GRAPH AUTOMORPHISMS OF SEMIMODULAR LATTICES

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Abstract. This paper deals with the relations between graph automorphisms and direct factors of a semimodular lattice of locally finite length.

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1. INTRODUCTION

Each lattice dealt with in the present paper is assumed to be of locally finite length (i.e., all its bounded chains are finite).

For a lattice $L$ let $G(L)$ be the corresponding unoriented graph.

An automorphism of the graph $G(L)$ is called also a graph automorphism of the lattice $L$. The graph isomorphism of lattices is defined analogously.

We denote by $C$ the class of all finite lattices $L$ such that each automorphism of $G(L)$ turns out to be a lattice automorphism.

In connection with Birkhoff’s problem 6 from [1], the following result has been proved in [5] (by using the results of [2] and [6]):

(*) Let $L$ be a finite modular lattice. Then the following conditions are equivalent:

(i) $L$ belongs to $C$.

(ii) No direct factor of $L$ having more than one element is self-dual.

The natural question arises whether in (*) the assumption of modularity can be replaced by the assumption that $L$ is semimodular.

In Section 3 we show by an example that the answer is “No”.

We define the notions of an interval of type (C) in $L$ and of a graph automorphism of type (C) (cf. Definitions 2.1 and 2.2).
Let $A$ be a direct factor of a lattice $L$ and $\emptyset \neq X \subseteq L$. We say that $A$ is orthogonal to $X$ if for any $x_1, x_2 \in X$, the components of $x_1$ and $x_2$ in the direct factor $A$ are equal.

Let $\mathcal{C}$ be the class of all lattices $L$ such that each graph automorphism of type (C) of $L$ is a lattice automorphism.

We prove (by applying the results and the methods of [3], [5] and [6]):

$(\ast)$ Let $L$ be a semimodular lattice. Then the following conditions are equivalent:

(i) $L$ belongs to $\mathcal{C}$.

(ii) If $A$ is a direct factor of $L$ such that $A$ is self-dual and orthogonal to each interval of type (C) in $L$, then $A$ is trivial (i.e., $\text{card } A = 1$).

2. Preliminaries

In what follows, $L$ is a lattice. For the notion of the unoriented graph $G(L)$ of $L$ cf., e.g., [1], [2].

If $x, y \in L$, $x < y$ and if the interval $[x, y]$ of $L$ is a two-element set, then we write $x \preceq y$ or $y \succeq x$.

Hence a graph automorphism of $L$ is a one-to-one mapping $\varphi$ of $L$ onto $L$ such that, whenever $x, y \in L$ and $x < y$, then

(i) $\varphi(x) \preceq \varphi(y)$ or $\varphi(y) \preceq \varphi(x)$,

(ii) either $\varphi^{-1}(x) \preceq \varphi^{-1}(y)$ or $\varphi^{-1}(y) \preceq \varphi^{-1}(x)$.

2.1. Definition. Let $L_0$ be a sublattice of $L$ such that $L_0$ is isomorphic to the lattice in Fig. 1; then the convex closure $\overline{L_0}$ of $L_0$ in $L$ is said to be an interval of type (C) in $L$.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{fig1.png}
\caption{Fig. 1}
\end{figure}

2.2. Definition. A graph automorphism $\varphi$ of $L$ is said to be of type (C) if, whenever $L_1$ is an interval of type (C) in $L$ and $x, y \in L_1$, $x < y$, then $\varphi(x) \preceq \varphi(y)$ and $\varphi^{-1}(x) \preceq \varphi^{-1}(y)$.
It is easy to verify that if $L$ is modular, then it has no sublattice of type (C); consequently, in this case each graph automorphism of $L$ is of type (C). Therefore (°) is a corollary of (°1).

We denote by $L^\sim$ the lattice dual to $L$. If $L$ and $L^\sim$ are isomorphic, then $L$ is said to be self-dual.

