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NOTES ON AN APPROXIMATE DERIVATIVE

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D. Preiss has recently proved the following theorem ([4]):

Theorem 1. (D. Preiss) *Any approximate derivative on an interval is in the first Baire class.*

In this theorem he considered not only finite approximate derivatives.

In the first part of this article some notes relating to Z. Zahorski's article ([7]) are given. In the second part a proof of Preiss' theorem is given which is a modification of the proof for approximate derivatives given by L. E. Snyder ([5]).

1.

If we consider only finite approximate derivatives, then the assertion — *any approximate derivative is in the first Baire class* — has been known for a very long time. In 1938 the following theorem was presented: *Any approximate derivative of an approximate continuous function is in the first Baire class* ([6]). Approximate derivatives which also can obtain ∞ and $-\infty$ as their values are considered there. G. Tolstov proved his assertion by the help of the well known Baire's theorem on functions in the first Baire class.

In 1948 Z. Zahorski ([7], pp. 321—323) gave two examples of functions and as he asserts they have an approximate derivative not in the first Baire class. Unfortunately his assertion is not true. In his examples the functions fail to have an approximate derivative at the points of an uncountable set. We give here the proof of this assertion.

First we recall the definition of these two functions on the interval $\langle 0, 1$. Let C be the Cantor set and (a_i, b_i) be any component of the set $\langle 0, 1\rangle - C$. Let $b_i - a_i = 3^{-n}$. Now, Zahorski defines the function f^* on $\langle 0, 1\rangle$ as follows: $f^*(x) = 0$ at the left endpoints of the components of $\langle 0, 1\rangle - C$, $f^*(x) = 1$ at other points of C and $f^*(x) = -1$ for $a_i < x \leqq a_i + 3^{-2n}$, $f^*(x) = (x - a_i - 3^{-2n}) . 2.3^{2n} - 1$ for $a_i + 3^{-2n} < x < a_i + 2.3^{-2n}$ and $f^*(x) = 1$ for $a_i + 2.3^{-2n} \leqq x < b_i$. He obtains the function f from the function f^* by modifying the graph of f^* in the neighbourhood of the points $a_i + 3^{-2n}$ and $a_i + 2.3^{-2n}$. He substitutes here the graph of f^* by an arc of the circle with the radius 3^{-2n}. The substitution will be made in such a way that the derivative of f exists at all points of (a_i, b_i). He defines the second function F as follows: $F(x) = 1 - 3^{-n}$ for all left endpoints of the components of $\langle 0, 1\rangle - C$, $F(x) = 1$ for all other points of the Cantor set C and $F(x) = - Af^2(x) + Bf(x) + C - \sqrt{(x - a_i)(x - b_i)^2}$ for all $x \in (a_i, b_i)$, where $A = 1 - 2^{-2} . 3^{-n+1} + \sqrt{1 - 2^{-1} . 3^{-n+1} + 2^{-1} . 3^{-2n}}$, $B = 2^{-1} . 3^{-n}$ and $C - 1 - 2^{-1} . 3^{-n} - A$.

Let $\{(a_i, b_i)\}_{i=1}^{\infty}$ be the sequence of all components of the set $\langle 0, 1\rangle - C$. In his first example Z. Zahorski asserts ([7], p. 322) that $f'_{\mathrm{ap}}(x) = 0$ for all such $x \in C$ in which $f(x) = 1$. That is not true. We shall prove that $f'^{-}_{\mathrm{ap}}(x) \geqq 0$ for such a point $x \in C$ ($f'^{-}_{\mathrm{ap}}(x)$ means a left approximate derivative of f in x).

For $x = b_i$ where $i = 1, 2, 3, \ldots$ we have obviously $f'^{-}_{\mathrm{ap}}(x) = 0$. Let $x \in C$ be a two-sided limit point of C. There exists an increasing sequence $\{b_{i_k}\}_{k=1}^{\infty}$ converging to x. Then the following holds: $\left\{u : \dfrac{f(u) - f(x)}{u - x} \geqq 0,\ b_{i_k} \leqq u < x\right\} \supset \bigcup \{a_i + 3(b_i - a_i)^2, b_i) : b_{i_k} \leqq a_i < x\}$. Therefore we conclude:

