \int_{Q} f \varphi \, dx \, dt + \int_{\Gamma_{1}} g_{1} \varphi \, d\sigma \, dt
\end{array} \right.
\end{array}
$$

for any $\varphi \in C^{r}(\Omega)$, $\varphi(x,T) = 0$ in $\Omega$

$u(x,0) = g_{2}$

The problem (2.4) is formally equivalent to the problem

$$
\begin{array}{c}
\left\{ \begin{array}{l}
\left( \sum_{i=1}^{m} \frac{\partial u}{\partial x_{i}} + \sum_{i=1}^{m} d_{i} u \right) + \left( \sum_{i=1}^{m} b_{i} \frac{\partial u}{\partial x_{i}} + cu \right) + \frac{\partial u}{\partial t} = f \quad \text{in} \quad Q \\
\frac{\partial u}{\partial \nu} + a u + \sum_{i=1}^{m} d_{i} u \cos n_{x_{i}} = g_{1} \quad \text{on} \quad \Gamma_{1} \\
u = g_{2} \quad \text{on} \quad \Gamma_{2} \\
u(x,0) = 0 \quad \text{in} \quad \Omega.
\end{array} \right.
\end{array}
$$
3. Hypotheses on Coefficients

Let us denote by $A$ the set of pairs $(\alpha^*, \alpha)$ with $2 \leq \alpha^* \leq \alpha < +\infty$, such that there exists a positive constant $\beta$ for which

$$\|u\|_{\alpha^*, \alpha} \leq \beta (\|u\|_{2, \infty} + \|u\|_{1, \alpha})$$

for any $u \in L^{2, \infty}(Q) \cap \wedge^{1, 0}(\nu \psi, Q)$. The set $A$ obviously contains the pair $(2, +\infty)$.

Let us observe that, under the hypotheses on $\Omega$, we have

$$\|u\|_{\alpha^*, \alpha} \leq \gamma \|u\|_{1, \alpha}$$

for any $u \in \wedge^1(\nu, \Omega)$.

Consequently, we obtain:

$$\left( \int_0^T \psi(t) \|u\|_{\alpha^*, \alpha} \nu \psi \, dt \right)^{1/\alpha} \leq \gamma (\|u\|_{2, \infty} + \|u\|_{1, \alpha})$$

for any $u \in L^{2, \infty}(Q) \cap \wedge^{1, 0}(\nu \psi, Q)$.

The constant in (3.2) and (3.3) depends on $\Omega$.

**Hypothesis 3.1.** The functions $a_{ij}(x, t)$, $b_i(x, t)$, $c(x, t)$, $d_i(x, t)$ ($1 \leq i, j \leq m$) are defined and measurable in $Q$;

$$a_{ij} \in L^\infty(Q), \quad b_i \in L^{p^*}(Q), \quad c \in L^{q^*}(Q), \quad d_i \in L^{r^*}(Q),$$

where

$$\frac{1}{p^*} + \frac{1}{a_1^*} = \frac{1}{2}, \quad \frac{1}{p} + \frac{1}{a_1} = \frac{1}{2}, \quad \frac{1}{q^*} + \frac{2}{a_2^*} = 1,$$

$$\frac{1}{q} + \frac{2}{a_2} = 1, \quad \frac{1}{r^*} + \frac{1}{a_3^*} = \frac{1}{2}, \quad \frac{1}{r} + \frac{1}{a_3} = \frac{1}{2}$$

with $(\alpha_1^*, \alpha_1)$, $(\alpha_2^*, \alpha_2)$ and $(\alpha_3^*, \alpha_3)$ belonging to $A$.

