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ON THE CONNECTIONS ON THE PROLONGATIONS OF PRINCIPAL FIBRE BUNDLES

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The prolongations of a principal fibre bundle were studied from different points of view by Kolář [5], Gollek [4], Virsík [7]. For our purpose the approach by Kolář seems to be the most suitable. The non-holonomic prolongation $\tilde{W}^r(P)$ of a principal fibre bundle $P(B, G)$ has the structure of a principal fibre bundle of the symbol $\tilde{W}^r(P)\,(B, \tilde{G}^r_n)$, $n = \dim B$, and can be identified with the fibre product $\tilde{H}^r(B) \oplus \tilde{J}^r(P)$. The structure group $\tilde{G}^r_n$ coincides with the semi-direct product $\tilde{L}^r_n \bar{\mathrm{x}} \tilde{T}^r_n(G)$ with respect to the jet-action of $\tilde{L}^r_n$ on $\tilde{T}^r_u(G)$. The canonical projection $j^s_r (s < r)$ of r-jets into the underlying s-jets determines on $\tilde{W}^r(P)$ the structure of a principal fibre bundle over $\tilde{W}^s(P)$. In the present paper we study the connections on the latter bundle. The standard terminology and notations of the theory of jets, (see [3]), are used throughout the paper. Our considerations are in the category C^∞.

1. $\tilde{T}^r_n(M)$ denotes the set of the all non-holonomic r-jets of R^n into M with the source $0 \in R^n$. Let $1 \leqq s < r$ and $Y \in \tilde{T}^s_n(M)$. Let t_z denote the translation of R^n from $0 \in R^n$ into $z \in R^n$. We put $j(Y) = j^{r-s}_0(Yt_z^{-1})$. We have an injection $j\colon \tilde{T}^s_n(M) \to \tilde{T}^r_n(M)$. We will dedenote by $(\tilde{T}^r_n(G))^{[s]}$ the submanifold $j(\tilde{T}^s_n(M))$.

Let us recall that G^1_n is the set of the all 1-jets of local isomorphisms $\psi\colon R^n \mathrm{x} G \to R^n \mathrm{x} G$ with the source $(0, e)$, where e denotes the unit of G, and that $\tilde{G}^r_n = (\tilde{G}^s_n)^{r-s}_n$. It is well known that $\tilde{L}^r_n$ coincides with $(\tilde{L}^s_n)^{r-s}_n$, $1 \leqq s < r$. Then we can identify

$$\tilde{L}^r_n = \tilde{L}^{r-s}_n \,\bar{\mathrm{x}}\, \tilde{T}^{r-s}_n\,(\tilde{L}^s_n).$$

Let now e be the unit of $\tilde{L}^{r-s}_n$. Denote by i the mapping

$$i\colon \tilde{L}^s_n \to \tilde{L}^r_u \equiv \tilde{L}^{r-s}_u \mathrm{x}\, \tilde{T}^{r-s}_n(\tilde{L}^s_n), \quad i(g) = (e, j^{r-s}_0 \hat{g}),$$

where $\hat{g}$ denotes the constant mapping $R^n \to g \in \tilde{L}^s_n$. We have

$$i(g_1 g_2) = (e, j^{r-s}_0 g_1 g_2) = (e, (j^{r-s}_0 \hat{g}_1) \,.\, (j^{r-s}_0 \hat{g}_2)) = (e, j^{r-s}_0 \hat{g}_1)\,(e, j^{r-s}_0 g_2),$$

where the dot denotes the group composition on $\tilde{T}^{r-s}_n(\tilde{L}^s_n)$ and thus i is a group monomorphism. The subgroup $i(\tilde{L}^s_n) \subset \tilde{L}^r_n$ will be denoted by $(\tilde{L}^r_n)^{[s]}$. It is easy to see that $(\tilde{T}^r_n(G))^{[s]}$ is a subgroup of $\tilde{T}^r_n(G)$ and that $j: \tilde{T}^s_n(G) \to (\tilde{T}^r_n(G))^{[s]}$ is an isomorphism. Let $h \in (\tilde{L}^r_n)^{[s]}$, $h = i(g) = (e, j_0^{r-s}\hat{g})$. Let $u \in (\tilde{T}_0^{s-r}(G))^{[s]}$, $u = j(Y) = j_0^{r-s}(Yt_z^{-1})$. Let uh denote the jet composition. It is obvious

$$uh = j_0^{r-s}(Ygt_z^{-1}) \in (\tilde{T}^r_n(G))^{[s]},$$

where Yg is the jet composition of $g \in \tilde{L}^s_n$ and of $Y \in \tilde{T}^s_n(G)$. According to this jet action of $(\tilde{L}^r_n)^{[s]}$ on $(\tilde{T}^r_n(G))^{[s]}$ we put

$$(\tilde{G}^r_n)^{[s]} \stackrel{\text{def}}{=} (\tilde{L}^r_n)^{[s]} \text{x} (\tilde{T}^r_n(G))^{[s]} \subset \tilde{G}^r_n = \tilde{L}^r_n \text{x} \tilde{T}^r_n(G) .$$