$$(x - b_{i_k})^{-1} \left|\left\{u : \frac{f(u) - f(x)}{u - x} \geqq 0,\ b_{i_k} \leqq u < x\right\}\right| \geqq (x - b_{i_k})^{-1} |\bigcup \{a_i + 3(b_i - a_i)^2, b_i) : b_{i_k} \leqq a_i < x\}| \geqq (x - b_{i_k})^{-1} \sum \{(b_i - a_i)(1 - 3(b_i - a_i)) : b_{i_k} \leqq a_i < x\} > (x - b_{i_k})^{-1} \sum \{(b_i - a_i)(1 - 3(x - b_{i_k})) : b_{i_k} \leqq a_i < x\} = 1 - 3(x - b_{i_k})^{(1)}.$$

It is therefore evident that $f'^{-}_{\mathrm{ap}}(x) \geqq 0$.

Now, let $\{(a_{i_k}, b_{i_k})\}_{k=1}^{\infty}$ and $\{x_k\}_{k=1}^{\infty}$ be two sequences with the following properties:

a) For $k = 1, 2, 3, \ldots$ (a_{i_k}, b_{i_k}) is a component of the set $\langle 0, 1\rangle - C$ of the length 3^{-n_k},

b) For $k = 1, 2, 3, \ldots$ x_k is a two-sided limit point of C,

c) For $k = 1, 2, 3, \ldots$ the following holds $x_k < x_{k+1} < a_{i_{k+1}} < b_{i_{k+1}} < a_{i_k}$ and $a_{i_k} - x_{k+1} < \dfrac{1}{k} 3^{-2n_k}$.

From the properties of the Cantor set C it follows that we can construct such sequences by induction. The existence of $\lim\limits_{k \to \infty} x_k$ is obvious. We denote this limit by u. Then u is a two-sided limit point of C. Therefore $f(u) = 1$.

(1) $|A|$ denotes the outer Lebesgue measure of the set A.

Now, we shall prove that $f'^{+}_{\mathrm{ap}}(x) = -\infty$. We obtain it in the following way:

Let $k = 1, 2, 3, \ldots$. Then we have $\dfrac{f(y) - f(u)}{y - u} \leqq -3^{-2n_k}\left(1 + \dfrac{1}{k}\right)^{1}$ for $y \in \langle a_{i_k}, a_{i_k} + 3^{-2n_k}\rangle$, because $f(y) \leq 0$, $y - u \leq a_{i_k} - u + 3^{-2n_k} < a_{i_k} - x_{k+1} + 3^{-2n_k} < \left(1 + \dfrac{1}{k}\right)^{-1} 3^{-2n_k}$ and $-\dfrac{1}{y - u} < 3^{-2n_k}\left(1 + \dfrac{1}{k}\right)^{-1}$.

From that we get:

$$(a_{i_k} + 3^{-2n_k} - u)^{1}\left|\left\{y : \frac{f(y) - f(u)}{y - u} < -3^{-2n_k}\left(1 + \frac{1}{k}\right)^{-1}, u < y < a_i + 3^{-2n_k}\right\}\right| > \frac{3^{-2n_k}}{3^{-2n_k}\left(1 + \frac{1}{k}\right)} = \frac{k}{k + 1}.$$

It is obvious from this that $f'^{+}_{\mathrm{ap}}(u) = -\infty$.

Now, we see the existence of points in C in which f'_{ap} does not exist. Denote by D the set of all points at which an approximate derivative does not exist. We shall show that D is an uncountable set. Suppose D is a countable set.

Let $D = \{\xi_1, \xi_2, \xi_3, \ldots\}$. We pick out an interval (a_{i_1}, b_{i_1}) of the length 3^{-n_1} and a two-sided limit point $x_1 < a_{i_1}$ of C. Let ξ_{r_1} be the first point from the sequence $\{\xi_1, \xi_2, \xi_3, \ldots\}$ belonging to the interval $\langle x_1, a_{i_1})$. Let x_2 be a two-sided limit point of C with the property $a_{i_1} - x_2 < 3^{-2n_1}$. Then we pick out an interval (a_{i_2}, b_{i_2}) of the length 3^{-n_2} for which $x_2 < a_{i_2} < a_{i_1}$. Let ξ_{r_2} be the first point from the sequence $\{\xi_1, \xi_2, \xi_3, \ldots\}$ belonging to the interval $\langle x_2, a_{i_2})$. We choose from $(\max(\xi_{r_2}, a_{i_2} - \frac{1}{2}\, 3^{-2n_2}), a_{i_2})$ a two-sided limit point x_3 of C. Now, we again pick out an interval (a_{i_3}, b_{i_3}) of the length 3^{-n_3} for which $x_3 < a_{i_3} < a_{i_2}$. It is now obvious that in such a way we can construct three sequences $\{(a_{i_k}, b_{i_k})\}_{k=1}^{\infty}$, $\{x_k\}_{k=1}^{\infty}$ and $\{r_k\}_{k=1}^{\infty}$ which satisfy the conditions a) — c) and the following one:

