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LOCALLY REGULAR GRAPHS

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Abstract. A graph \( G \) is called locally \( s \)-regular if the neighbourhood of each vertex of \( G \) induces a subgraph of \( G \) which is regular of degree \( s \). We study graphs which are locally \( s \)-regular and simultaneously regular of degree \( r \).

Keywords: regular graph, locally regular graph

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At the Czechoslovak Symposium on Graph Theory in Smolenice in 1963 [1] A. A. Zykov suggested the problem to characterize graphs \( H \) with the property that there exists a graph \( G \) in which the neighbourhood of any vertex induces a subgraph isomorphic to \( H \). This problem inspired many mathematical works and led also to a certain generalization, namely the study of local properties of graphs. A graph \( G \) is said to have locally a property \( P \), if the neighbourhood of each vertex of \( G \) induces a subgraph having the property \( P \). For locally connected graphs let us mention e.g. [2] and [4], for locally linear graphs e.g. [3]. A survey paper on local properties of graphs was written by J. Sedláček [5].

Here we will study locally \( s \)-regular graphs. A graph \( G \) is called locally \( s \)-regular, where \( s \) is a non-negative integer, if the neighbourhoods of all vertices of \( G \) induce subgraphs which are regular of degree \( s \), shortly \( s \)-regular. We consider finite undirected graphs without loops and multiple edges. The vertex set of a graph \( G \) is denoted by \( V(G) \), the complement of \( G \) by \( \overline{G} \). If \( A \subseteq V(G) \), then \( G(A) \) is the subgraph of \( G \) induced by \( A \). The symbol \( G_1 + G_2 \) denotes the disjoint union of two graphs \( G_1, G_2 \); the symbol \( G_1 \circ G_2 \) denotes the Zykov sum of \( G_1 \) and \( G_2 \), i.e. the graph obtained from \( G_1 + G_2 \) by joining each vertex of \( G_1 \) with each vertex of \( G_2 \) by an edge. By \( G_1 \times G_2 \) the Cartesian product of \( G_1 \) and \( G_2 \) is denoted; its vertex set is \( V(G_1) \times V(G_2) \) and two vertices \((u_1, u_2), (v_1, v_2)\) are adjacent in it if and only if either \( u_1 = v_1 \) and \( u_2, v_2 \) are adjacent in \( G_2 \), or \( u_1, v_1 \) are adjacent in \( G_1 \) and...
The symbol $N_G(V)$ denotes the (open) neighbourhood of a vertex $v$ in a graph $G$, i.e. the set of all vertices which are adjacent to $v$ in $G$. By $C_n$, we denote the circuit of length $n$.

By $\text{Locreg}(r,s)$, where $r$ is a positive integer and $s$ a non-negative integer, we denote the class of graphs which are simultaneously $r$-regular and locally $s$-regular.

**Proposition 1.** If $\text{Locreg}(r,s) \neq \emptyset$, then $r \geq s + 1$ and at least one of the numbers $r,s$ is even.

This assertion is evident, because the conditions mentioned are the well-known necessary conditions for the existence of an $s$-regular graph with $r$ vertices.

**Proposition 2.** Let $s,k$ be positive integers, let $k$ be a divisor of $s$. Then $\text{Locreg}(s+k,s) \neq \emptyset$.

**Proof.** Let $G$ be the complement of the disjoint union of $s+2$ copies of the complete graph $K_k$ with $k$ vertices. Then $G \in \text{Locreg}(s+k,s)$. \hfill $\square$

**Corollary 1.** $\text{Locreg}(s+1,s) \neq \emptyset$ for each integer $s \geq 0$.

**Corollary 2.** $\text{Locreg}(s+2,s) \neq \emptyset$ for each even integer $s \geq 0$.

**Proposition 3.** If $\text{Locreg}(r_1,s) \neq \emptyset$ and $\text{Locreg}(r_2,s) \neq \emptyset$, then also $\text{Locreg}(r_1 + r_2,s) \neq \emptyset$.

**Proof.** If $G_1 \in \text{Locreg}(r_1,s)$ and $G_2 \in \text{Locreg}(r_2,s)$, then the Cartesian product $G_1 \times G_2 \in \text{Locreg}(r_1 + r_2,s)$. \hfill $\square$

Now we state two lemmas.

**Lemma 1.** Let $p,q$ be positive integers such that $q \geq p^2 - 1$. Then there exist non-negative integers $a,b$ such that

$$q = ap + b(p + 1).$$

**Proof.** Let $a$ be an integer such that $0 \leq a \leq p - 1$ and $b \equiv q(\text{mod } p)$. Let $a = (q-b)/p - b$. We have $ap + b(p + 1) = q$. The number $a$ is an integer, because $q \equiv b(\text{mod } p)$. Further we have $q \geq p^2 - 1 = (p + 1)(p - 1) \geq b(p + 1)$ and thus

$$a = (q-b)/p - b \geq (b(p + 1) - b)/p - b = 0.$$ 

This proves the assertion. \hfill $\square$

**Lemma 2.** Let $p,q$ be positive integers such that $q \geq (2p-1)(p-1)$. Then there exist non-negative integers $a,b$ such that

$$q = ap + b(2p-1).$$
Proof. Let b be an integer such that 0 < b ≤ p - 1 and b + q ≡ 0 (mod p). Let a = (q + b)/p - 2b. The proof that a is a non-negative integer is analogous to the proof of Lemma 1. □

Now we prove some theorems.