It is not difficult to see that

$$\psi \equiv (i, j): \tilde{L}^s_n \bar{\text{x}} \tilde{T}^s_n(G) \to (\tilde{G}^r_n)^{[s]}$$

is an isomorphism of the groups $\tilde{G}^s_n$ and $(\tilde{G}^r_n)^{[s]}$.

Lemma 1. *The restriction of the homomorphism j^r_s: $\tilde{G}^r_n \to \tilde{G}^s_n$ to the subgroup $(\tilde{G}^r_n)^{[s]}$ is an isomorphism.*

The proof is obvious.

Denote by ${}^s\tilde{G}^r_n$ the kernel of the homomorphism j^s_r: $\tilde{G}^r_n \to \tilde{G}^s_n$. The isomorphism $\psi = (i, j)$ is a splitting of the exact sequence

$$O \to {}^s\tilde{G}^r_n \to \tilde{G}^r_n \underset{\psi}{\rightleftarrows} \tilde{G}^s_n \to O .$$

Now we can identify

$$\tilde{G}^r_n = (\tilde{G}^r_n)^{[s]} \text{x}\, {}^s\tilde{G}^r_n . \tag{1}$$

If $1 \leq s \leqq r - 1$, then the group ${}^{r-1}\tilde{G}^r_n$ is a subgroup of ${}^s\tilde{G}^r_n$. Since ${}^{r-1}\tilde{G}^r_n = \ker j^{r-1}_r$, ${}^{r-1}\tilde{G}^r_n$ is normal in ${}^s\tilde{G}^r_n$.

Denote by $({}^s\tilde{G}^r_n)^{[r-1]}$ the group ${}^s\tilde{G}^r_n \cap (\tilde{G}^r_n)^{[r-1]}$. Then

$$j^{r-1}_r : ({}^s\tilde{G}^r_n)^{[r-1]} \to j^{r-1}_r({}^s\tilde{G}^r_n)$$

is an isomorphism. By the procedure used in (1) we get the identification

$${}^s\tilde{G}^r_n \equiv ({}^s\tilde{G}^r_n)^{[r-1]} \text{ x } {}^{r-1}\tilde{G}^r_n, \tag{2}$$

where on the right side of (2) there is the semi-direct product of the groups with respect to the action $\tilde{h}(g) = h^{-1}gh$ of the group $({}^s\tilde{G}^r_n)^{[r-1]}$ on the group ${}^{r-1}\tilde{G}^r_n$.

We shall denote by the $\tilde{\mathscr{G}}^r_n$, $\tilde{\mathscr{G}}^{r[s]}_n$, $({}^s\tilde{\mathscr{G}}^r_n)^{[r-1]}$, ${}^{r-1}\tilde{\mathscr{G}}^r_n$ Lie-algebras of the groups $\tilde{G}^r_n$, $(\tilde{G}^r_n)^{[s]}$, $({}^s\tilde{G}^r_n)^{[r-1]}$, ${}^{r-1}\tilde{G}^r_n$. It follows immediately from (1) and (2) that

$$\tilde{\mathscr{G}}^r_n = (\tilde{\mathscr{G}}^r_n)^{[s]} \oplus ({}^s\tilde{\mathscr{G}}^r_n)^{[r-1]} \oplus {}^{r-1}\tilde{\mathscr{G}}^r_n . \tag{3}$$

We shall use the identification

$$(4) \qquad \tilde{\mathscr{G}}_n^{r-1} \equiv (\tilde{\mathscr{G}}_n^r)^{[s]} \oplus ({}^s\tilde{\mathscr{G}}_n^r)^{[r-1]} ,$$

which is induced by the isomorphism

$$j_r^{r-1} : ({}^s\tilde{G}_n^r)^{[r-1]} \times (\tilde{G}_n^r)^{[s]} \equiv (\tilde{G}_n^r)^{[r-1]} \to \tilde{G}_n^{r-1} .$$