Moreover, if $p = +\infty$ $[q = +\infty$, $r = +\infty]$ and $p^* < +\infty$ $[q^* < +\infty$, $r^* < +\infty]$, then there exists a function $\eta_1(\sigma)$ $[\eta_2(\sigma)$, $\eta_3(\sigma)]$ defined for $\sigma \geq 0$, non decreasing,

\[\text{of order } \frac{1}{(q+1)T} \leq \frac{1}{(p+1)T} \leq \frac{1}{(r+1)T}.\]
vanishing for $\sigma$ approaching zero and satisfying for almost all $t$ in the interval $[0,T]$ the inequalities
\[
\sum_{i=1}^{m} \left( \int_{\Omega} \left( \frac{|b(x,t)|}{\sqrt{\nu(x)}} \right)^r \, dx \right)^{\frac{1}{r}} \leq \eta_i(\sigma) \sqrt{\nu(t)},
\]
\[
\left( \int_{\Omega} \left( |c(x,t)| - c(x,t) \right)^r \, dx \right)^{\frac{1}{r}} \leq \eta_i(\sigma),
\]
\[
\sum_{i=1}^{m} \left( \int_{\Omega} \left( \frac{|d(x,t)|}{\sqrt{\nu(x)}} \right)^r \, dx \right)^{\frac{1}{r}} \leq \eta_i(\sigma) \sqrt{\nu(t)}
\]
for all measurable subsets $E$ of $\Omega$ such that $\text{meas} E \leq \sigma$.

The function $\alpha(x,t)$ is defined and measurable on $\Gamma_1$ and
\[
\frac{\alpha}{\sqrt{\nu}} \in L^\infty \left(0,T; L^{\frac{2m}{m-2}}(\partial\Omega_1) \right).
\]

**Hypothesis 3.2.** The functions $f(x,t), g(x,t)$ are defined and measurable respectively in $Q$ and in $\Gamma_1$, moreover
\[
f \in L^2(Q), \quad \frac{\partial f}{\sqrt{\nu}} \in L^2 \left(0,T; L^{\frac{2(m-1)}{m-2}}(\partial\Omega_1) \right).
\]

The function $g_2(x,t)$ is defined and measurable in $Q$ and
\[
g_2 \in H^{1,0}(\nu \psi, Q), \quad \frac{\partial g_2}{\sqrt{\psi}} \in L^2(Q), \quad g_2(x,t) \leq 0$ on $\Gamma_1$.

Finally, the functions $f^*(x,t), g^*(x,t)$ are defined and measurable respectively in $Q$ and in $\Gamma_1$, moreover
\[
f^* \in L^2(Q), \quad \frac{\partial f^*}{\sqrt{\psi}} \in L^2 \left(0,T; L^{\frac{2(m-1)}{m-2}}(\partial\Omega_1) \right).
\]

**Hypothesis 3.3.** The following inequality holds for a.e. $(x,t)$ in $Q$ and for all real numbers $\chi_1, \chi_2, \ldots, \chi_m$:
\[
\sum_{i=1}^{m} a_{ij}(x,t) \chi_i \chi_j \geq \nu(x) \psi(t) \sum_{i=1}^{m} \chi_i^2.
\]
4. PRELIMINARY LEMMAS

Lemma 4.1. Let us assume that hypotheses (2.1), (2.2), (3.1) hold and let \( u(x, t) \) be an \( L^r \)-subsolution of the problem (2.1) bounded from above on \( \partial \Omega_2 \times [0, T] \). Then if \( 0 \leq \tau_1 < \tau < T \) and \( k > \sup u \), we get

\[
\int_{Q(\tau_1, \tau)} \left( \sum_{j=1}^m a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial u}{\partial x_i} v + cw + \sum_{i=1}^m d_i u \frac{\partial v}{\partial x_i} \right) \, dx \, dt + \frac{1}{2} \int_\Omega v^2(x, \tau) \, dx + \int_{\tau_1}^\tau \int_{\partial \Omega} au \, dv \, dt \leq \frac{1}{2} \int_\Omega v^2(x, \tau) \, dx,
\]

where \( v = u - \min(u, k) \) in \( \Omega \); moreover, \( v \in H^{1,0}(\nu \psi, Q) \).

Proof. Let \( \bar{\tau}_1, \tau \) be such that \( 0 < \bar{\tau}_1 < \tau < T \); setting \( \tau_1 = \frac{\tau_1 + \tau}{2} \), \( \tau_2 = T - \tau_1 \), we denote by \( C_{0}^m(\Omega) \) the set of nonnegative functions from \( C^m(\Omega) \) equal to zero for \( t \geq \tau_1 \). Let \( \varphi \) be a function from \( C_{0}^m(\Omega) \). We extend \( u, \varphi \) and the coefficients of (2.1) to \( \Omega \times (-\infty, +\infty) \), assuming that these functions are equal to zero in those points where they are not defined.

