Ján Jakubík
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ON PROJECTIVE INTERVALS IN A MODULAR LATTICE

Ján Jakubík, Košice

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Summary. In this paper a combinatorial result concerning pairs of projective intervals of a modular lattice will be established.

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1. PRELIMINARIES

The recent papers dealing with combinatorial questions concerning partially ordered sets are rather frequent (cf., e.g., [2], [3], [4]).

Let $L$ be a modular lattice. We denote by $\mathcal{D}$ the collection of all systems $D = (a_1, a_2, a_3, u, v)$ of distinct elements of $L$ such that

$$u = a_1 \land a_2 = a_1 \land a_3 = a_2 \land a_3, \quad v = a_1 \lor a_2 = a_1 \lor a_3 = a_2 \lor a_3.$$ 

An interval $[a_1, a_2]$ of $L$ will be said to be an $m$-interval if there is $D \in \mathcal{D}$ such that (under the above notation), $[a_1, a_2]$ is projective to $[u, a_1]$.

Let $\alpha = [b_1, b_2]$ and $\beta = [c_1, c_2]$ be distinct projective intervals of $L$. Assume that $\alpha$ is nontrivial (i.e. $b_1 \neq b_2$); then $\beta$ is nontrivial as well.

There exists a least positive integer $n$ such that for some $\alpha_0, \alpha_1, \ldots, \alpha_n$ in $L$ the following conditions are satisfied:

(i) $\alpha_0 = \alpha$ and $\alpha_n = \beta$;
(ii) for each $i \in \{1, 2, \ldots, n\}$, the interval $\alpha_i$ is transposed to the interval $\alpha_{i-1}$. We denote $\mu(\alpha, \beta) = n$.

Let $S(\alpha)$ be the collection of all systems $(y_0, y_1, y_2, \ldots, y_m)$ with $b_1 = y_0 < y_1 < y_2 < \ldots < y_m = b_2$. The collection $S(\beta)$ is defined analogously. For each $i \in \{1, 2, \ldots, m\}$ let $k(i)$ be a positive integer.
A system of distinct intervals

\[(1) \quad (\beta_{ij})(i = 1, 2, \ldots, m; j = 1, 2, \ldots, k(i))\]

will be said to be a \(p\)-system for the intervals \(\alpha\) and \(\beta\) if the following conditions are satisfied:

(i) there are \(Y = (y_0, y_1, \ldots, y_n) \in S(\alpha)\) and \(Z = (z_1, z_2, \ldots, z_n) \in S(\beta)\) such that for each \(i \in \{1, 2, \ldots, m\}\) we have \(\beta_i = [y_{i-1}, y_i]\) and \(\beta_{i,j} = [z_{i-1}, z_i]\);

(ii) for each \(i \in \{1, 2, \ldots, m\}\) and each \(j \in \{1, 2, \ldots, k(i)\}\) the interval \(\beta_{i,j-1}\) is transposed to \(\beta_{i,j}\). The collection of all \(p\)-systems for \(\alpha\) and \(\beta\) will be denoted by \(p(\alpha, \beta)\). For \(A \in P(\alpha, \beta)\) (where \(A\) is as in (1)) let \(A_0\) be the of all \(\beta_{ij} \in A\) such that \(\beta_{ij}\) fails to be an \(m\)-interval. We put

\[\nu(A) = \text{card } A_0,\]

\[\nu_0(\alpha, \beta) = \min\{\nu(A): A \in P(\alpha, \beta)\}.\]

In this note it will be proved that we always have

\[(2) \quad \nu_0(\alpha, \beta) \leq 3\]

and this estimate cannot be sharpened in general.