**Theorem 1.** Let r, s be positive integers such that r ≥ s + 2. Then Locreg(r, s) ≠ ∅.

Proof. According to Lemma 1 there exist non-negative integers a, b such that r = a(s + 1) + b(s + 2). Then the assertion follows from the Corollaries 1 and 2 and from Proposition 3. □

**Theorem 2.** Let r, s be positive integers such that r is even, s is odd and r ≥ s + 1. Then Locreg(r, s) ≠ ∅.

Proof. Put p = 1/2(s + 1); then 1/2r ≥ (2p - 1)(p - 1) and, as r is even, according to Lemma 2 there exist non-negative integers a, b such that 1/2r = ap + b(2p - 1) = 1/2a(s + 1) + bs and thus r = a(s + 1) + 2bs. According to Proposition 2 we have Locreg(s + 1, s) ≠ ∅ and Locreg(2s, s) ≠ ∅ and thus, by Proposition 3, also Locreg(r, s) ≠ ∅. □

Now we turn our attention to small values of s.

**Proposition 4.** Let 0 ≤ s ≤ 2 and r ≥ s + 1 and in the case of s = 1 let r be even. Then Locreg(r, s) ≠ ∅.

Proof. A graph from Locreg(r, 0) is an arbitrary r-regular graph without triangles, e.g. the complete bipartite graph $K_{r,r}$. A graph from Locreg(2, 1) is $C_3$ and Locreg(n, 1) ≠ ∅ follows from Theorem 1. Examples of graphs from Locreg(3, 2), Locreg(4, 2) and Locreg(5, 2) are successively the graphs of regular polyhedra tetrahedron, octahedron, icosahedron. Every s ≥ 6 is a sum of numbers from {3, 4, 5} and thus Proposition 3 implies Locreg(r, s) for every r ≥ 6. □

From these results it may seem that Locreg(r, s) ≠ ∅ for any r, s which satisfy the condition of Proposition 1. We will show an example for which this is not true.

**Theorem 3.** The class Locreg(7, 4) = ∅.

Proof. Suppose the contrary and let $G \in$ Locreg(7, 4). Let u be a vertex of G. The graph $G(N_G(u))$ is a 4-regular graph with seven vertices. Its complement is a 2-regular graph and therefore it is isomorphic either to $C_7 + C_4$, or to $C_7$. Hence $G(N_G(u)) \cong C_7 \oplus C_4$ or $G(N_G(u)) \cong C_7$. Suppose the first case occurs. Denote the vertices of $G(N_G(u))$ by $v_1, v_2, v_3, w_1, w_2, w_3, w_4$ so that
Consider the graph $G(NG(VI))$. It contains the graph $\langle w_1, w_2, w_3 \rangle \cong C_3$ as an induced subgraph and therefore it cannot be isomorphic to $C_7$. We have $G(NG(VI)) \cong C_5 \cup C_3$ and there exist vertices $x_1, x_2, x_3$ outside $NG(u)$ which are pairwise non-adjacent and each of them is adjacent to $w_1, w_2, w_3$. One of them is $w_4$. But now $G(NG(w_4))$ contains two disjoint independent triples $\{v_1, v_2, v_3\}$ and $\{x_1, x_2, x_3\}$ and hence it is isomorphic neither to $C_3 \cup C_4$ nor to $C_7$, which is a contradiction. As $w_4$ was chosen arbitrarily, we have proved that the neighbourhood of any vertex of $G$ cannot induce $C_3 \cup C_4$ and thus it must induce $C_7$.

Thus let $G(NG(u)) \cong C_7$. The vertices of $G(NG(u))$ will be denoted by $v_1, v_2, v_3, v_4, v_5, v_6, v_7$ in such a way that $v_iv_{i+1}$ for $i = 1, \ldots, 7$ are edges of the complement of $G(NG(u))$; here and everywhere in the sequel the subscripts are taken modulo 7. Consider the graph $G(NG(v_i))$ for an arbitrary $i \in \{1, \ldots, 7\}$. It contains the vertices $u, v_1, v_2, v_3, v_4, v_5, v_6$ and does not contain $v_7$.

As $G(NG(v_i)) \cong C_7$, it contains vertices $w_i, x_i$ outside $NG(u)$ such that $w_i$ is adjacent to $v_i, v_{i+1}, v_{i+2}, v_{i+4}, v_{i+6}$ and non-adjacent to $x_i$. If $i = 1, \ldots, 7$, consider the vertices $w_i$ for $i = 1, \ldots, 7$. Suppose that $w_i = w_j$ for some $i$ and $j$. This vertex is non-adjacent to $v_{i+2}$ and adjacent to $v_i, v_{i+3}, v_{i+4}, v_{i+5}$ and thus $j + 2 \notin \{i - 1, i + 3, i + 4, i + 5\}$, which implies $j \notin \{i + 1, i + 2, i + 3, i + 5\}$. Further, this vertex is non-adjacent to $v_{i+2}$ and adjacent to $v_{i+1}, v_{i+3}, v_{i+4}, v_{i+5}$; we have $i \notin \{j - 1, j + 2, j + 3, j + 5\}$, which implies $j \notin \{i + 2, i + 4, i + 5, i + 6\}$. Therefore $w_i = w_j$ implies $i = j$ and the vertices $w_1, \ldots, w_7$ are pairwise distinct. As $w_i$ is adjacent to $v_i, v_{i+3}, v_{i+4}, v_{i+5}$ for $i = 1, \ldots, 7$, the vertex $v_i$ is adjacent to $w_i, w_{i+2}, w_{i+3}, w_{i+4}$ and thus its degree in $G$ is at least 9, which is a contradiction. This proves the assertion. 

References


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