d) for $k = 1, 2, 3, \ldots$ and $1 \leq j < r_k$ ξ_j is not contained in $\langle x_k, a_{i_k})$.

Let $u = \lim\limits_{k\to\infty} x_k$. Then $u \notin D$ and $f'_{\mathrm{ap}}(u)$ does not exist.

In the second of Zahorski's examples, as in the first one, we can prove that $F'_{\mathrm{ap}}(x) \geq 0$ for every such point of C which is not a left endpoint of some component of $\langle 0, 1\rangle - C$ ($F'^{-}_{\mathrm{ap}}(x)$ means the upper left approximate derivative of F in x). The definition of F implies the existence of such an I that $F(u) < \frac{1}{2}$ holds for all $u \in \langle a_i + \frac{1}{2}\, 3^{-2n}, a_i + 3^{-2n})$ and $i \geqq I$, where 3^{-n} is the length of (a_i, b_i). Now, we can construct, as in the case of the first Zahorski's example, two sequences $\{(a_{i_k}, b_{i_k})\}_{k=1}^{\infty}$ and $\{x_k\}_{k=1}^{\infty}$ satisfying the conditions a)—c).

Then $F(u) = 1$ for $u = \lim_{k \to \infty} x_k$. For $i_k \geqq I$ and $y \in \langle a_{i_k} + \frac{1}{2} 3^{-2n_k}, a_{i_k} + 3^{2n_k} \rangle$ it holds $\frac{F(y) - F(u)}{y - u} < -\frac{1}{2} \frac{k}{k+1} . 3^{2n_k}$. Therefore $(a_{i_k} + 3^{2n_k} - u)^{1} \left| \left\{ y : \frac{F(y) - F(u)}{y - u} < -\frac{1}{2} \frac{k}{k+1} . 3^{2n_k} \right\} \right| \geqq \frac{k}{k+1}$ for $i_k \geqq I$. Therefore $F'^{+}_{\mathrm{ap}}(u) \geq 0$ cannot obviously hold ($F'^{+}_{\mathrm{ap}}(u)$ means the lower right approximate derivative of F in u).

In a similar way as in the case of the first Zahorski function, we can prove that the set of all points in which an approximate derivative does not exist is an uncountable set.

The paper [7] refers to the known Khintchine theorem — *every finite approximate derivative f'_{ap} satisfying $f'_{\mathrm{ap}}(x) \geqq \varphi'(x)$* for *some function φ is a derivative* — and states that it is true for infinite approximate derivatives. This statement is not true. In [2] it is proved that Khintchine theorem is also true for infinite approximate derivatives.

2.

L. E. Snyder uses his theorem on an approximate Stolz angle boundary function ([5], p. 417) to prove the theorem — *the finite approximate derivative f'_{ap} is in the first Baire class.* He considers only real finite functions; we shall prove his theorem for the case when the approximate Stolz angle boundary function also obtains ∞ and $-\infty$.

Lemma 1. *Let f be a real (not necessary finite) function defined on a perfect set P which is not continuous on P. Then there exists some segment Q of P and two numbers α and β, $\alpha < \beta$ such that $Q \subset \overline{\{x : x \in Q, f(x) \leq \alpha\}} \cap \overline{\{x : x \in Q, f(x) > \beta\}}$.*

Proof. We can suppose that $-1 \leqq f(x) \leqq 1$ for all $x \in P$. From the assumption regarding the function f we can conclude the existence of such an integer $k > 0$ and such a segment Q_0 of P that the set Q_0 is a subset of $\left\{x : x \in P, \delta(x) > \frac{1}{k}\right\}$, where $\delta(x)$ is the oscillation of f in x. But the following holds $Q_0 \subset \bigcup \{\overline{\{x : x \in Q_0, f(x) \leqq p\}} \cap \overline{\{x : x \in Q_0, f(x) \geqq q\}} : -1 \leq p < q \leqq 1$, p and q are rational numbers$\}$. Therefore there exists a segment Q and two numbers α and β, $\alpha < \beta$ such that $Q \subset \overline{\{x : x \in Q, f(x) \leqq \alpha\}} \cap \overline{\{x : x \in Q, f(x) > \beta\}}$.