The estimate (2) is a consequence of the following result:

(A) Let \(\alpha = [b_1, b_2]\) and \(\beta = [c_1, c_2]\) be nontrivial intervals of a modular lattice \(L\). Assume that \(\alpha\) is projective to \(\beta\). Then there exist elements \(x_0, x_1, \ldots, x_m, y_0, y_1, \ldots, y_m\) in \(L\) such that the following conditions are satisfied:

(i) \(b_1 = x_0 < x_1 < \ldots < x_m = b_2, c_1 = y_0 < y_1 < \ldots < y_m = c_2\) and for each \(i \in \{1, 2, \ldots, m\}\) the interval \([x_{i-1}, x_i]\) is projective to \([y_{i-1}, y_i]\);

(ii) there is \(i(1) \in \{1, 2, \ldots, m\}\) such that \([x_{i-1}, x_i]\) is an \(m\)-interval for each \(i \in \{1, 2, \ldots, m\} \setminus \{i(1)\}\), and either \([x_{i(1)-1}, x_{i(1)}]\) is an \(m\)-interval, or there is an interval \([t_1, t_2] \subseteq L\) such that \([x_{i(1)-1}, x_{i(1)}]\) is transposed to \([t_1, t_2]\) and \([t_1, t_2]\) is transposed to \([y_{i(1)-1}, y_{i(1)}]\).

**The proof of (A)**

We will apply the notation from Section 1. Again, let \(\alpha\) and \(\beta\) be distinct nontrivial intervals of a modular lattice \(L\). Assume that \(\alpha\) and \(\beta\) are projective. A \(p\)-system \(A\) for \(\alpha\) and \(\beta\) will be said to be reduced if (under the notation as above), whenever \(i \in \{1, 2, \ldots, m\}\) and \(j \in \{1, 2, \ldots, k(i) - 1\}\), then \(\beta_{i,j-1}\) fails to be transposed to \(\beta_{i,j+1}\).

The following lemma is easy to verify.
2.1. Lemma. Let $A \in P(\alpha, \beta)$. Then there exists $A' \in P(\alpha, \beta)$ such that $A' \subseteq A$ and $A'$ is reduced.

Let $[c_1, c_2]$ and $[d_1, d_2]$ be transposed intervals of $L$; then we have either

(i) $c_2 \land d_1 = c_1$, \quad $c_2 \lor d_1 = d_2$,

or

(ii) $d_2 \land c_1 = d_1$, \quad $d_2 \lor c_1 = c_2$.

If (i) is valid, then we write $[c_1, c_2] \not\preceq [d_1, d_2]$; the validity of (ii) will be recorded by writing $[c_1, c_2] \setminus [d_1, d_2]$.

2.2. Lemma. Let $A \in P(\alpha, \beta)$ and assume that $A$ is reduced. Let $A$ be as in (1). If $i \in \{1, 2, \ldots, m\}$, $j \in \{1, 2, \ldots, k(i) - 1\}$, $\alpha_{i,j-1} \not\preceq \alpha_{i,j}$, then $\alpha_{i,j} \setminus \alpha_{i,j+1}$ (and dually).

The proof is trivial.

Let $A \in P(\alpha, \beta)$ be as in (1). Let $i \in \{1, 2, \ldots, m\}$, $z_{i1} \in L$, $z_{i-1,1} < z_{i1} < x_{i1}$. We define elements $z_{i2}, z_{i3}, \ldots, z_{i,k(i)}$ by induction as follows: if $z_{i,j-1}$ ($j \in \{2, \ldots, k(i) - 1\}$) is already defined and if $\alpha_{i,j-1} \not\preceq \alpha_{i,j}$, then we put $z_{ij} = z_{i,j-1} \lor d_1$, where $d_1$ is the least element of $\alpha_{i,j}$; on the other hand, if $\alpha_{i,j-1} \setminus \alpha_{i,j}$, then we set $z_{ij} = z_{i,j-1} \land d_2$, where $d_2$ is the largest element of $\alpha_{ij}$.