2. The space $\tilde{W}^r(P)$ has the structure of the principal fibre bundle $\tilde{W}^r(P)\,(\tilde{W}^s(P),\ {}^s\tilde{G}_n^r, j_r^s)$ with the base $\tilde{W}^s(P)$, with the structure group ${}^s\tilde{G}_n^r$ and with the fibre projection j_r^s. This structure will be denoted by $\tilde{W}_s^r(P)(\tilde{W}^r(P))$ always denotes the structure $\tilde{W}^r(P)(B, \tilde{G}_n^r))$. Let Γ be a connection on $\tilde{W}_s^r(P)$, i.e. Γ is a ${}^s\tilde{G}_n^r$-invariant mapping $\tilde{W}_s^r(P) \to J^1\tilde{W}_s^r(P)\colon \Gamma(wh) = \Gamma(w)h$ for any $w \in \tilde{W}_s^r(P)$ and $h \in {}^s\tilde{G}_n^r$. Then the canonical decomposition

$$(5) \qquad T(\tilde{W}_s^r(P)) = T_0 \oplus T_1 \oplus T_2 ,$$

where T_0 or T_1 at $w \in \tilde{W}_s^r(P)$ denotes the tangent subspace of the orbit ${}^{r-1}\tilde{G}_n^r(w)$ or $({}^s\tilde{G}_n^r)^{[r-1]}(w)$, respectively, and T_2 is the horizontal tangent subspace determined by $\Gamma(w)$, is given at any point $w \in \tilde{W}_s^r(P)$.

Let $g \in \tilde{G}_n^r$. Then $g = j_{(o,e)}^1\psi$, where ψ is such a local bundle isomorphism of $R_n \times \tilde{G}_n^{r\,1}$ that $\psi(o, e) = (0, q)$, e is the unit of $\tilde{G}_n^{r-1}$. Let $X = j_o^1\gamma(t) \in T_{(o,e)}(R_n \times \tilde{G}_n^{r-1})$. Put

$$(6) \qquad \varrho(g)X = j_o^1\, \psi(\gamma(t))q^{-1}] .$$

ϱ is a representation of $\tilde{G}_n^r$ on $R^n \oplus \tilde{\mathscr{G}}_n^{r-1}$ (see [5]). ϱ induces the following bilinear mapping $\bar{\varrho}$. If $X \in R^n \oplus \tilde{\mathscr{G}}_n^{r-1}$ and $Y \in \tilde{\mathscr{G}}_n^r$, $Y = j_0^1\gamma(t)$, then

$$(7) \qquad \bar{\varrho}(Y)X = j_o^1\varrho(\gamma(t))X .$$

Let Θ^r be the canonical form on $\tilde{W}_s^r(P)$ (see [5] or [2]). Let us recall that Θ^r is a 1-form on $\tilde{W}_s^r(P)$ with values in $R^n \oplus \tilde{\mathscr{G}}_n^{r-1}$ and that $\Theta^r(T_0) = 0$, $\Theta^r R_{g*}(X) = \varrho(g^{-1})\Theta^r(X)$, $X \in T(\tilde{W}^r(P))$, $g \in \tilde{G}_n^r$. Let $U \in ({}^s\tilde{\mathscr{G}}_n^r)^{[r-1]}$. Let $Y = -O + U + O \in (\tilde{\mathscr{G}}_n^r)^{[s]} \oplus ({}^s\tilde{\mathscr{G}}_n^r)^{[r-1]} \oplus {}^{r-1}\tilde{\mathscr{G}}_n^r = \tilde{\mathscr{G}}_n^r$. Let Y be the fundamental vector field on $\tilde{W}_s^r(P)$ determined by $\overline{Y}$. It follows from the definition of Θ^r that according to (4)

$$(6) \qquad \Theta^r(Y) = O + O + U \in R^n \oplus (\tilde{\mathscr{G}}_n^r)^{[s]} \oplus ({}^s\tilde{\mathscr{G}}_n^r)^{[r-1]} .$$

If Y_1, Y_2 are fundamental vector fields determined by $U_1, U_2 \in ({}^s\tilde{\mathscr{G}}_n^r)^{[r-1]}$, then the field $[Y_1, Y_2]$ is determined by $[U_1, U_2]$ and thus (6) yields

$$\Theta^r\, [Y_1, Y_2] = [U_1, U_2] .$$

Let $p_i\colon T(\tilde{W}_s^r(P)) \to T_i$ be the natural projection, $i = 0, 1, 2$. Then

$$\Theta_1^r = \Theta^r p_1$$

is a 1-form on $\tilde{W}_s^r(P)$ with values in $0 \oplus 0 \oplus ({}^s\tilde{\mathscr{G}}_n^r)^{[r-1]} \subset R^n \oplus \tilde{\mathscr{G}}_n^{r-1}$.