We define in \( \Omega \times (-\infty, +\infty) \) and for any integer \( \nu \):

\[
\Phi_\nu(x, t) = \frac{\nu}{\tau_1} \int_{-\infty}^{t} \varphi(x, \lambda) \, d\lambda,
\]

\[
U_\nu(x, t) = \frac{\nu}{\tau_2} \int_{t}^{\infty} u(x, \lambda) \, d\lambda,
\]

\[
B_\nu(x, t) = \frac{\nu}{\tau_2} \int_{t}^{\infty} \sum_{i=1}^m b_i(x, \lambda) \frac{\partial u(x, \lambda)}{\partial x_i} \, d\lambda,
\]

\[
C_\nu(x, t) = \frac{\nu}{\tau_2} \int_{t}^{\infty} c(x, \lambda) u(x, \lambda) \, d\lambda,
\]

\[
A_\nu(x, t) = \frac{\nu}{\tau_2} \int_{t}^{\infty} \sum_{i=1}^m a_{ij}(x, \lambda) \frac{\partial u(x, \lambda)}{\partial x_j} \, d\lambda,
\]

\[
D_\nu(x, t) = \frac{\nu}{\tau_2} \int_{t}^{\infty} d_i(x, \lambda) u(x, \lambda) \, d\lambda,
\]

\[
\alpha_\nu(x, t) = \frac{\nu}{\tau_2} \int_{t}^{\infty} \alpha(x, \lambda) u(x, \lambda) \, d\lambda.
\]

\[\text{Let us observe that, for a.e. in } [0, T], v(x, t) \in H^s(\nu, \Omega).\]
From (2.2), in virtue of \( \varphi = \Phi_\alpha(x,t) \), via an exchange of the order of integration with respect to \( t \) and \( \lambda \), we get

\[
(4.1) \quad \int_{Q} \left( \sum_{i=1}^{m} A_{1i} \frac{\partial \varphi}{\partial x_i} + B_{1i} \varphi + C_{1i} \varphi + \sum_{i=1}^{m} D_{1i} \frac{\partial \varphi}{\partial x_i} + \frac{\partial U_2}{\partial t} \right) \, dz \, dt + \int_{\Omega} \alpha \varphi \, d\lambda \leq 0
\]

for all \( \varphi \) belonging to the functional class \( C^\infty_0(Q) \).

Let \( \{u_n\} \) be a sequence of functions of \( C^1(Q) \) such that \( u_n < \sup^\star u \) on \( \Gamma_2 \) and satisfying (*). For all pairs of positive integers \( \nu \) and \( n \), we define

\[
U_{\nu,n}(x,t) = \frac{\nu}{\tau_2} \int_{t_0}^{T_2} u_n(x,\lambda) \, d\lambda,
\]

the function \( U_{\nu,n}(x,t) \) belongs to \( C^1(Q(0,T)) \).

Let us now introduce the function

\[
V_{\nu,n}(x,t) = \begin{cases} 
U_{\nu,n} - \min(U_{\nu,n}, k) & \text{in } Q(\bar{\tau}, \tau), \\
0 & \text{in } Q \setminus Q(\bar{\tau}, \tau).
\end{cases}
\]

Let \( \{\Phi_n\}_{n\in\mathbb{N}} \) be a sequence of nonnegative equibounded functions from \( C^\infty_0(Q) \) converging to \( V_{\nu,n} \) in \( \tilde{H}^1_0(\nu, Q(\bar{\tau}, \tau)) \); moreover, let also the functions in the sequence \( \{\Phi_n\}_{n\in\mathbb{N}} \) be equibounded.