Consider the system $A'$ which we obtain from the system $A$ if the $i$-th row $(\alpha_{i,1}, \alpha_{i,2}, \ldots, \alpha_{i,i(k)})$ of $A$ is replaced by the rows

$$
\alpha'_{i,1}, \alpha'_{i,2}, \ldots, \alpha'_{i,i(k)}
$$

$$
\alpha''_{i,1}, \alpha''_{i,2}, \ldots, \alpha''_{i,n},
$$

where

$$
\alpha'_{i,j} = \{t \in \alpha_{ij} : t \leq z_{i,j}\}, \quad \alpha''_{i,j} = \{t \in \alpha_{ij} : t \geq z_{i,j}\}.
$$

Then we obviously have:

2.3. Lemma. $A'$ is a $p$-system for the intervals $\alpha$ and $\beta$.

The system $A'$ will be said to be generated by the system $A$ and by the element $z_{i1}$.

Let $y, z \in L$, $b_1 < y < b_2$, $c_1 < z < c_2$. Suppose that $[b_1, y]$ is projective to $[c_1, z]$ and that $[y, b_2]$ is projective to $[z, c_2]$.

2.4. Lemma. Let $A \in p([b_1, y], [c_1, z])$. (We apply the same notation as in (1) with the distinction that we now have $y$ and $z$ instead of $b_2$ and $c_2$.) Let $\beta_{m+1,i}$ ($i = 1, 2, \ldots, k(m + 1)$) be intervals of $L$ such that $\beta_{m+1,1} = [y, b_2]$, $\beta_{m+1,k(m+1)} = [z, c_2]$.
and for each \( i \in \{2, 3, \ldots, k(m + 1)\} \) the interval \( \beta_{m+1,i-1} \) is transposed to \( \beta_{m+1,i} \). Let \( A' \) be the system

\[
(\beta_{ij} (i = 1, 2, \ldots, m + 1; j = 1, 2, \ldots, k(i))).
\]

Then \( A' \in p(\alpha, \beta) \).

Proof. This is an immediate consequence of the definition of \( p(\alpha, \beta) \).

The assertion dual to 2.4. is also valid. \( \square \)

2.5. Lemma. Let \( x, y \) and \( z \) be elements of a modular lattice \( L \). Assume that the relations

\[
[x \land y, x] \not\rightarrow [y, x \lor y] \quad \text{and} \quad [y, x \lor y] \setminus [y \land z, z]
\]

are valid. Then the sublattice \( L_1 \) of \( L \) generated by the elements \( x, y \) and \( z \) is a homomorphic image of the lattice on Fig. 1.

![Diagram](image)

Fig. 1

Proof. If we consider the free modular lattice with three free generators (cf. e.g. [1], Chap. III, Theorem 8) \( x, y \) and \( z \), and if we take into account that in our case we have \( x \lor y = y \lor z \), then we obtain the assertion of the lemma. \( \square \)

Theorem. Let \( \alpha \) and \( \beta \) be nontrivial distinct intervals of a modular lattice \( L \). Assume that \( \alpha \) is projective to \( \beta \). Then there is \( A \in P(\alpha, \beta) \) such that (under the notation as in (1) the following condition is satisfied: there is \( i(1) \in \{1, 2, \ldots, m\} \).
such that, whenever \( i \in \{1, 2, \ldots, m\} \setminus \{i(1)\} \) and \( j \in \{1, 2, \ldots, k(i)\} \), then \( \beta_{ij} \) is an \( m \)-interval; next, either \( \beta_{i(1), 1} \) is an \( m \)-interval, or \( k(i(1)) \leq 3 \).

**Proof.** Under the notation as in Section 1, let \( \mu(\alpha, \beta) = n \). We have \( n \geq 1 \). If \( n = 1 \), then the assertion obviously holds (it suffices to consider the system \( (\alpha_0, \alpha_1) \)).

Suppose that \( n \geq 2 \) and let us apply induction with respect to \( n \). First we consider the system

\[
(\alpha_k) \quad (k = 0, 1, 2, \ldots, n)
\]

which obviously belongs to \( p(\alpha, \beta) \). Without loss of generality we may assume that this system is reduced. Next, we can suppose that \( \alpha_0 \neq \alpha_1 \setminus \alpha_2 \) is valid (in the case \( \alpha_0 \setminus \alpha_1 \neq \alpha_2 \) we apply a dual procedure).