Remark 1. If ξ, ζ are $({}^s\tilde{G}_n^r)^{[r-1]}$-fundamental vector fields on $\tilde{W}_s^r(P)$, then

$$d\Theta^r(\xi, \zeta) = d\Theta_1^r(\xi, \zeta) .$$

Let ω be a 1-form on a manifold M with values in a vector space V. Let ψ be a 1-form on M with values in the vector space of the linear transformations of V. We shall denote by $[\psi \wedge \omega]$ a 2-form on M with values in V defined by

$$[\psi \wedge \omega]\,(X, Y) = \psi(X)\omega(Y) - \psi(Y)\omega(X) .$$

As we recalled above, $\bar{\varrho}(X)$, $(X \in \tilde{\mathscr{G}}_n^1)$, is a linear transformation of $R^n \oplus \tilde{\mathscr{G}}_n^{r-1}$. Let φ be the canonical form of a connection Γ on $\tilde{W}_s^r(P)$. Let $X \in T(\tilde{W}_s^r(P))$. Then $\bar{\varrho}(\varphi(X))$ is a linear transformation of $R^n \oplus \tilde{\mathscr{G}}_n^{r-1}$. We shall write $\varphi(X)\Theta^r(Y)$ instead of $\bar{\varrho}(\varphi(X))\Theta^r(Y)$.

Theorem 1. *(The structure equations of the connection Γ.) Let φ be the form of the connection Γ on $\tilde{W}_s^r(P)$, $1 \leq s < r$. Then*

$$d\Theta^r = -[\varphi \wedge (\Theta^r - \Theta_1^r)] - 1/2\,[\Theta_1^r, \Theta_1^r] + D\Theta^r \tag{8}$$

$$d\varphi = -1/2[\varphi, \varphi] + \Phi , \tag{9}$$

where $D\Theta^r = d\Theta^r p_2$ and Φ is the curvature form of Γ.

Proof. The equation (9) is known from the theory of connection. To prove (8) we use the standard procedure. Denote by X or Y a fundamental vector field on $\tilde{W}_s^r(P)$ determined by an element of Lie algebra ${}^{r-1}\tilde{\mathscr{G}}_n^r$ or $({}^s\tilde{\mathscr{G}}_n^r)^{[r-1]}$ respectively; further the letter Z will denote a horizontal ${}^s\tilde{G}_n^r$-invariant vector field on $\tilde{W}^r(P)$. Our problem is local. There is locally on $\tilde{W}_s^r(P)$ such a basis of $T(\tilde{W}_s^r(P))$ determined by the vector fields of the types X, Y, Z that $(Y, Z] = 0$, $[X, Z] = 0$. It is sufficient to prove (8) for the elements of this basis. The definition of $d\Theta^r$ yields

$$d\Theta^r(\xi, \zeta) = \xi\Theta^r(\zeta) - \zeta\Theta^r(\xi) - \Theta^r[\xi, \zeta] .$$

Denote by Ω the form on the right-hand side of (8). There are the following cases:

a. $\xi = Z_1, \zeta = Z_2$. Then $d\Theta^r(Z_1, Z_2) = D\Theta^r(Z_1, Z_2) = \Omega(Z_1, Z_2)$.

b. $\xi = Y, \zeta = Z$. Then $[Y, Z] = 0$ and $\Theta^r(Y)$ is constant and thus $X\Theta^r(Y) = 0$. Therefore $d\Theta^r(Y, Z) = Y\Theta^r(Z)$. Let Y be generated by $\bar{Y} \in ({}^s\tilde{\mathscr{G}}_n^r)^{[r-1]}$. Since Z is ${}^s\tilde{G}_n^r$-invariant, Lemma 3 of [1] (p. 111) yields

$$Y\Theta^r(Z) = -\bar{\varrho}(\bar{Y})\Theta^r(Z) = -\varphi(Y)\Theta^r(Z) .$$

On the other hand $\Omega(Y, Z) = -\varphi(Y)\Theta^r(Z)$.

