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RECONSTRUCTING A GRAPH FROM THE INCIDENCE RELATION ON ITS EDGE SET

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We consider a connected undirected graph G without loops and multiple edges. Our goal is to reconstruct G if we know its edge set E and the relation ϱ of incidence on this set. (This means that $(e_1, e_2) \in \varrho$, where $e_1 \in E$, $e_2 \in E$, if and only if the edges e_1, e_2 have a common end vertex.) We suppose that G has at least two edges; the reverse case is trivial. The theorem of Whitney [2] asserts that this reconstruction is possible for any finite graph without loops and multiple edges which is not isomorphic to any of the graphs in Fig. 1.

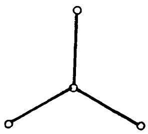
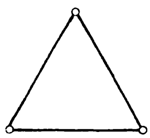


Fig. 1



H. A. Jung [1] has proved this theorem also for infinite graphs. Here we shall describe the algorithm of such a reconstruction.

For this purpose we need to construct the subfamily $\mathscr{R}$ of the family $\exp E$ of all subsets of E such that any $R \in \mathscr{R}$ is the set of all edges incident with some vertex of G and any such set is in $\mathscr{R}$. Then to every $R \in \mathscr{R}$ we can assign a vertex $u(R)$ and we join two vertices $u(R_1)$, $u(R_2)$ for $R_1 \in \mathscr{R}$, $R_2 \in \mathscr{R}$ by an edge if and only if $R_1 \cap R_2 \neq \emptyset$. The graph thus obtained is evidently that graph G.

By $\mathscr{R}_n$, where n is a positive integer less than four, we denote the family of all sets $R \subset \mathscr{R}$ which have the cardinality n. The family $\mathscr{R} \dot{-} (\mathscr{R}_1 \cup \mathscr{R}_2 \cup \cup \mathscr{R}_3)$ will he denoted by $\mathscr{R}^*$.

At first we construct the subfamily $\mathscr{P}$ of $\exp E$ defined as follows.

If $P \subset E$, we put $P \in \mathscr{P}$ if and only if the following two conditions are satisfied:

(a) Any two edges of P are incident to each other.

(b) There does not exist any set $P' \subset E$ such that the condition (a) would be satisfied in P' and P would be a proper subset of P'.

By $\mathscr{P}_n$, where n is a positive integer less than four, we denote the family of all sets $P \in \mathscr{P}$ which have the cardinality n. The family $\mathscr{P} \dot{-} (\mathscr{P}_1 \cup \mathscr{P}_2 \cup \cup \mathscr{P}_3)$ will be denoted by $\mathscr{P}^*$.

Now we shall give some lemmas. Most of them will be given without proofs, because the proofs are simple and may be left to the reader.

Lemma 1. *Let $P \in \mathscr{P}^*$. Then there exists a vertex v incident with all edges of P and only with these edges.*

This lemma says that $\mathscr{P}^* \subset \mathscr{R}^*$. Evidently also $\mathscr{R}^* \subset \mathscr{P}^*$, therefore $\mathscr{P}^* - \mathscr{R}^*$.

Lemma 2. *Let $P \in \mathscr{P}_3$. Then either there exists a vertex v which is incident with all edges of P and only with these edges, or the edges of P with their end vertices form a triangle.*

Lemma 3. *Let $P \in \mathscr{P}_2$. Then P is a pair of incident edges which do not belong to any triangle.*

Lemma 4. *Let $P \in \mathscr{P}_1$. Then the graph G consists of one edge and its end vertices.*

This is a trivial case; therefore in the following we shall assume that $\mathscr{P}_1 \quad \emptyset$.