Let W be the open upper half-plane. The symbol $l(x, \Theta)$ denotes the half-line from the point $(x, 0)$ whose angle of inclination is Θ. Let S_x be the Stolz angle with the vertex $(x, 0)$ which consists of the angular sector between $l\left(x, \frac{\pi}{4}\right)$ and $l\left(x, \frac{3\pi}{4}\right)$. The symbol R denotes the real line.

Theorem (L. E. Snyder) *Suppose $\Phi : W \to R$ a function and for each $x \in R$ there is a set $E_x \subset W$ such that*

(i) $(x, 0)$ *is a point of density of E_x relative to S_x and*

(ii) $f(x) = \lim \Phi(u, v)$ *exists finite or infinite as $(u, v) \to (x, 0)$ relative to the set E_x for each $x \in E$.*

Then the boundary function f of Φ determined by the family of sets $\{E_x\}_{x \in R}$ is in the first Baire class.

Proof. Let P be a perfect set, $P \neq \emptyset$. Let the function f/P (this function is the restriction of f to P) have no point of continuity. It follows from Lemma 1 that there exists some segment Q of P and two numbers α and β which satisfy $\alpha < \beta$, $Q \subset \overline{A^\beta}$ and $Q \subset \overline{A_\alpha}$, where $A^\beta = \{x : x \in Q, f(x) \geqslant \beta\}$ and $A_\alpha = \{x : x \in Q, f(x) \leqq \alpha\}$. It is easy to see that we can choose Q such that $Q \subset \left\{x : x \in P, |E_x^n| > \frac{7}{8} |S_x^n| \text{ for all } n \leqq k\right\}$, where $S_x^n = \left\{(u, v) : (u, v) \in S_x, v \leqq \frac{1}{n}\right\}$ and $E_x^n = S_x^n \cap E_x$.

Let $x \in Q$. Let $r \geqq k$ and $\varepsilon > 0$. Since A^β is dense in Q there exists in $A^\beta \cap Q$ an element y satisfying the inequality $|x - y| < \frac{1}{r}$. Then we have $E_y^r > \frac{7}{8} |S_y^r|$ and $|E_x^r| > \frac{7}{8} |S_x^r|$. It is obvious that $|S_x^r \cap S_y^r| > \frac{1}{4r^2} = \frac{1}{4} |S_x^r|$.
But this implies that $E_x^r \cap E_y^r \neq \emptyset$. Therefore there exists a sequence $\{(u_n, v_n)\}_{n=1}^\infty$ of points in E_x satisfying the inequality $\Phi(u_n, v_n) \geqq \beta - \varepsilon$ for all n. Hence it follows: $f(x) \geqq \beta - \varepsilon$. Therefore $f(x) \geqq \beta$.

In a similar way we prove that $f(x) \leqq \alpha$ for all $x \in Q$. But this is a contradiction.

We note that *Snyder's Corollary* 1 ([5], p. 419) *and the author's Theorem* 1 ([3], p. 188) *remain valid also for functions which can obtain infinite values as well.*

To prove the Preiss theorem on an approximate derivative by Snyder's method ([5], p. 421) we should add to Snyder's proof of $\lim_{n\to\infty} \Phi(x_n, r_n) = \infty$ ($\lim_{n\to\infty} \Phi(x_n, r_n) = -\infty$), where $\{(x_n, r_n)\}_{n-1}^\infty$ is a sequence of points in E_{x_0} with $(x_0, 0)$ as limit, if $f'_{ap}(x_0) = \infty$ ($f'_{ap}(x_0) = -\infty$). We shall use the symbols introduced by L. E. Snyder in the proof of his Theorem 3 ([5], p.