c. $\xi = X$, $\zeta = Z$. Now $[X, Z] = 0$ and $\Theta^r(X) = 0$. Further as in the case *b.*

d. $\xi = Y_1, \zeta = Y_2$. Then $\Theta^r(Y_1)$, $\Theta^r(Y_2)$ are constant and $Y_2\Theta^r(Y_1) = 0$, $Y_1\Theta^r(Y_2) = 0$. Now $d\Theta^r(Y_1, Y_2) = -\Theta^r[Y_1, Y_2] = -[\Theta^r(Y_1), \Theta^r(Y_2)] = -[\Theta^r_1(Y_1), \Theta^r_1(Y_2)]$. On the other hand $\Omega(Y_1, Y_2) = 1/2[\Theta^r_1, \Theta^r_1](Y_1, Y_2) = -[\Theta^r_1(Y_1), \Theta^r_1(Y_2)]$.

e. $\xi = X$, $\zeta = Y$. Then $\Theta^r(X) = 0$ and $\Theta^r(Y)$ is constant. Because ${}^{r-1}\tilde{\mathscr{G}}^r_n$ is an ideal in $\tilde{\mathscr{G}}^r_n$, $[X, Y] \in T_0$ and $\Theta^r[X, Y] = 0$. Therefore $d\Theta^r(X, Y) = 0$. Since $\Theta^r(Y) = \Theta^r_1(Y)$, we have $\Omega(X, Y) = 0$.

f. $\xi = X_1, \zeta = X_2$. In this case, the values of the forms on the left and right-hand sides of (8) are 0 . QED.

Remark 2. Let $g \in {}^{r-1}\tilde{G}^r_n$ and $Y \in \tilde{\mathscr{G}}^{r-1}$. Then (6) yields

$$\varrho(g)Y = Y.$$

Therefore $\bar{\varrho}(X)(Y) = 0$ for any $X \in {}^{r-1}\tilde{\mathscr{G}}^r_n$. Let $X, Y \in ({}^s\tilde{\mathscr{G}}^r_n)^{[r-1]}$. It follows from (7) that

$$\bar{\varrho}(X)(Y) = adX(Y) = [X, Y].$$

Now one can prove easily the following relation

$$[\varphi \wedge \Theta^r_1] = [\Theta^r_1, \Theta^r_1].$$

Then the structure equation can be modified as follows

$$(8') \qquad d\Theta^r = -[\varphi \wedge \Theta^r] + 1/2[\Theta^r_1, \Theta^r_1] + D\Theta^r.$$

Remark 3. We can extend our considerations to the case $s = 0, r = 2, 3, \ldots$ putting

$$(\tilde{G}^r_n)^{[o]} = \{e\}\bar{x}\,{}^oG \subset \tilde{L}^r_n\,\bar{x}\tilde{T}^r_n(G) = \tilde{G}^r_n\,,$$

$${}^o\tilde{G}^r_n = \ker j^o_r\,,$$

where e is the unit of $\tilde{L}^r_n$, ${}^oG = \{j^r_o\hat{g} : g \in G, \hat{g} : R^n \to g\}$ and j^o_r is a group homomorphism $\tilde{G}^r_n \to \bar{G} = \{(0, g) \in R^n\mathrm{x}G : g \in G\}$. $\tilde{W}^r_o(P)$ is a principal fibre bundle of the symbol

$$\tilde{W}^r_o(P)\,(P, {}^o\tilde{G}^r_n, j^o_r)\,.$$

It is not difficult to see that (8), (9) are the structure equations of a connection Γ on $\tilde{W}^r_o(P)$.

Remark 4. In the case of the principal fibre bundle $\tilde{W}^r(P)$,

$$\tilde{G}^r_n = (\tilde{G}^r_n)^{[r-1]}\mathrm{x}\;{}^{r-1}\tilde{G}^r_n\,.$$

Let Γ be a connection on $\tilde{W}^r(P)$. Then T_1 at $w \in \tilde{W}^r(P)$ in the decomposition

(5) is the tangent subspace of the orbit $(\tilde{G}^r_n)^{[r-1]}(w)$ and the notations of T_0, T_2 do not change. Using

$$\tilde{\mathscr{G}}^r_n = (\tilde{\mathscr{G}}^r_n)^{[r-1]} \oplus {}^{r-1}\tilde{\mathscr{G}}^r_n ,$$
$$\tilde{\mathscr{G}}^{r-1}_n = (\tilde{\mathscr{G}}^r_n)^{[r-1]}$$

instead of (3), (4) and replacing $({}^s\tilde{G}^r_n)^{[r-1]}$ by $(\tilde{G}^r_n)^{[r-1]}$ we can repeat any of our considerations in Theorem 1 and Remark 2. Therefore (8) (or (8')) and (9) are the structure equations of a connection Γ on $\tilde{W}^r(P)$.