Lemma 5. *Let $P_1 \in \mathscr{P}^*$, $P_2 \in \mathscr{P}^*$, $P_1 \neq P_2$. Then the intersection $P_1 \cap P_2$ either is empty, or contains only one element.*

Lemma 6. *Let $P_1 \in \mathscr{P}_3$, $P_2 \in \mathscr{P}^* \cup \mathscr{P}_2$. Then the intersection $P_1 \cap P_2$ either is empty, or contains one or two elements. If it contains two elements, then P_1 is the set of edges of a triangle. If it contains only one element, then P_1 is the set of edges incident to some vertex.*

Lemma 7. *Let $P_1 \in \mathscr{P}_3$, $P_2 \in \mathscr{P}_3$, $P_1 \neq P_2$. Then the intersection $P_1 \cap P_2$ either is empty, or contains one or two elements. If it contains two elements, then one of the sets P_1, P_2 is the set of edges of a triangle, the other is the set of all edges incident with some vertex.*

Lemma 8. *Let $P \in \mathscr{P}_3$. Then there exist at most three sets of $\mathscr{P}$ such that their intersections with P contain two elements each.*

Lemma 9. *Let $P \in \mathscr{P}_3$, $P' \in \mathscr{P}_3$, $P'' \in \mathscr{P}_3$, $P''' \in \mathscr{P}_3$ be pairwise different. Let $|P \cap P'| - |P \cap P''| = |P \cap P'''| = 2$. Then either at least one of the sets P', P'', P''' has a non-empty intersection with a set of $\mathscr{P}^*$, or P is the set of edges of a triangle, or G is the complete graph with four vercites* (Fig. 2).

Proof. Assume that P is the set of edges incident with some vertex. Then P', P'', P''' are the sets of vertices of triangles. The edges of $P \cup P' \cup P'' \cup P'''$ with their end vertices form a subgraph of G isomorphic to the complete graph with four vertices. Thus either the third case occurs, or this subgraph is proper. If this subgraph is proper, then at least one of its vertices must have a greater degree in G than in this subgraph, i.e. greater than three, and the first case occurs.

Lemma 10. *Let $P \in \mathscr{P}_3$, $P' \in \mathscr{P}_3$, $P'' \in \mathscr{P}_3$. Let $|P \cap P'| = |P \cap P''$ $\;\;2$ and let P have a non-empty intersection with no other set of $\mathscr{P}$ except for P' and P''. Then P is the set of edges of a triangle.*

Lemma 11. *Let $P \in \mathscr{P}_3$, $P' \in \mathscr{P}_3$, $P'' \in \mathscr{P}_3$, $P''' \in \mathscr{P}_3$. Let these sets be pairwise different and $|P \cap P'| = |P \cap P''| = 2$, $|P \cap P'''| = 1$. Let P have a non-empty intersection with no other set of $\mathscr{P}$ except for P', P'', P'''. Then either P is a set of edges of a triangle, or G is isomorphic to the graph in* Fig. 3.

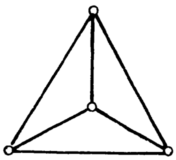


Fig. 2

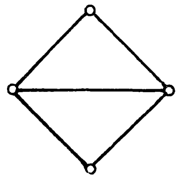


Fig. 3

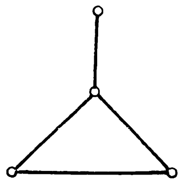


Fig. 4

Proof. Assume that P is the set of edges incident with some vertex. Let $P = \{e_1, e_2, e_3\}$. Then without the loss of generality $P \cap P' - \{e_1, e_2\}$, $P \cap P'' = \{e_2, e_3\}$. Let e' (or e'', respectively) be the edge of P' (or P'', respectively) not belonging to P. We have $e' \neq e''$, because P' and P'' can have at most one common element. The edges e_2, e', e'' are pairwise incident (because according to Lemma 7 the sets P' and P'' are sets of edges of triangles), thus the set $\{e_2, e', e''\}$ is a subset of some set P_0 of P. This set P_0 must be equal to P''', because P has a non-empty intersection with no other set of $\mathscr{P}$ except for P', P'', P'''. As the cardinality of P''' is three, we have $\{e_2, e', e''\}$ $\;\;P'''$ and the common end vertex of e_2, e', e'' is not incident with any other edge. The same holds evidently for the common end vertex of e_1, e_2, e_3 (because $P \in \mathscr{P}$). The common end vertex of e_1 and e' must be of the degree two; otherwise the set of the edges incident with it would form a set of $\mathscr{P}$ with a non-empty intersection with P. The same holds for the common end vertex of e_3 and e''. Thus we obtain a graph isomorphic to the graph in Fig. 3.