From (4.1), in virtue of \( \varphi = \Phi_\alpha(x,t) \), as \( \mu \) diverges to \( +\infty \), we obtain the following relation:

\[
(4.2) \quad \int_{Q(\bar{\tau}, \tau)} \left( \sum_{i=1}^{m} A_{1i} \frac{\partial V_{\nu,n}}{\partial x_i} + B_{1i} V_{\nu,n} + C_{1i} V_{\nu,n} + \sum_{i=1}^{m} D_{1i} \frac{\partial V_{\nu,n}}{\partial x_i} + \frac{\partial U_2}{\partial t} V_{\nu,n} \right) \, dz \, dt + \int_{\Omega} \alpha V_{\nu,n} \, d\lambda \leq 0.
\]

Setting now in \( Q \):

\[
V_\alpha = \begin{cases} 
U_\alpha - \min(U_\alpha, k) & \text{in } Q(\bar{\tau}, \tau), \\
0 & \text{in } Q \setminus Q(\bar{\tau}, \tau);
\end{cases}
\]

\(^a\)See [10], p. 141.

\(^b\)Since \( k > \sup^\star u \) there exists a neighbourhood of \( \partial \Omega_2 \) such that \( V_{\nu,n}(x,t) = 0 \) for any \( t \in [0,T] \) (see Lemma 4.2 of [9]).

\(^c\)See remark 4.1 of [8].
the sequence \{V_{e,n}\} converges to \(V_e\) in \(H^{1,0}(\nu, Q(\hat{t}, \tau)) \cap L^{2,\infty}(Q(\hat{t}, \tau))\) and satisfies the relation
\[
\lim_{n \to \infty} \left\| (V_{e,n} - V_e) \right\|_{H^{1,0}(\nu, Q(\hat{t}, \tau)) \cap L^{2,\infty}(Q(\hat{t}, \tau))} = 0.
\]

On the other hand, the functions of the sequence \{V_{e,n}\} belong to \(H^{1,0}(\nu, Q(\hat{t}, \tau))\) and so also \(V_e\) belongs to this space.

From (3.1)-(3.6) we deduce, as \(n\) goes to \(+\infty\), the following inequality:\(^9\)
\[
\int_Q \left\{ \sum_{\alpha=1}^m A_{\alpha} \frac{\partial V_e}{\partial x_\alpha} + B_{\alpha} V_e + C_{\alpha} V_e + \sum_{\alpha=1}^m D_{\alpha} \frac{\partial V_e}{\partial t} + \frac{\partial U_e}{\partial t} V_{e,n} \right\} \, dx \, dt
+ \frac{1}{2} \int_{\Omega_{\alpha}(t_1,k)} |V_{e}(x,\tau) - k|^2 \, dx
- \frac{1}{2} \int_{\Omega_{\alpha}(t_0,k)} |V_{e}(x,0) - k|^2 \, dx + \int_{T_1} \int_{\partial\Omega_{\alpha}} \alpha_e V_e \, d\sigma \, dt \leq 0.
\]

Let us remark that the sequence \{V_{e,n}\} converges in \(H^{1,0}(\nu, Q) \cap L^{2,\infty}(Q)\) to the function equal to \(v\) in \(Q(t_1, \tau)\) and equal to zero in \(Q \setminus Q(t_1, \tau)\).

From (4.3), the conclusion follows via another passage to the limit. For example, we prove that
\[
\lim_{n \to \infty} \int_{T_1} \int_{\partial\Omega_{\alpha}} \alpha_e V_e \, d\sigma \, dt = \int_{T_1} \int_{\partial\Omega_{\alpha}} \alpha v \, d\sigma \, dt.
\]
We get
\[
\left| \int_{T_1} \int_{\partial\Omega_{\alpha}} \alpha_e V_e \, d\sigma \, dt \right|
\leq \gamma \frac{\psi(t)}{\psi(t_1)} \frac{\alpha}{\alpha(t)} \left(\|u\|_{2,\infty} + \|u\|_{1,0}\right)
\times \left(\|V_e - v\|_{2,\infty}(\tau, \tau) + \|V_e - v\|_{1,0}(\tau, \tau) + \left(\frac{1}{\psi(t_1)}\right)^{\frac{1}{2}}\right).
\]
for any \(\alpha \in N.\)\(^{10}\)

\(^9\) For a fixed \(t \in [0, T],\) we set \(\Omega_{\alpha}(t, k) = \{\nu \in \Omega: U_{\alpha}(x,t) > k\}.
\]
\(^{10}\) Let us remark that, by the properties of Steklov averages, it follows that \(\alpha_e\) converges to \(\alpha v\) in \(L^2(\tau_1, \tau; L^{\infty}(\partial\Omega_{\alpha}))\).
Next, it is easy to verify that the restriction of the function $v$ to $Q(\bar{\tau}, \tau)$ belongs to $\tilde{H}^{1,0}(\psi, Q)$ for any $0 < \bar{\tau} < \tau < T$ and, therefore, since $v$ by definition belongs to $\tilde{H}^{1,0}(\psi, Q(\bar{\tau}, \tau))$, it belongs to $\tilde{H}^{1,0}(\psi, Q)$, too.