Consider now a connection Γ on $\tilde{W}^r_{r-1}(P)$, $r \geqq 1$. Since $({}^{r-1}\tilde{G}^r_n)^{[r-1]} = {}^{r-1}\tilde{G}^r_n \cap (\tilde{G}^r_n)^{[r-1]} = e$, then $V_1 = 0$ and $\Theta^r_1 = 0$. We obtain for (8)

$$d\Theta^r = - [\varphi \wedge \Theta^r] + D\Theta^r .$$

That is the well-known equation from the theory of the linear connections. The connection Γ on $\tilde{W}^r_{r-1}(P)$ will be said to be the r-linear connection on P. The form $D\Theta^r$ will be called the torsion form of the r-linear connection Γ. The group ${}^{r-1}\tilde{G}^r_n$ is the set of all 1-jets of the local isomorphisms of the space $R^n \text{x} \tilde{G}^{r-1}_n$ with the source and the target $(0, e)$. Let dim $\tilde{G}^{r-1}_n = k$. We can identify locally $R^n \text{x} \tilde{G}^{r-1}_n$ with R^{n+k} and then ${}^{r-1}\tilde{G}^r_n$ is a subgroup of L^1_{n+k}. It follows from the definition of $\tilde{W}^r(P)$ that $\tilde{W}^r(P)$ can be considered in the sense of the local identification $R^n \text{x} \tilde{G}^{r-1}_1 = R^{n+k}$ as a reduction of $H^1(\tilde{W}^{r-1}(P))$ to the subgroup ${}^{r-1}\tilde{G}^r_n \subset L^1_{n+k}$. Now, well-known the result from the theory of linear connections yields.

Assertion. *The r-linear connection Γ is without torsion if and only if $\Gamma(\tilde{W}^r(P)) \subset \subset H^2(\tilde{W}^{r-1}(P))$.*

3. We first recall a construction by Kolář (see [6]). Let $M(N, G, \pi)$ and $N(B, H, p)$ be two principal fibre bundles. Assume

a) H acts as a homomorphism on the right on G: $(g, h) \to gh$,

b) H acts on the right on M through $(U, h) \to U\tilde{h}$ in such a way that $\pi(U\tilde{h}) = (\pi U)h$,

c) $(Ug)\tilde{h} = (U\tilde{h})(gh)$ for every $U \in M$, $g \in G$, $h \in H$.

Let $H\bar{x}G$ be the semi-direct product with multiplication

$$(h_1, g_1)(h_2, g_2) = (h_1h_2, (g_1h_2)g_2) .$$

The action $U(h, g) = (U\tilde{h})g$ of $H\bar{x}G$ on M and the projection $p_0\pi$ impart to M a structure of a principal fibre bundle over B with structure group $H\bar{x}G$. Denote this structure by $\bar{M}$. We are going to study some properties of connections on N, M and $\bar{M}$.

Let Γ_1 be a connection on N and Γ_2 be a connection on $M(N, G, \pi)$. Let $U \in M$, $u = \pi U$. Let $\Gamma_1(u) = j^1_x\gamma_1(y)$, $\Gamma_2(U) = j^1_u\gamma_2$. We can define

$$\Gamma_2 \circ \Gamma_1(U) = j^1_x\gamma_2(\gamma_1(y)) .$$

We obtain a global cross-section $\Gamma_2 \circ \Gamma_1 : M \to J^1(\bar{M})$, which is $G \equiv e\bar{x}G$-invariant because of

$$\Gamma_2 \circ \Gamma_1(Ug) = j^1_x[\gamma_2(\gamma_1(y))]g = [\Gamma_2 \circ \Gamma_1(U)]g, \quad g \in G.$$

Definition. *The connection Γ_2 will be called H-conjugate with the connection Γ_1 if the cross-section $\Gamma_2 \circ \Gamma_1$ is $(H\bar{x}\bar{e})$-invariant.*

Remark 5. If a connection Γ_2 on M is H-invariant, i.e.

$$\Gamma_2(U\tilde{h}) = \Gamma_2(U)\tilde{h},$$

then Γ_2 is H-conjugate with any connection Γ_1 on N.

Lemma 2. *The connection Γ_2 on M is H-conjugate with a connection Γ_1 on N if and only if $\Gamma_2 \circ \Gamma_1$ is a connection on M.*

The proof is obvious.