Lemma 12. *Let $P \in \mathscr{P}_3$, $P' \in \mathscr{P}_3$, $P'' \in \mathscr{P}_3$, $P''' \in \mathscr{P}_3$, $P'''' \in \mathscr{P}_3$, $P' \neq P''$, $P''' \neq P''''$. Let $|P \cap P'| = |P \cap P''| = 2$, $|P \cap P'''| = |P \cap P''''| = 1$ and let P have an intersection of the cardinality two with no other set of $\mathscr{P}$ than P', P''. Then P is the set of edges incident with some vertex.*

Lemma 13. *Let $P \in \mathscr{P}_3$, $P' \in \mathscr{P}_3$. Let $|P \cap P'| = 2$ and let P have a non-*

empty intersection with no set of $\mathscr{P}$ except for P'. Then either P is the set of edges of a triangle, or the graph G is isomorphic to the graph on Fig. 4.

Lemma 14. *Let $P \in \mathscr{P}_3$, $P' \in \mathscr{P}_3$, $P'' \in \mathscr{P}_3$. Let $|P \cap P'| = 2$, $|P \cap P''|$ 1 and let P have an intersection of the cardinality two with no set of $\mathscr{P}$ except for P'. Then P is the set of edges incident with some vertex.*

Lemma 15. *If $P \in \mathscr{P}_3$ and P has an intersection of the cardinality two with no set of $\mathscr{P}$, then either P is the set of edges incident with some vertex, or the graph G is a triangle.*

The result of these fifteen lemmas will be summarized in the following lemma.

Lemma 16. *Let a graph G be connected and not isomorphic to any of the graphs in Figs.* 1, 2, 3, 4. *Let its edge set E and the relation ϱ of incidence on it be given. Then we can find out, which sets of $\mathscr{P}$ are the sets of edges of triangles and which are the sets of edges incident with some vertex.*

Proof. If $P \notin \mathscr{P}_3$, it evidently cannot be a set of edges of a triangle. For $P \in \mathscr{P}_3$ the recognizing algorithm is presented by the block scheme in Fig. 5. The steps of this algorithm follow from the preceding lemmas. The symbol $\mathscr{P}_3'$ means the subfamily of $\mathscr{P}_3$ consisting of the sets of edges incident with some