Let \( x, y \) and \( z \) be the greatest element of \( \alpha_0 \), the least element of \( \alpha_1 \) and the greatest element of \( \alpha_2 \), respectively. (Cf. Fig. 1.) Then

\[
\alpha_0 = [x \wedge y, x], \quad \alpha_1 = [y, x \vee y], \quad \alpha_2 = [x \wedge z, z].
\]

At the same time, \( x \vee y = y \vee z \). Put \( x' = (x \wedge y) \vee (x \wedge z) \). We have obviously

\[
x \wedge y \leq x' \leq x.
\]

From \( x \wedge y < x \) we infer that either \( x \wedge y < x' \) or \( x' < x \).

Let us distinguish the following cases.

(a) Let \( x \wedge y = x' \). Then \( \alpha = \alpha_0 = [x', x] \). In view of Fig. 1, \( \alpha_0 \) is an \( m \)-interval; therefore \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are \( m \)-intervals as well. Now it suffices to put \( a = (\alpha_i) \) \((i = 0, 1, 2, \ldots, n)\).

(b) Let \( x' = x \). Then \( \alpha = \alpha_0 = [x \wedge y, x] \). Next, \( \alpha_2 = [y \wedge z, t] \), where \( t = (x \wedge z) \vee (y \wedge z) \). Denote \( \alpha_1' = [x \wedge y \wedge z, x \wedge z] \). We have (cf. Fig. 1)

\[
\alpha_0 \setminus \alpha_1' \neq \alpha_2.
\]

Thus the system \( A' \) consisting of the intervals

\[
\alpha_0, \alpha_1', \alpha_2, \alpha_3, \ldots, \alpha_n
\]

belongs to \( P(\alpha, \beta) \). Since \( \alpha_2 \neq d_3 \), according to 2.2 the system \( A' \) fails to be reduced. Thus in view of 2.1 there exists a system

\[
\beta_0, \beta_1, \ldots, \beta_l
\]

which belongs to \( P(\alpha, \beta) \) such that \( l < n \). Therefore by the induction hypothesis, the assertion of the theorem is valid for \( \alpha \) and \( \beta \).
(c) Let \( x \land y < x' < x \). Let \( A_1 \) be the system

\[
(\alpha_i) \quad (i = 0, 1, 2, \ldots, n)
\]

and let \( A_2 \) be the system generated by \( A_1 \) and the element \( x' \). Then (under the notation as in Lemma 2.3) the system \( A_2 \) consists of intervals

\[
\alpha'_0, \alpha'_1, \ldots, \alpha'_{n},
\]

\[
\alpha''_0, \alpha''_1, \ldots, \alpha''_{n},
\]

where

\[
\alpha'_0 = [x \land y, x'], \quad \alpha'_n = [y \land z, t], \\
\alpha''_0 = [x', x], \quad \alpha''_n = [t, z].
\]

Since \( \alpha'_0 \) is an \( m \)-interval, all \( \alpha'_i \) \((i = 1, 2, \ldots, n)\) must be \( m \)-intervals. Next, by the same argument as in (b) we can verify that there exists a system \( A_3 \) consisting of intervals

\[
\beta_0, \beta_1, \ldots, \beta_l
\]

with \( 1 < n \) such that \( A_3 \in p([x', x], [t, z]) \). Hence by the induction hypothesis, the assertion of the theorem is valid for the intervals \([x', x]\) and \([t, z]\). Now it suffices to apply Lemma 2.3. \( \square \)

Theorem (A) in Section 1 is obviously a consequence of (in fact, equivalent to) Theorem 2.6.

2.7. Example. Let \( L \) be as in Fig. 1 Consider the intervals \( \alpha = [x \land y, x'] \) and \( \beta = [y \land z, t] \). It is easy to verify that \( \mu_0(\alpha, \beta) = 3 \). Hence the estimate (2) cannot be sharpened in general.

References


Author's address: Matematický ústav SAV, dislokované pracovisko Grešákova 6, 040 01 Košice.