421), e. g. $\Phi(u, v) = \frac{1}{v}\left(f\left(u+\frac{v}{2}\right) - f\left(u-\frac{v}{2}\right)\right)$, B_{x_0} is a set, the density of which at x_0 is 1 and $\lim\limits_{x \to x_0} \frac{1}{x - x_0}\,(f(x) - f(x_0)) = f'_{ap}(x_0)$ for $x \in B_{x_0}$ and E_{x_0} is a set $\left\{(x, r) : (x, r) \in W, x_0 - \frac{r}{2} \leqq x \leqq x_0 + \frac{r}{2}, x - \frac{r}{2} \text{ and } x + \frac{r}{2} \in B_{x_0}\right\}$. Let $f'_{ap}(x_0) = \infty$ and $\{(x_n, r_n)\}_{n=1}^{\infty}$ be a sequence of points of E_{x_0} with $(x_0, 0)$ as limit. Since the points $x_n + \frac{r_n}{2}$ and $x_n - \frac{r_n}{2}$ are in B_{x_0} and $x_n + \frac{r_n}{2} \to x_0$ and $x_n - \frac{r_n}{2} \to x_0$ (2) we have:

$$\lim_{n\to\infty} \frac{f\left(x_n + \frac{r_n}{2}\right) - f(x_0)}{x_n + \frac{r_n}{2} - x_0} = \infty$$

and

$$\lim_{n\to\infty} \frac{f\left(x_n - \frac{r_n}{2}\right) - f(x_0)}{x_n - \frac{r_n}{2} - x_0} = \infty .$$

Thence we get:

$$\lim_{n\to\infty} \Phi(x_n, r_n) = \lim_{n\to\infty} \frac{f\left(x_n + \frac{r_n}{2}\right) - f\left(x_n - \frac{r_n}{2}\right)}{r_n} = \lim_{n\to\infty}\left(\frac{x_n + \frac{r_n}{2} - x_0}{r_n} \times \right.$$

$$\left. \times \frac{f\left(x_n + \frac{r_n}{2}\right) - f(x_0)}{x_n + \frac{r_n}{2} - x_0} + \frac{x_0 - x_n + \frac{r_n}{2}}{r_n} \; \frac{f(x_0) - f\left(x_n - \frac{r_n}{2}\right)}{x_0 - x_n + \frac{r_n}{2}}\right) = \infty .$$

In the case $f'_{\mathrm{ap}}(x_0) = -\infty$ the proof of $\lim\limits_{n\to\infty} \Phi(x_n, r_n) = -\infty$ is similar.

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(2) It is obvious that $x_n + \frac{r_n}{2} - x_0 \geqq 0$ and $x_0 \quad x_n + \frac{r_n}{2} \geqq 0$.

C. Goffman and C. J. Neugebauer have also proved ([1]) that the finite approximate derivative is in the first Baire class. They set $A(I, k) = \left\{(x, y) : \right.$ $\left. : x, y \in I, \dfrac{f(x) - f(y)}{x - y} > k \right\}$ and $F(I) = \sup \left\{ k : \dfrac{|A(I, k)|}{|I|^2} > \dfrac{1}{2} \right\}$ for every interval I and real number k. Further they have proved that $f'_{\text{ap}}(x_0) = \lim_{n\to\infty} F(I_n)$, where $x_0 \in \cap \{I_n : n = 1, 2, 3, \ldots\}$ and $|I_n| \to 0$, if the approximate derivative $f'_{\text{ap}}(x_0)$ exists and is finite. The equality would not hold, if $f'_{\text{ap}}(x_0) = \infty$. For instance: let $f(x) = \operatorname{sign} x$. Then $f'_{\text{ap}}(0) = f'(0) = \infty$. Let $I_n = \left(-\dfrac{1}{n}, \dfrac{1}{n}\right)$ and ε any positive number less than 1. Then $A(I_n, \varepsilon) \subset$ $\subset \{(x, y) : x, y \in I_n, xy < 0\}$. $|A(I_n, \varepsilon)| \leqq 2\left(\dfrac{1}{n}\right)^2 = \dfrac{1}{2}|I_n|^2$ holds for any n. Therefore $F(I_n) \leqq 0$ for any n and $f'_{\text{ap}}(0) \neq \lim_{n\to\infty} F(I_n)$.

For these reasons we cannot complete the proof of C. Goffman and C. J. Neugebauer for the cases $f'_{\text{ap}}(x) = \infty$ and $f'_{\text{ap}}(x) = -\infty$ and give in this way the proof of Preiss's theorem.

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