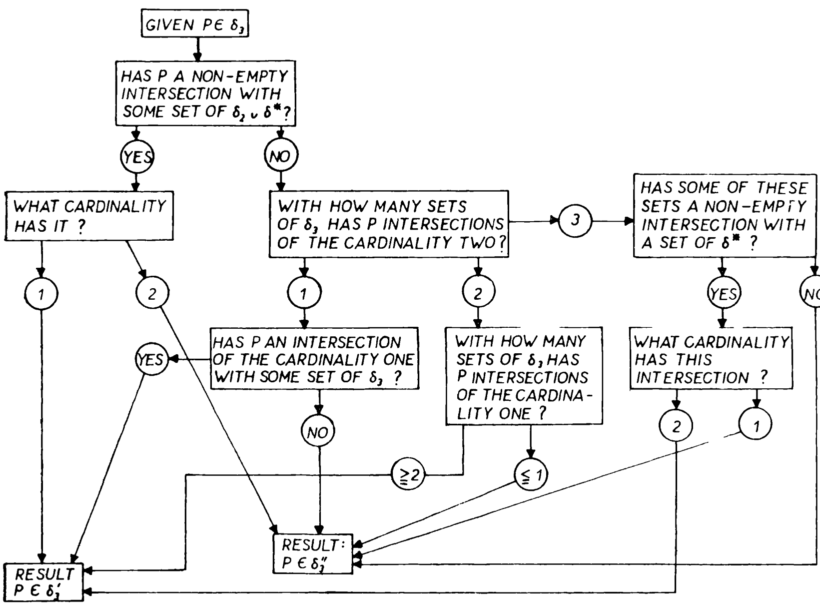


Fig. 5 (Read $\mathscr{P}$ for δ.)

vertex, the symbol $\mathscr{P}_3''$ means the subfamily of $\mathscr{P}_3$ consisting of the sets of edges of triangles. Evidently $\mathscr{P}_3' = \mathscr{R}_3$, $\mathscr{P}_3'' \cap \mathscr{R} = \emptyset$.

Now let us mention also the vertices of the degrees two and one.

Lemma 17. *Let e be an edge contained only in one of the sets of* $\mathscr{P}_2 \cup \mathscr{P}_3' \cup \mathscr{P}^*$. *Then e is incident with a vertex of the degree one.*

Lemma 18. *Let e be an edge contained only in one of the sets of* $\mathscr{P}_3''$ *and not contained in any of the sets of* $\mathscr{P}_2 \cup \mathscr{P}_3' \cup \mathscr{P}^*$. *Then e is incident with two vertices of the degree two.*

Lemma 19. *Let e be an edge not contained in any intersection of two sets of* $\mathscr{P}_2 \cup \mathscr{P}_3' \cup \mathscr{P}^*$, *but contained in an intersection* $P \cup P'$, *where* $P \in \mathscr{P}_2 \cup \cup \mathscr{P}_3' \cup \mathscr{P}^*$, $P' \in \mathscr{P}_3''$. *Then the edge e is incident with exactly one vertex of the degree two.*

Now a theorem can be expressed.

Theorem. *Let G be a connected undirected graph with at least two edges and without loops and multiple edges. If we know its edge set E and the relation* ϱ *of incidence on it, we can reconstruct G in the following way:*

(1) *We construct the family* $\mathscr{P}$ *of subsets of E such that* $P \subset E$ *is in* $\mathscr{P}$ *if and only if the following two conditions are satisfied:*

(a) *Any two edges of P are incident to each other.*

(b) *There does not exist any set* $P' \subset E$ *such that the condition* (a) *would be satisfied in P' and P would be a proper subset of P'.*

(2) *We decompose* $\mathscr{P}$ *into the subfamilies* $\mathscr{P}_2$, $\mathscr{P}_3$, $\mathscr{P}^*$ *such that* $\mathscr{P}_2$ *means the family of sets of* $\mathscr{P}$ *of the cardinality* 2, $\mathscr{P}_3$ *the family of sets of* $\mathscr{P}$ *of the cardinality* 3 *and* $\mathscr{P}^* = \mathscr{P} \dot{-} (\mathscr{P}_2 \cup \mathscr{P}_3)$.

(3) *If* $\mathscr{P}_2 \cup \mathscr{P}^* \neq \emptyset$, *then G is not isomorphic to any of the graphs on* Figs. 1, 2, 3, 4. *We go to* (5). *If* $\mathscr{P}_2 \cup \mathscr{P}^* = \emptyset$, *we proceed to* (4).

(4) *With the help of* Lemmas 9, 11, 13 *and* 15 *we find out, whether G is isomorphic to some graph in* Figs. 1, 2, 3, 4. *If it is, the procedure is finished. If not, we proceed to* (5).

(5) *We construct* $\mathscr{P}_3'$ *by the algorithm described by the block scheme in* Fig. 5.