Finally, if $\tau = 0$, as $\tau > 0$ is assumed, it suffices to consider $\tau = \frac{\tau}{n+1}$ for $n \in \mathbb{N}$ recalling that the function $v(x, t)$ is continuous in $[0, T]$ with values in $L^2(\Omega)$.

**Lemma 4.2.** Let us assume the hypotheses (2.1), (2.2), (3.1), (3.3) hold and let $u(x, t)$ be an $L^r_T$-subsolution of the problem (2.1) satisfying the conditions

$$\text{ess sup } u(x, 0) \leq 0, \quad \text{sup } u \leq 0.$$ 

Then, we have:

$$\text{ess sup } u(x, t) \leq 0.$$

**Proof.** For any integer $n$, we consider the functions

$$v = u - \min(u, 0), \quad v_n = u - \min(u, \frac{1}{n}).$$

From Lemma 4.1 we deduce that $v_n$ belongs to $\tilde{H}^{1,0}(\psi, Q)$ and that, provided $\tau \in [0, T]$, we have

$$\int_{Q(\tau)} \left( \sum_{i,j=1}^m a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + \sum_{i=1}^m b_i \frac{\partial u}{\partial x_i} v_n + c u_n + \sum_{i=1}^m d_{ii} \frac{\partial u}{\partial x_i} \right) \, dx \, dt$$

$$+ \frac{1}{2} \int_0^\tau \int_\Omega v_n^2 \, dx + \int_0^\tau \int_{\partial \Omega} a u_n \, d\sigma \, dt \leq 0.$$

On the other hand, we obtain

$$\lim_{n \to \infty} v_n(x, t) = v(x, t), \quad |v_n(x, t)| \leq |u(x, t)| \text{ in } Q$$

and

$$\lim_{n \to \infty} \frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial x_i}, \quad \left| \frac{\partial u}{\partial x_i} \right| \leq \left| \frac{\partial v}{\partial x_i} \right| \text{ a.e. in } Q.$$

Furthermore, also $v$ belongs to $\tilde{H}^{1,0}(\psi, Q)$ and so, as $n$ goes to $+\infty$ in (4.4), we get

$$\int_{Q(\tau)} \left( \sum_{i,j=1}^m a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^m b_i \frac{\partial v}{\partial x_i} v + c v^2 + \sum_{i=1}^m d_{ii} \frac{\partial v}{\partial x_i} \right) \, dx \, dt$$

$$+ \frac{1}{2} \int_0^\tau \int_\Omega v^2 \, dx + \int_0^\tau \int_{\partial \Omega} a v \, d\sigma \, dt \leq 0.$$

From (4.5) we deduce that $|v|_{2, \infty} = 0$ and the conclusion easily follows.
The proof is similar to that given in Lemma 4.1 of [13]; let us remark that since \( v \in \dot{H}^{1,0}(\psi \omega, Q) \cap L^{2,n} - (Q) \) we can apply the relations (3.1) and (3.3) instead of the hypothesis A) of [10].

\[ \text{Proof.} \]

5. A GENERALIZED MAXIMUM PRINCIPLE

We will prove

**Theorem 5.1.** Let us assume the hypotheses (2.1), (2.2), (3.1), (3.3) hold and let \( w(x, t) \) be an \( L_{\Gamma_1} \)-supersolution of the problem (2.1) satisfying the conditions

\[
\begin{align*}
& w(x, t) > 0 \quad \text{a.e. in } Q, \\
& w(x, 0) > 0 \quad \text{a.e. in } \Omega, \\
& w(x, t) \geq 0 \quad \text{on } \Gamma_2.
\end{align*}
\]

Then

\[
(5.1) \quad \esssup_{Q} \frac{w(x, t)}{w(x, t)} \leq \max \left( 0, \esssup_{\Omega} \frac{u(x, 0)}{w(x, 0)} \right.
\]

for any \( L_{\Gamma_1} \)-subsolution \( u(x, t) \) of the problem (2.1).