### 3. An Example

Let us recall that if $L$ can be expressed as a direct product $L_1 \times L_2$ and if $x = (x_1, x_2) \in L$, $y = (y_1, y_2) \in L$, then $x \prec y$ if and only if either $x_1 \prec y_1$ and $x_2 = y_2$, or $x_1 = x_2$ and $y_1 \prec y_2$.

From this we immediately obtain

3.1. **Lemma.** Let $L_1, L_2$ be lattices and let $\psi$ be a graph isomorphism of $L_1$ onto $L_2$. Put $L = L_1 \times L_2$. For each $x = (x_1, x_2) \in L$ we set

$$\psi(x) = (\psi^{-1}(x_2), \psi(x_1)).$$

Then $\psi$ is a graph automorphism of $L$.

Consider the lattices $L_1$ and $L_2$ in Fig. 2 or Fig. 3, respectively. Both $L_1$ and $L_2$ are semimodular.

![Fig. 2](image1.png)

![Fig. 3](image2.png)

3.2. **Lemma.** Both $L_1$ and $L_2$ are directly indecomposable.

**Proof.** The assertion for $L_1$ was proved in [5], pp. 164–165. The proof for $L_2$ is similar. \[ \square \]

3.3. **Lemma.** Let $i \in \{1, 2\}$. Then the lattice $L_i$ fails to be self-dual.

**Proof.** It is easy to verify that $L_i^\sim$ fails to be semimodular. Therefore $L_i^\sim$ is not isomorphic to $L_i$. \[ \square \]
Put $L = L_1 \times L_2$.

Since any two direct product decompositions of $L$ have a common refinement and since $L_1, L_2$ are directly indecomposable by 3.3, we conclude

3.4. Lemma. Let $A$ be a direct factor of $L$ having more than one element. Then the lattice $A$ is isomorphic to some of the lattices $L, L_1, L_2$.

By the same argument as in 3.3 we obtain

3.5. Lemma. The lattice $L$ is not self-dual.

Now, 3.3, 3.4 and 3.5 yield

3.6. Corollary. The lattice $L$ satisfies the condition (ii) from (*).

It is easy to verify that there exists a graph isomorphism $\varphi$ of $L_2$ onto $L_2$ such that $\varphi$ fails to be a lattice isomorphism. Hence there are $x_1, y_1$ in $L_1$ such that $x_1 < y_1$ and $\varphi(x_1) > \varphi(y_1)$. Consequently, if $\psi$ is defined as above, then $\psi$ is not a lattice automorphism of $L$.

In view of 3.1 we conclude that in (*), the assumption of modularity cannot be replaced by the assumption of semimodularity of the lattice $L$.

We also remark that $\psi$ is an example of a graph automorphism on a semimodular lattice such that $\psi$ is not of type (C).

4. Proof of (*).

In this section we assume that the lattice $L$ is semimodular.

4.1. Lemma. Suppose that $B$ is a direct factor of $L$ such that

(i) $B$ is self-dual;
(ii) $B$ is orthogonal to each interval of type (C) in $L$;
(iii) card $B > 1$.

Then $L$ does not belong to $C_1$.

Proof. There is a lattice $A$ such that there exists an isomorphism $\psi$ of $L$ onto $A \times B$. Further, in view of (i), there is an isomorphism $\chi$ of the lattice $B$ onto $B^{-}$.

For each $x \in L$ we put $\varphi(x) = y$, where

$$\varphi(x) = (a, b), \quad y = \psi^{-1}(\langle a, \chi(b) \rangle).$$

Then $\varphi$ is a graph automorphism of the lattice $L$ (cf. [5], Lemma 1.1). Moreover, (ii) yields that $\varphi$ is of type (C). By applying Lemma 1.2 of [5] we conclude that $\varphi$ fails to be a lattice automorphism. Therefore $L$ does not belong to $C_1$. \qed
Let $L_1$ and $L_2$ be semimodular lattices. Suppose that $\varphi$ is a graph isomorphism of $L_1$ onto $L_2$ such that

(a) if $X$ is an interval of type $(C)$ in $L_1$ and $x_1, x_2 \in X$, $x_1 \prec x_2$, then $\varphi(x_1) \prec \varphi(x_2)$;

(b) if $Y$ is an interval of type $(C)$ in $L_2$ and $y_1, y_2 \in Y$, $y_1 \prec y_2$, then $\varphi^{-1}(y_1) \prec \varphi^{-1}(y_2)$.

