Ján Jakubík
The Interval Topology of an $l$-Group


Persistent URL: [http://dml.cz/dmlcz/126327](http://dml.cz/dmlcz/126327)
THE INTERVAL TOPOLOGY OF AN L-GROUP

JÁN JAKUBÍK, Košice

Let $G$ be an $l$-group, $a, c \in G$. We shall call $a$ an archimedean element, if $a > 0$ and if for each $b \in G$ there exists a positive integer $n$ such that $na \geq b$. The sets

$$I_1(c) = \{x \in G, x \leq c\}, \quad I_2(c) = \{x \in G, x \geq c\}$$

are called infinite intervals (in $G$). The interval topology of $G$ is defined by taking as a sub-basis for the closed sets all infinite intervals and the set $G$. We will consider the following condition:

(t) $G$ is a topological group in its interval topology.

G. Birkhoff [1, p. 233, problem 104] has asked the question: Does any $l$-group satisfy the condition (t)? It is a rather trivial fact that any ordered (= linearly ordered) $l$-group satisfies (t). E. S. Northam [4, proposition 6] proved that the additive group $A$ of all continuous real-valued functions defined on the closed unit interval (using the natural ordering) is an $l$-group which does not satisfy (t).

T. H. Choe [3] has shown: If each non-empty subset $M \subset G^+$ has a minimal element and if $G$ satisfies (t), then $G$ is ordered. In the recent paper [2] P. Conrad studies $l$-groups which fulfill the condition $(F)$: Each $a \in G$, $a > 0$ is greater than or equal to at most a finite number of disjoint elements. (The elements $c, d \in G$ are called disjoint if $c \cap d = 0$.) It is proved in [2, theorem 6.3]: If $G$ satisfies the conditions $(F)$ and (t) then $G$ is ordered. (Evidently this theorem includes the result of Choe but not that of Northam.) In this note we prove the following

**Theorem.** If there exist disjoint archimedean elements $a, b \in G$ then $G$ does not satisfy (t).

**Corollary.** Any archimedean $l$-group satisfying (t) is ordered.

Clearly this implies the result of Northam. Since an $l$-group in which each non-empty subset $M \subset G^+$ has a minimal element is archimedean (this follows easily from [1, p. 236, Theorem 21]) the result of Choe is also a consequence of the corollary.

1. Let $a, b \in G$, $a > 0$, $b > 0$, $a \cap b = 0$. Let $I$ be the set of all integers, $A = \{x | x = ma, m \in I\}$, $B = \{y | y = nb, n \in I\}$, $C = \{z | z = x + y, x \in A, y \in B\}$. Then

a) $C$ is an $l$-subgroup of $G$, and

b) $C$ is isomorphic with the direct product ([1, p. 222]) of $l$-groups $A, B$. 

14 Matematicko-fyz. čas. 3 209
Proof. Let \( m, n \in I, m > 0, n > 0, m_i, n_i \in I, i = 1, 2 \). From \( a \cap b = 0 \) follows (cf. [1, p. 219]) \( ma \cap nb = 0, ma + nb = ma \cup nb = nb + ma \), hence \( m_i a + n_i b = n_i b + m_i a \). Therefore \( C \) is a subgroup of the group \( G \). Let \( m_3 = \max (m_1, m_2) \), \( n_3 = \max (n_1, n_2) \), \( m_4 = \min (m_1, m_2) \), \( n_4 = \min (n_1, n_2) \), \( z_i = m_i a + n_i b \). If \( m_i, n_i \) \((i = 1, 2)\) are non-negative, then \( z_i = m_i a \cup n_i b \), hence (because of the distributivity of \( G \))

\[
\begin{align*}
z_1 \cup z_2 &= m_3 a + n_3 b, \\
z_1 \cap z_2 &= m_4 a + n_4 b.
\end{align*}
\]

If \( m_i, n_i \) are arbitrary, we choose \( m, n \) such that \( m + m_i \geq 0, n + n_i \geq 0, i = 1, 2 \); let \( \circ \in \{ \cap, \cup \} \). From

\[
z_1 \circ z_2 = ((z_1 + z) \cap (z_2 + z)) - z
\]

follows that in this case (1) also holds. Thus the assertion a) is proved. It is now immediate that the mapping \( C \to A \times B \) defined by \( ma + nb \to (ma, nb) \) is an isomorphism.

In the following \( C \) has the same meaning as above.

2. Let \( a, b \) be archimedean elements. Let \( u \in G, A = I_1(u) \cap C \neq 0 \). Then \( A \) is an infinitive interval in \( C \).

Proof. Let \( m_o a + n_o b \in A \). Put \( M = \{ m | m \in I, ma + n_o b \leq u \} \). Since \( a \) is archimedean, there exists the greatest element \( m_1 \) in \( M \). Denote \( N = \{ n | n \in I, m_1 a + nb \leq u \} \); there exists the greatest element \( n_1 \) in \( N \). If \( m_1 a + n_1 b \leq ma + nb \leq u \), then \( n_1 \leq n \), \( m_1 a + nb \leq u \), hence \( n_1 = n \); moreover \( m_1 \leq m \), \( ma + n_0 b \leq u \), thus \( m = m_1 \). This shows that \( c_1 = m_1 a + n_1 b \) is the greatest element of \( A \). Evidently each element \( c \in C, c + c_1 \) belongs to \( A \).

A similar result holds for \( I_2(u) \cap C \).

3. Let \( A, B \) be nonzero ordered groups, \( D = A \times B \). Then \( D \) is not Hausdorff in its interval topology.

This assertion is proved (though not explicitly stated) in [2, proof of the lemma 6.2].

4. Proof of the theorem. Let \( a, b \) be disjoint archimedean elements of \( G \). Let \( p, q \in C, p \neq q \). Suppose that \( G \) satisfies \((t)\). Then there exist infinite intervals \( I^1, \ldots, I^n \) such that \( \cup I^i = G \) and no \( I^i \) contains both \( p \) and \( q \) (this follows easily from the definition of the sub-basis; cf. also [2, proof of the lemma 6.5, and 6.4]). It follows from 2 that the set \( \cup J_i = C \) is an infinite interval in \( C \) or \( J_i = \emptyset \); clearly \( \cup J_i = C \) and no \( J_i \) contains both \( p \) and \( q \). Hence \( C \) in Hausdorff in its interval topology. But from 1 and 3 we obtain that \( C \) is not Hausdorff, and we have a contradiction.

210
ИНТЕРВАЛЬНАЯ ТОПОЛОГИЯ В L-ГРУППАХ

Ян Якубик

Резюме

Пусть $G$ — l-группа; $a, c \in G$. Элемент $a$ называется архимедовым, если $a > 0$ и если для каждого $b \in G$ существует натуральное число $n$ такое, что $na \leq b$. Множества

$$I_1(c) = \{x \mid x \in G, x \leq c\}, \quad I_2(c) = \{x \mid x \in G, x \geq c\}$$

называются бесконечными интервалами в $G$. Интервальная топология в $G$ определена так, что в качестве суббазы замкнутых множеств берется система, состоящая из всех бесконечных интервалов и из множества $G$. Мы говорим, что $G$ обладает свойством $(t)$, если $G$ — топологическая группа в интервальной топологии. Доказана следующая

Теорема. Если в $G$ существуют архимедовы элементы $a, b, a \cap b = 0$, то $G$ не обладает свойством $(t)$.

Следствие. Архимедова l-группа, обладающая свойством $(t)$, является упорядоченной.

Из этого вытекают как частные случаи теоремы Нортгама [4] и Чои [3], касающиеся интервальной топологии в l-группе.