**Proof.** The conclusion is obvious if the second term of (5.1) is equal to \(+\infty\). Let us suppose, now, that this term is finite and let us denote by \( h \) some real number greater than its value. Consequently, the function \( u(x, t) - hw(x, t) \) is an \( L_{\Gamma_1} \)-subsolution of the problem (2.1) such that \( \esssup_{Q} \left[ w(x, 0) - hw(x, 0) \right] \leq 0, \) \( \sup \left( u - hw \right) \leq 0. \) From Lemma 4.2 we can see that \( u(x, t) - hw(x, t) \leq 0 \) a.e. in \( Q. \) So, we obtain \( \esssup_{Q} \frac{w(x, t)}{w(x, t)} \leq h \) and the conclusion easily follows. \( \Box \)

6. A COMPARISON THEOREM

Let us define the following closed convex sets:

\[ K^* = \{ z \in \dot{H}^{1,0}(\psi \omega, Q), \ z \in C([0, T]; L^2(\Omega)), \ z(x, 0) = 0, \ z \geq g_2 \ \text{on } \Gamma_2 \}, \]

\[ \Phi^* = \{ \varphi \in \dot{H}^{1,0}(\psi \omega, Q), \ \frac{\partial \varphi}{\partial t} \in L^2(Q), \ \varphi(x, T) = 0, \ \varphi \geq g_2 \ \text{on } \Gamma_2 \}, \]

and let us suppose that there exists a solution \( z \in K^* \) of the variational inequality

\[
\begin{align*}
& \int_Q \left( \sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial (\varphi - z)}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i} (\varphi - z) + cz (\varphi - z) \\
& + \sum_{i=1}^m d_i z \frac{\partial (\varphi - z)}{\partial x_i} - \frac{\partial \varphi}{\partial t} \right) \, dx \, dt + \int_{\Gamma_1} a z (\varphi - z) \, d\sigma \, dt \\
& \geq \int_Q f^*(\varphi - z) \, dx \, dt + \int_{\Gamma_1} g_1^*(\varphi - z) \, d\sigma \, dt
\end{align*}
\]

49
for any $\varphi \in \Phi^*$. The problem (6.1) is formally equivalent to the problem
\[
\begin{align*}
-\sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{m} a_{ij} \frac{\partial z}{\partial x_j} + d_i z \right) + \left( \sum_{i=1}^{m} b_i \frac{\partial z}{\partial t} + c_i \right) + \frac{\partial z}{\partial t} &= f^* & \text{in Q} \\
\frac{\partial z}{\partial \nu} + \alpha z + \sum_{i=1}^{m} d_i z \cos nx_i &= g_i^* & \text{on } \Gamma_1 \\
z \geq g_2, \quad \frac{\partial z}{\partial \nu} + \sum_{i=1}^{m} d_i z \cos nx_i \geq 0, \quad (z - g_2) \left( \frac{\partial z}{\partial \nu} + \sum_{i=1}^{m} d_i z \cos nx_i \right) &= 0 & \text{on } \Gamma_2 \\
z(x,0) &= 0 & \text{in } \Omega.
\end{align*}
\]

We will prove

**Theorem 6.1.** Let us assume the hypotheses (2.1), (2.2), (3.1), (3.2), (3.3) hold and let $w(x,t)$ be an $L^\infty$ supersolution of the problem (2.1) satisfying the conditions
\[
w(x,t) > 0 \text{ a.e. in Q}, \\
w(x,0) > 0 \text{ a.e. in } \Omega, \quad w(x,t) \geq 0 \text{ on } \Gamma_2.
\]

Let $z(x,t)$ be a solution of the problem (6.1) with $f^* \geq f$ in Q, $g_i^* \geq g_i$ on $\Gamma_i$. Then, we have the inequality
\[
u(x,t) \leq z(x,t) \text{ a.e. in Q}
\]
for any solution $u(x,t)$ of the problem (2.4).