We apply similar steps as in Section 2 of [5]. For the sake of completeness, we recall the corresponding notation.

Let $A_1$ be the set of all intervals $[x, y]$ of $L_1$ such that

$$x \prec y \quad \text{and} \quad \varphi(x) \prec \varphi(y).$$

Further, let $B_1$ be the set of all intervals $[u, v]$ of $L_1$ such that

$$u \prec v \quad \text{and} \quad \varphi(u) \succ \varphi(v).$$

Similarly we define the sets $A_2$ and $B_2$ of intervals of $L_2$ (with $\varphi^{-1}$ instead of $\varphi$).

Choose $x_1^1 \in L_1$, $x_2^1 \in L_2$. We denote by $A_1^1$ the set of all elements $x \in L_1$ such that either $x = x_1^1$, or there exist $y_1, y_2, \ldots, y_n \in L_1$ such that

(i) $y_1 = x_1^1$, $y_n = x$,
(ii) for each $i \in \{1, 2, \ldots, n - 1\}$, the elements $y_i, y_{i+1}$ are comparable and the corresponding interval belongs to $A_1$.

Similarly we define the set $B_1^1$ (taking $B_1$ instead of $A_1$). The subsets $A_1^2$ and $B_1^2$ are defined analogously (taking $x_2^1$ and $\varphi^{-1}$ instead of $x_1^1$ and $\varphi$).

We apply the notion of the internal direct product decomposition of a lattice $L$ with the central element $x^0$ in the same sense as in [5] (cf. also [6]). By using this notion and by applying the assumption given above we conclude that the results of [3] (cf. Theorem 2 in [3] and the lemmas applied for proving this Theorem) yield

4.2. Proposition. Under the assumptions as above, there exist internal direct product decompositions

\[ \psi_1 : L_1 \to A_1^0 \times B_1^0 \quad \text{(with the central element $x_1^0$)} \]
\[ \psi_2 : L_2 \to A_2^0 \times B_2^0 \quad \text{(with the central element $x_2^0$)} \]

such that

(i) the lattices $A_1^0$ and $A_2^0$ are isomorphic,
(ii) the lattice $B_1^0$ is isomorphic to $(B_2^0)^\sim$.

Now suppose that the lattice $L$ satisfies the condition (ii) of (*i).

Let $\varphi$ be a graph automorphism of type (C) of the lattice $L$. 

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Choose $x^0 \in L$. We put $L = L_1 = L_2$ and $x^0 = x^0_1 = x^0_2$. The fact that $\varphi$ is of type (C) yields that the conditions (a) and (b) are satisfied. Hence we can apply Proposition 4.2.

The further steps are the same as in Part 3 of [5]. By using them we obtain

4.3. Lemma. Let $L$ be a semimodular lattice satisfying the condition (ii) of $(*)_1$. Then the condition (i) of $(*)_1$ is valid.

In view of 4.1 and 4.3, we infer that $(*)_1$ holds.

If $L_1$ is a sublattice of $L$ and $a, b \in L_1$, $a < b$, then we denote by $[a, b]_1$ the corresponding interval of $L_1$. We put $a \prec_1 b$ if $[a, b]_1$ is a two-element set.

We say that $L_1$ is a $c$-sublattice of $L$ if, whenever $a, b \in L_1$ and $a \prec_1 b$, then $a < b$.

We remark that Theorem 2 in the paper [7] by Ratanaprasert and Davey (this theorem solved a problem proposed in [4]) implies that in Definition 2.1 above it suffices to consider only those sublattices $L_0$ of $L$ which are $c$-sublattices of $L$.

References


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