Let φ_1 or φ_2 be the connection form of Γ_1 or Γ_2, respectively. Let $\mathscr{H}$ or $\mathscr{G}$ be the Lie algebra of H or G, respectively. Then $\mathscr{H} \oplus \mathscr{G}$ is the Lie algebra of $H \bar{x} G$. Denote by i and j the following injections

$$i : H \to H\bar{x}G, \quad i(h) = (h, e_2),$$

$$j : G \to H\bar{x}G, \quad j(g) = (e_1, g),$$

where e_1 or e_2 is the unit of H or G, respectively.

Denote by

$$\bar{\varphi}_1 \equiv i_* \,.\, \pi^*\varphi_1, \quad \text{i.e.}$$

if $X \in T_U(\bar{M})$ then $\bar{\varphi}_1(X) = i_*\, \varphi_1(\pi_*(X))$.

Lemma 3. *The form $\bar{\varphi}_1$ is a π-horizontal $(H\bar{x}e_2)$-equivariant vector 1-form on $\bar{M}$ with values in $\mathscr{H} \oplus 0 \subset \mathscr{H} \oplus \mathscr{G}$.*

Proof. Let $X \in T_U(\bar{M}_{\pi U})$. Then $\pi_*(X) = 0$ and thus $\bar{\varphi}_1$ is π-horizontal. Further, being commutative, the diagram

$$\begin{array}{ccc} T(N) & \xrightarrow{\;h_*\;} & T(N) \\ \downarrow \varphi_1 & & \downarrow \varphi_1 \\ \mathscr{H} & \xrightarrow{\;Ad(h^{-1})\;} & \mathscr{H} \end{array}$$

induces the commutability of the diagram

$$\begin{array}{ccc} T(\bar{M}) & \xrightarrow{\;(h,e_2)\;} & T(\bar{M}) \\ \bar{\varphi}_1 \downarrow & & \searrow \bar{\varphi}_1 \\ \mathscr{H} \oplus 0 & \xrightarrow{\;Ad(h,e)^{-1}\;} & \mathscr{H} \oplus 0, \; h \in H. \end{array}$$

But it means that $\bar{\varphi}_1$ is $(H\bar{x}e_2)$-equivariant.

The cross-section $\Gamma_2 \circ \Gamma_1$ determines on $\bar{M}$ a distribution of n-dimensional tangent subspaces V_2. We have the decomposition

$$T(\bar{M}) = V_0 \oplus V_1 \oplus V_2 ,$$

where V_0 or V_1 at $U \in \bar{M}$ is the tangent subspace of the orbit $(e_1\bar{x}G)(U)$ or $(H\bar{x}e_2)(U)$, respectively. It is obvious that

$$\bar{\varphi}_1(V_2) = 0, \quad \varphi_2(V_2) = 0, \quad \bar{\varphi}_1(V_0) = 0 .$$

Denote by $\bar{\varphi}_2$ the form $j_*\varphi_2 p_0$, where p_0 is the natural projection $V_0 \oplus \oplus V_1 \oplus V_2 \to V_0$, i.e. if $X \in T_U(\bar{M})$, then

$$\bar{\varphi}_2(X) = j_*(\varphi_2(p_0(X))) \in 0 \oplus \mathscr{G} \subset \mathscr{H} \oplus \mathscr{G} .$$

Then the form

$$\varphi = \bar{\varphi}_2 + \bar{\varphi}_1$$

is a 1-form on M with values in $\mathscr{H} \oplus \mathscr{G}$.

Lemma 4. *Let X be a fundamental vector field on $\bar{M}$ generated by $\bar{X} \in \mathscr{H} \oplus \mathscr{G}$. Then*

$$\varphi(X_U) = \bar{X} \quad \textit{for any} \quad U \in \bar{M} .$$

Proof. It is sufficient to consider two cases:

a. $X_U \in V_0 \subset T_U(\bar{M})$ for any $U \in \bar{M}$, i.e. $\bar{X} \in 0 \oplus \mathscr{G}$. Then $\varphi(X_u) = = \bar{\varphi}_2(X_U) = \bar{X}$.

b. $X_U \in V_1 \subset T_U(\bar{M})$ for any $U \in \bar{M}$, i.e. $\bar{X} \in \mathscr{H} \oplus 0$. Now, $\varphi(X_U) = = \bar{\varphi}_1(X_U) = \bar{X}$.