**Proof.** Let us extend $z(x,t)$ to $\mathbb{R}^{m+1}$ assuming that it vanishes at points not belonging to $Q$; for a fixed $\tau \in [0,T]$ and for any pairs of integers $\rho, n$ we introduce the functions
\[
\theta_n(t) = \begin{cases} 
0 & \text{if } t < \tau - \frac{2}{n}, \\
n(t + \frac{2}{n} - \tau) & \text{if } \tau - \frac{2}{n} \leq t \leq \tau - \frac{1}{n}, \\
1 & \text{if } t > \tau - \frac{1}{n}.
\end{cases}
\]

\[
z_{n,e}(x,t) = \theta_n(t) \int_{-\frac{1}{2}}^{\frac{1}{2}} z(x,y) \theta_n(y) \, dy.
\]
We have
\[ \frac{\partial z_{n,t}}{\partial t} = \phi'(t) \int_{-\frac{t}{2}}^{t} z(x,y) \theta_n(y) \, dy \]
\[ + \phi'(t) \left( z\left(x,t + \frac{1}{2t}\right) \theta_n\left(t + \frac{1}{2t}\right) - z\left(x,t - \frac{1}{2t}\right) \theta_n\left(t - \frac{1}{2t}\right) \right). \]

Choosing \( \psi = z_{n,a} + \beta, 0 \leq \beta \in C^\infty(\bar{Q}), \beta(x,t) = 0 \) \text{a.e. in} \ Q, \text{we get from (6.1)}
\begin{equation}
(6.2)
\end{equation}
\[ \int_Q \left( \sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_i} \frac{\partial (\varepsilon_{n,a} + \beta - z)}{\partial x_j} \right) \, dx \, dt + \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i} (\varepsilon_{n,a} + \beta - z) \, dx \, dt \]
\[ - \int_Q z \frac{\partial z_{n,a}}{\partial t} \, dx \, dt - \int_Q \frac{\partial \varepsilon}{\partial t} \, dx \, dt + \int_{\Gamma_1} \alpha z (\varepsilon_{n,a} + \beta - z) \, d\sigma \, dt \]
\[ \geq \int_Q f (\varepsilon_{n,a} + \beta - z) \, dx \, dt + \int_{\Gamma_1} g (\varepsilon_{n,a} + \beta - z) \, d\sigma \, dt, \]
now, taking into account the relation
\[ \int_Q z(x,t) \phi(t) \theta_n(t) \theta_n(t) \left( t + \frac{1}{2t} \right) z\left(x,t + \frac{1}{2t} \right) \, dx \, dt \]
\[ = \int_Q \phi(t) \int_{-\infty}^{+\infty} z(x,t) \phi(t) \theta_n(t) \theta_n(t) \left( t + \frac{1}{2t} \right) z\left(x,t + \frac{1}{2t} \right) \, dx \, dt \]
\[ = \int_Q \phi(t) \int_{-\infty}^{+\infty} z(x,t) \phi(t) \theta_n(t) \theta_n(t) \left( t - \frac{1}{2t} \right) z\left(x,t \right) \, dx \, dt \]
\[ = \int_Q \phi(t) \theta_n(t) \theta_n(t) \left( t - \frac{1}{2t} \right) z\left(x,t \right) \, dx \, dt, \]
we obtain from (6.2)
\[ \int_Q \left( \sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_i} \frac{\partial (\varepsilon_{n,a} + \beta - z)}{\partial x_j} \right) \, dx \, dt + \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i} (\varepsilon_{n,a} + \beta - z) \, dx \, dt \]
\[ + \int_Q \left( z \phi'(t) \int_{-\frac{t}{2}}^{\frac{t}{2}} z(x,y) \theta_n(y) \, dy \right) \, dx \, dt - \int_Q z \frac{\partial \varepsilon}{\partial t} \, dx \, dt \]
\[ + \int_{\Gamma_1} \alpha z (\varepsilon_{n,a} + \beta - z) \, d\sigma \, dt \]
\[ \geq \int_Q f (\varepsilon_{n,a} + \beta - z) \, dx \, dt + \int_{\Gamma_1} g (\varepsilon_{n,a} + \beta - z) \, d\sigma \, dt, \]
and therefore, letting $Q$ tend to $+\infty$, we find that