(6) *We construct* $\mathscr{R}_2$ *as the family consisting of all sets of* $\mathscr{P}_2$, *and further of all sets* $\{e_1, e_2\}$, *where* e_1 *is contained in some set* $P \in \mathscr{P}_3''$ *and not contained in any set of* $\mathscr{P}_2 \cup \mathscr{P}_3' \cup \mathscr{P}^*$ *and* e_2 *is contained in the same set P.*

(7) *We costruct* $\mathscr{R}_1$ *as the family of all sets* $\{e\}$, *where e is contained only in one of the sets of* $\mathscr{P}_2 \cup \mathscr{P}_3' \cup \mathscr{P}^*$ *and in none of the sets of* $\mathscr{P}_3''$.

(8) *We put* $\mathscr{R} = \mathscr{R}_1 \cup \mathscr{R}_2 \cup \mathscr{P}_3' \cup \mathscr{P}^*$.

(9) *To any set of* $\mathscr{R}$ *we assign a vertex in a one-to-one manner and join such pairs of vertices that the corresponding sets have non-empty (i.e. one-element) intersections.*

The proof of this theorem follows from the preceding lemmas.

Example. We have given $E = \{e_1, \ldots, e_{16}\}$ and the relation $\varrho = \{\{e_1, e_2\}, \{e_1, e_3\}, \{e_1, e_4\}, \{e_1, e_5\}, \{e_2, e_3\}, \{e_2, e_5\}, \{e_2, e_6\}, \{e_2, e_7\}, \{e_2, e_9\}, \{e_2, e_{10}\}, \{e_2, e_{12}\}, \{e_2, e_{15}\}, \{e_3, e_4\}, \{e_3, e_6\}, \{e_3, e_7\}, \{e_3, e_9\}, \{e_3, e_{10}\}, \{e_3, e_{12}\}, \{e_3, e_{15}\}, \{e_4, e_5\}, \{e_4, e_6\}, \{e_5, e_6\}, \{e_6, e_7\}, \{e_6, e_9\}, \{e_6, e_{10}\}, \{e_6, e_{12}\}, \{e_6, e_{15}\}, \{e_7, e_8\}, \{e_7, e_9\}, \{e_7, e_{10}\}, \{e_7, e_{12}\}, \{e_7, e_{15}\}, \{e_8, e_9\}, \{e_9, e_{10}\}, \{e_9, e_{12}\}, \{e_9, e_{15}\}, \{e_{10}, e_{11}\}, \{e_{10}, e_{12}\}, \{e_{10}, e_{15}\}, \{e_{12}, e_{13}\}, \{e_{12}, e_{15}\} \{e_{13}, e_{14}\}, \{e_{14}, e_{15}\}, \{e_{14}, e_{16}\}\}$.

1)(and (2). $\mathscr{P}_2 = \{\{e_{10}, e_{11}\}, \{e_{12}, e_{13}\}, \{e_{14}, e_{15}\}\}$, $\mathscr{P}_3 = \{\{e_1, e_2, e_3\}, \{e_1, e_2, e_5\}, \{e_1, e_3, e_4\}, \{e_1, e_4, e_5\}, \{e_2, e_5, e_6\}, \{e_3, e_4, e_6\}, \{e_4, e_5, e_6\}, \{e_7, e_8, e_9\}, \{e_{13}, e_{14}, e_{16}\}\}$, $\mathscr{P}^* \quad \{\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\}\}$.

(3) $\mathscr{P} \dot{-} \mathscr{P}_3 \neq \emptyset$, thus G is not isomorphic to any of the graphs in Figs. 1, 2, 3, 4.