Corollary. *If $X \in V_0 \oplus V_1$, then $\varphi(q_*X) = Ad(q^{-1})\,(\varphi(X))$ for any $q \in H\bar{x}G$.*

Lemma 5. *The form φ is G-equivariant, i.e. $\varphi(g_*(X)) = Ad(g^{-1})\varphi(X)$ for any $g \in e_1\bar{x}G$ and $X \in T_U(\bar{M})$.*

Proof. According to the Corollary of Lemma 4 it is sufficient to consider $X \in V_2$. Let $g \in e_1\bar{x}G$. Then X and g_*X are Γ^2-horizontal and $\varphi_2(X) = 0$, $\varphi_2(g_*(X)) = 0$. Since $\pi_*(X)$ and $\pi_*(g_*(X))$ are Γ_1-horizontal on N, $\bar{\varphi}_1(g_*(X)) = = 0$, $\bar{\varphi}_1(X) = 0$. It proves our assertion.

Theorem 2. *If the connection Γ_2 is H-conjugate with the connection Γ_1, then the form φ is the form of the connection $\Gamma_2 \circ \Gamma_1$.*

Proof. It is obvious that φ is C^∞-differentiable. Lemma 7 proves that φ is $(e_1\bar{x}G)$-equivariant. We shall prove that φ is $(H\bar{x}e_2)$-equivariant. This assertion is correct by the Corollary of Lemma 6 for $X \in V_0 \oplus V_1$. Let $X \in V_2$ and $h \in H\bar{x}e_2$. Since $\Gamma_2 \circ \Gamma_1$ is a connection on $\bar{M}$, the distribution of the tangent subspaces V_2 is $(H\bar{x}e_2)$-invariant. Therefore $\bar{\varphi}_1(X) = 0$, $\bar{\varphi}_2(X) = 0$, $\bar{\varphi}_1(h_*X) = 0$, $\varphi_2(h_*X) = 0$ (that immediately yields $\varphi(V_2) = 0$). It proves that φ is $(H\bar{x}e_2)$-equivariant. Then φ is $(H\bar{x}G)$-equivariant. The assertion of Lemma 4 completes the proof of Theorem 2.

Remark 6. Let $\bar{M}(B, F, \mu)$ and $N(B, H, p)$ be two principal fibre bundles over B. Let $\pi : \bar{M} \to N$ be a surjection and $\psi_1 : F \to H$ be a such epimorphism of the structure groups F and H that

$$\pi(uf) = \pi(u)\psi_1(f)$$

for any $u \in M$ and $f \in F$. Define a group G by

$$0 \to G \to F \xrightarrow{\psi_1} H \to 0 . \tag{10}$$

Then $\bar{M}$ has the principal fibre bundle structure of the symbol $M(N, G, \pi)$. Let $\psi_2 : H \to F$ be a splitting of (10). Then identifying $\psi_2(h)g = (h, g)$ we have $F = H\bar{x}G$ with respect to the action $gh = [\psi_2(h)]^{-1}g\psi_2(h)$. Define the action of H on $\bar{M}$ by

$$uh = u\psi_2(h) .$$

Then $u(h, g) = u\psi_2(h)g = (u\bar{h})g$. Now it is easy to show that $\bar{M}(B, F, \mu)$ follows from $M(N, G, \pi)$ and $N(B, H, p)$ by the Kolář construction.

The principal fibre bundles $\tilde{W}^r(P)$, $\tilde{W}^s(P)$ $(o \leqq s > r)$ together with the jet projections j^s_r: $\tilde{W}^r(P) \to \tilde{W}^s(P)$, j^s_r: $\tilde{G}^r_n \to \tilde{G}^s_n$ and the splitting $\psi = (i, j)$: $\tilde{G}^s_n \equiv \tilde{L}^s_n \bar{x} \tilde{T}^s_n(G) \to \tilde{L}^r_n \bar{x} \tilde{T}^r_n(G)$ of the sequence

$$0 \to {}^s\tilde{G}^r_n \to \tilde{G}^r_n \underset{\psi}{\overset{j^s_r}{\rightleftarrows}} \tilde{G}^s_n \to 0$$

naturally satisfy our above assumptions and thus one gets $\tilde{W}^r(P)$ from $\tilde{W}^r_s$ and $\tilde{W}^s(P)$. Then Theorem 2 can be used for the construction of the connection form of Γ on $\tilde{W}^r(P)$ determined by the connections Γ_2 and Γ_1 on $\tilde{W}^r_s$ and on $\tilde{W}^s(P)$, where Γ_2 is $(\tilde{G}^r_n)^{[s]} \equiv \tilde{G}^s_n$-conjugate with Γ_1. This construction can also be used in the case of the connection on $\tilde{W}^r(P)$ introduced by Gollek [4].

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