\begin{align}
(6.3) \quad \int_Q \left( \sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial (\theta_1^2(t)z + \beta - z)}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i} (\theta_2^2(t)z + \beta - z) \\
+ c z (\theta_3^2(t)z + \beta - z) + \sum_{i=1}^m d_i z \frac{\partial (\theta_4^2(t)z + \beta - z)}{\partial x_i} \right) \, dx \, dt \\
- \int_Q z^2 \theta_5(t) \theta_6(t) \, dx \, dt - \int_Q \frac{\partial \gamma}{\partial t} \, dx \, dt + \int_{\gamma_1} \alpha z (\theta_5^2(t)z + \beta - z) \, d\sigma \, dt \\
\geq \int_Q f^* (\theta_5^2(t)z + \beta - z) \, dx \, dt + \int_{\gamma_1} g_1^* (\theta_5^2(t)z + \beta - z) \, d\sigma \, dt.
\end{align}

Let us observe that $\theta_5(t) \theta_6(t) \geq 0$ a.e. in $[0, T]$ and $\theta_5(t) \theta_6(t) > \frac{3}{2}$ if $\tau - \frac{2}{\mu} < t < \tau - \frac{1}{2}$. Then, from (6.3) we have

\begin{align}
&\int_Q \left( \sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial (\theta_1^2(t)z + \beta - z)}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i} (\theta_2^2(t)z + \beta - z) \\
+ c z (\theta_3^2(t)z + \beta - z) + \sum_{i=1}^m d_i z \frac{\partial (\theta_4^2(t)z + \beta - z)}{\partial x_i} \right) \, dx \, dt \\
- \int_Q z^2 \theta_5(t) \theta_6(t) \, dx \, dt + \int_{\gamma_1} \alpha z (\theta_5^2(t)z + \beta - z) \, d\sigma \, dt \\
\geq \int_Q z^2 \theta_5(t) \theta_6(t) \, dx \, dt + \int_Q f^* (\theta_5^2(t)z + \beta - z) \, dx \, dt \\
+ \int_{\gamma_1} g_1^* (\theta_5^2(t)z + \beta - z) \, d\sigma \, dt \\
\geq \frac{n}{2} \int_{\tau - \frac{2}{\mu}}^{\tau - \frac{1}{2}} dt \int_Q z^2 (x, t) \, dx + \int_{\gamma_1} f^* (\theta_5^2(t)z + \beta - z) \, dx \, dt \\
+ \int_{\gamma_1} g_1^* (\theta_5^2(t)z + \beta - z) \, d\sigma \, dt.
\end{align}

Finally, as $n \to \infty$ and $\tau \to 0$, we get

\begin{align}
(6.4) \quad \int_Q \left( \sum_{i,j=1}^m a_{ij} \frac{\partial z}{\partial x_j} \frac{\partial \beta}{\partial x_i} + \sum_{i=1}^m b_i \frac{\partial z}{\partial x_i} \beta + c \beta + \sum_{i=1}^m d_i z \frac{\partial \beta}{\partial x_i} \frac{\partial \beta}{\partial t} \right) \, dx \, dt \\
+ \int_{\gamma_1} \alpha z \beta \, d\sigma \, dt \geq \int_Q f^* \beta \, dx \, dt + \int_{\gamma_1} g_1^* \beta \, d\sigma \, dt
\end{align}

for any $0 \leq \beta \in C^* (Q)$, $\beta (x, T) = 0$ a.e. in $Q$. 

52
By virtue of (6.4) and (2.4) we conclude that
\[ a(u-z, \phi) \leq \int_Q (f - f^*) \beta \, dx \, dt + \int_{\Omega} (g_1 - g_1^*) \beta \, ds \, dt \leq 0 \]
for any \( 0 \leq \beta \in C^\infty_c(Q) \), \( \beta(x,T) = 0 \) a.e. in \( Q \).

Applying the above maximum principle to the \( L^1 \)-subsolution \((u - z)\) and to the \( L^1 \)-supersolution \( w \), we obtain
\[ \text{ess sup}_{Q} \frac{u(x,t) - z(x,t)}{w(x,t)} \leq \text{max} \left( 0, \sup_{w} \frac{u - z}{w} \right) = 0. \]

This completes the proof. \( \square \)

References


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