(5) $\{e_1, e_2, e_3\} \in \mathscr{P}_3$, $\{e_1, e_2, e_3\} \cap \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} = \{e_2, e_3\}$ is of cardinality 2, $\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} \in \mathscr{P}^* \Rightarrow \{e_1, e_2, e_3\} \in \mathscr{P}_3''$ $\{e_1, e_2, e_5\} \in \mathscr{P}_3$, $\{e_1, e_2, e_5\} \cap \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} = \{e_2\}$ is of cardinality 1, $\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} \in \mathscr{P}^* \Rightarrow \{e_1, e_2, e_5\} \in \mathscr{P}_3'$. $\{e_1, e_3, e_4\} \in \mathscr{P}_3$, $\{e_1, e_3, e_4\} \cap \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} = \{e_3\}$ is of cardinality 1, $\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} \in \mathscr{P}^* \Rightarrow \{e_1, e_3, e_4\} \in \mathscr{P}_3'$. $\{e_1, e_4, e_5\} \in \mathscr{P}_3$, $\{e_1, e_4, e_5\} \cap \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} \quad \emptyset$, $\mathscr{P}^* = \{\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\}\}$; $\{e_1, e_4, e_5\}$ has intersections of cardinality 2 with two sets $\{e_1, e_2, e_5\}$, $\{e_4, e_5, e_6\}$ of $\mathscr{P}_3$. $\{e_1, e_4, e_5\}$ has intersections of cardinality 1 with three sets $\{e_1, e_2, e_3\}$, $\{e_2, e_5, e_6\}$, $\{e_3, e_4, e_6\}$ of $\mathscr{P}_3$, thus with more than two sets, which implies $\{e_1, e_4, e_5\} \in \mathscr{P}_3'$. $\{e_2, e_5, e_6\} \in \mathscr{P}_3$, $\{e_2, e_5, e_6\} \cap \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} = \{e_2, e_6\}$ is of cardinality 2, $\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} \in \mathscr{P}^* \Rightarrow \{e_2, e_5, e_6\} \in \mathscr{P}_3''$. $\{e_3, e_4, e_6\} \in \mathscr{P}_3$, $\{e_3, e_4, e_6\} \cap \cap \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} = \{e_3, e_6\}$ is of cardinality 2, $\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} \in \mathscr{P}^* \Rightarrow \{e_3, e_4, e_6\} \in \mathscr{P}_3''$. $\{e_4, e_5, e_6\} \in \mathscr{P}_3$, $\{e_4, e_5, e_6\} \cap \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} - \{e_6\}$ is of cardinality 1, $\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} \in \mathscr{P}^* \Rightarrow \Rightarrow \{e_4, e_5, e_6\} \in \mathscr{P}_3'$. $\{e_7, e_8, e_9\} \in \mathscr{P}_3$, $\{e_7, e_8, e_9\} \cap \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} - \{e_7, e_9\}$ is of cardinality 2, $\{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\} \in \mathscr{P}^* \Rightarrow \{e_7, e_8, e_9\} \in \mathscr{P}_3''$. $\{e_{13}, e_{14}, e_{16}\} \in \mathscr{P}_3$, $\{e_{13}, e_{14}, e_{16}\} \cap \{e_{12}, e_{13}\} = \{e_{13}\}$ is of cardinality 1, $\{e_{12}, e_{13}\} \in \in \mathscr{P}_2 \Rightarrow \{e_{13}, e_{14}, e_{16}\} \in \mathscr{P}_3'$.

(6) $\mathscr{R}_2$ consists of the elements $\{e_{10}, e_{11}\}$, $\{e_{12}, e_{13}\}$, $\{e_{14}, e_{15}\}$ of $\mathscr{P}_2$, further of the pairs $\{e_7, e_8\}$, $\{e_8, e_9\}$, because $e_8 \in \{e_7, e_8, e_9\} \in \mathscr{P}_3''$ and $e_8 \notin P$ for any $P \in \mathscr{P}_2 \cup \mathscr{P}_3' \cup \mathscr{P}^*$, e_7 and e_9 are contained in the same set $\{e_7, e_8, e_9\} \in \mathscr{P}_3''$ as e_8.

