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RADICALS IN QUASI-COMMUTATIVE SEMIGROUPS

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A semigroup S is called *quasi-commutative* [3] if to every $a, b \in S$ there is a positive integer $r = r(a, b)$ such that $ab = b^r a$. A commutative semigroup is clearly quasi-commutative. J. Bosák [1; p. 209] and R. Šulka [4; p. 221] proved that if S is a commutative semigroup and J any ideal of S, then the Clifford, McCoy, Ševrin, Schwarz, and Luh radicals with respect to J denoted by $R_J^*(S)$, $M_J(S)$, $L_J(S)$, $R_J(S)$, and $C_J(S)$, respectively are equal to $N_J(S)$, the set of all nilpotent elements of S with respect to J. For their definitions, we refer to [1] and [4]. Further, J. E. Kuczkowski [2] proved that if S is a C_2-semigroup, then $M_J(S) = L_J(S) = R_J^*(S) = N_J(S) = C_J(S)$ for any ideal J of S. The purpose of this note is to extend the results of [1] and [4] to the class of quasi-commutative semigroups.

Let x be any element of a semigroup S. The principal ideal of S generated by x will be denoted by $J(x)$. Before coming to the main result we first prove two lemmas.

Lemma 1. *Let S be a quasi-commutative semigroup. Then an ideal of S is prime if and only if it is completely prime* [1].

Proof. Clearly it suffices to prove that any prime ideal of S is completely prime. Let P be any prime ideal and $ab \in P$ $(a, b \in S)$. Let x be any element of S. Then $ax = x^r . a$, for some positive integer $r \geq 1$, since S is quasi-commutative. Now $axb = x^r . ab \in P$, for all $x \in S$. Hence $aSb \subseteq P$ so that $J(a)J(b) \subseteq P$, and as P is a prime ideal, we get $a \in P$ or $b \in P$, proving that P is a completely prime ideal.

Corollary. *Let S be a quasi-commutative semigroup and J any ideal of S. Then $M_J(S) = C_J(S)$.*

Lemma 2. *Let S be a quasi-commutative semigroup. Then for any x, y in S, $J(x) . J(y) = J(xy)$.*

Proof. Clearly $J(xy) \subseteq J(x) . J(y)$. Let $a \in J(x)$ and $b \in J(y)$. Then a is one of x, sx, xt or sxt and b is one of y, $s'y$, yt' or $s'yt'$, where $s, s', t, t' \in S$. Using the fact that S is quasi-commutative, we obtain $ab \in J(xy)$ in every case; so that $J(x) . J(y) \subseteq J(xy)$. Hence the lemma follows.

Theorem. *If S is a quasi-commutative semigroup and J any ideal of S, then* $R_J(S) = M_J(S) = L_J(S) = R_J^*(S) = N_J(S) = C_J(S)$.

Proof. J. Bosák [1; Theorem 2] proved that

$$R_J(S) \subseteq M_J(S) \subseteq L_J(S) \subseteq R_J^*(S) \subseteq N_J(S) \subseteq C_J(S) \tag{1}$$

for any semigroup S and any ideal J of S. Now $M_J(S) = C_J(S)$ by the above corollary. We next show that $R_J^*(S) \subseteq R_J(S)$. Let $a \in R_J^*(S)$; then $a^m \in J$ for some positive integer m. By Lemma 2, $[J(a)]^m = J(a^m) \subseteq J$, whence $J(a)$ is a nilpotent ideal with respect to J and hence $a \in R_J(S)$. Combining $R_J^*(S) \subseteq R_J(S)$ with $M_J(S) = C_J(S)$, we get equality everywhere in (1). This completes the proof of the theorem.

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