(7) $\mathscr{R}_1$ consists of the sets $\{e_{11}\}$, $\{e_{16}\}$, because e_{11} (or e_{16}, respectively) is contained in $\{e_{10}, e_{11}\} \in \mathscr{P}_2$ (or in $\{e_{13}, e_{14}, e_{16}\} \in \mathscr{P}_3'$, respectively) and in no other set of $\mathscr{P}_2 \cup \mathscr{P}_3' \cup \mathscr{P}^*$.

(8) We have $\mathscr{R} = \{\{e_{11}\}, \{e_{16}\}, \{e_{10}, e_{11}\}, \{e_{12}, e_{13}\}, \{e_{14}, e_{15}\}, \{e_7, e_8\}. \{e_8, e_9\}, \{e_1, e_2, e_5\}, \{e_1, e_3, e_4\}, \{e_1, e_4, e_5\}, \{e_4, e_5, e_6\}, \{e_{13}, e_{14}, e_{16}\}, \{e_2, e_3, e_6, e_7, e_9, e_{10}, e_{12}, e_{15}\}\}$.

(9) After assigning vertices to the sets of $\mathscr{R}$ and joining them in the described way, we obtain the graph in Fig. 6

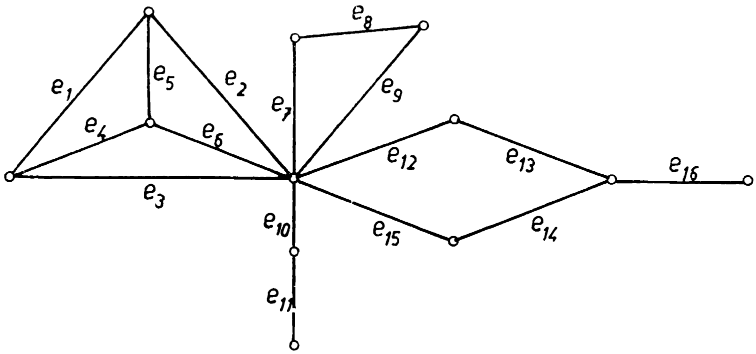


Fig. 6

We can generalize our considerations also for disconnected graphs. Assume that no connected component of G is isomorphic to any of the graphs in Figs. 1, 2, 3, 4 and any of them contains at least two edges. We can define the relation R on the set E such that two edges are in the relation R if and only if they are either equal or incident to one another (thus R is the reflexive closure of ϱ). Then we make the transitive closure TR of R. This relation is a relation of equivalence; each class of this equivalence is the set of edges of a connected component of G.

For graphs with loops or multiple edges the Theorem is not true. Fig. 7 shows us two graphs (one of them with a double edge), which have the same E and ϱ, but are not isomorphic. Such a case can occur in every graph, in which two edges exist, which join the same pair of vertices and one of their end vertices is incident only with those edges. Fig. 8 shows two graphs, one of

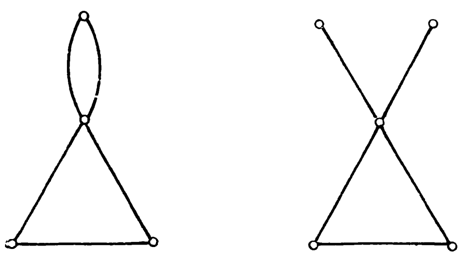


Fig. 7

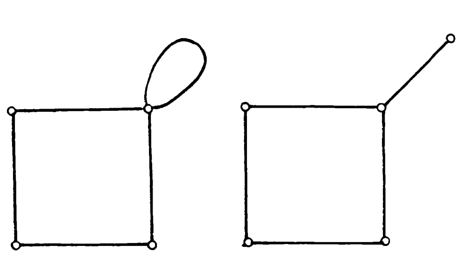


Fig. 8

them with a loop, which have the same E and ϱ but are not isomorphic. This case can evidently occur in each graph with a loop.

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