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Note on the Set of Nilpotent Elements and on Radicals of Semigroups


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NOTE ON THE SET OF NILPOTENT ELEMENTS
AND ON RADICALS OF SEMIGROUPS

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In the present paper we consider some properties of nilpotent elements and radicals in semigroups.

Let $S$ be a semigroup. Under an *ideal* of $S$ we understand a two-sided ideal of $S$. Let $x(T) [J, I]$ be an element (subsemigroup) [ideals] of $S$.

An element $x$ [subsemigroup $T$] is called *nilpotent* with respect to the ideal $J$ if there exists a natural number $n$ such that $x^n \in J$.

An ideal $I$ is called *locally nilpotent* with respect to $J$ if every finitely generated subsemigroup $T \subseteq I$ is nilpotent with respect to $J$.

An ideal $I$ is called *nil-ideal* with respect to $J$ if every element $x \in I$ is nilpotent with respect to $J$.

An ideal $P$ of $S$ is called *prime* if $S\setminus P$ is an $m$-system of $S$ (a set $H \subseteq S$ is called an $m$-system of $S$, if for every two elements $a, b \in H$ there exists such an element $x \in S$ that $axb \in H$; we take the empty set also as an $m$-system).

An ideal $P$ of $S$ is called *completely prime* if $S\setminus P$ is a face of $S$ (the non-empty subset $T$ of $S$ is called a *face* of $S$ if $ab \in T$ if and only if $a \in T$, $b \in T$; the empty set is also considered a face).

The set of all the nilpotent elements of $S$ with respect to $J$ will be denoted by $N(J)$.

The ideal $R(J) [L(J)]$, which is the union of all the nilpotent [locally nilpotent] ideals of $S$ with respect to $J$ is called the *Schwarz [Sevrin] radical* of $S$ with respect to $J$.

The ideal $R^*(J)$, which is the union if all nil-ideals of $S$ with respect to $J$ is called the *Clifford radical* of $S$ with respect to $J$.

Let $M$ be a non-empty subset of $S$. By $C(M) [M(M)]$ we denote the set of all such elements $r \in S$ that the intersection of every face [of every $m$-system] of the semigroup $S$ which contains $r$ with $M$ is non-empty.

It is known (see [4]) that $M(M) [C(M)]$ is the intersection of all prime ideals [complete prime ideals] of $S$ which contain $M$.

The set $M(J) [C(J)]$ is called the *McCoy [Jiang Luh] radical* of $S$ with respect to $J$.

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The direct product of the semigroups $S_1$ and $S_2$ will be denoted by $S_1 \times S_2$. We use the remaining notions in this paper in their current sense.

**Theorem 1.** Let $I$ be the minimal ideal of the semigroup $S_1 \times S_2$.

Then we have:

(a1) $N(I) = N(I') \times N(I'')$, 
(a2) $R(I) = R(I') \times R(I'')$, 
(a3) $M(I) = M(I') \times M(I'')$, 
(a4) $L(I) = L(I') \times L(I'')$, 
(a5) $R^*(I) = R^*(I') \times R^*(I'')$, 
(a6) $C(I) = C(I') \times C(I'')$, 

where $I' [I'']$ is the projection of $I$ into $S_1 [S_2]$.

**Proof.** For every minimal ideal of $S_1 \times S_2$ we have:

$I = I' \times I''$, 

where $I'[I'']$ is the projection of $I$ into $S_1[S_2]$ (see [2]). Wherefrom with respect to Theorem 3 of [1] we obtain the assertion of Theorem 1.

**Theorem 2.** Let $M_i (i = 1, 2)$ be an arbitrary non-empty subset of the semigroup $S_i$. Then the following holds

(b1) $M(M_1 \times M_2) = M(M_1) \times M(M_2)$, 
(b2) $C(M_1 \times M_2) = C(M_1) \times C(M_2)$.

The proof can be given in the same way as the proof of Theorem 3, (c) and (f) in [1].

**Theorem 3.** Let $J_1, J_2$ be ideals of the semigroup $S$. Then we have:

(c1) $N(J_1J_2) = N(J_1) \cap N(J_2)$, 
(c2) $R(J_1J_2) = R(J_1) \cap R(J_2)$, 
(c3) $M(J_1J_2) = M(J_1) \cap M(J_2)$, 
(c4) $L(J_1J_2) = L(J_1) \cap L(J_2)$, 
(c5) $R^*(J_1J_2) = R^*(J_1) \cap R^*(J_2)$, 
(c6) $C(J_1J_2) = C(J_1) \cap C(J_2)$. 


Proof. I. Let $J_1, J_2$ be arbitrary ideals of $S$. We know that the following holds:

$J_1 \subseteq \mathcal{I}(J_1)$ and $\mathcal{I}(J_1 \cap J_2) = \mathcal{I}(J_1) \cap \mathcal{I}(J_2),$

where instead of $\mathcal{I}$ we can put any of the signs $N, R, L, M, R^*, C$ (see [3], [5]).

As $J_1J_2 \subseteq J_1 \cap J_2$, then from (a) we have $\mathcal{I}(J_1J_2) \subseteq \mathcal{I}(J_1) \cap \mathcal{I}(J_2)$, where $\mathcal{I} = N, R, M, L, R^*, C$.

II. (c1) Let $x$ be an element of $N(J_1) \cap N(J_2)$, then $x$ is nilpotent with respect to $J_1(x^{n_1} \in J_1)$ and $J_2(x^{n_2} \in J_2)$. Let $n = n_1 + n_2$, then $x^n \in J_1J_2$. This means that $N(J_1) \cap N(J_2) \subseteq N(J_1J_2)$.

(c2) Let $x$ be an element of $R(J_1) \cap R(J_2)$, then $x$ is the element of a nilpotent ideal $I_2$ with respect to $J_2(I_2^{n_2} \in J_2)$ and of a nilpotent ideal $I_1$ with respect to $J_1(I_1^{n_1} \in J_1)$. The ideal $I_1 \cap I_2$ is nilpotent with respect to $J_1J_2$, because $(I_1 \cap I_2)^n \subseteq J_1J_2$, where $n = n_1 + n_2$. This means that $R(J_1) \cap R(J_2) \subseteq R(J_1J_2)$.

(c3) Let $x$ be an element of $M(J_1) \cap M(J_2)$. An arbitrary m-system $H$, which contains $x$, contains also an element $x_1 \in J_1$ and an element $x_2 \in J_2$. Because $H$ is an m-system of $S$, there exists at least one element $h \in S$ such that $x_1hx_2 \in H$, but the element $x_1hx_2 \in J_1J_2$. It follows that $M(J_1) \cap M(J_2) \subseteq M(J_1J_2)$.

(c4) Let $x \in L(J_1) \cap L(J_2)$; then the element $x$ is from a locally nilpotent ideal $I_1$ with respect to $J_1$ and from a locally nilpotent ideal $I_2$ with respect to $J_2$. Let $H$ be an arbitrary finitely generated subsemigroup of $I_1 \cap I_2$; then there exist natural numbers $n_1$ and $n_2$ such that $H^{n_1} \subseteq J_1$ and $H^{n_2} \subseteq J_2$. Therefore for $n = n_1 + n_2$ we have $H^n \subseteq J_1J_2$. Then $L(J_1) \cap L(J_2) \subseteq L(J_1J_2)$.

(c5) Let $x$ be an arbitrary element of $R^*(J_1) \cap R^*(J_2)$. This means that $x$ is in a nil-ideal $I_1$ with respect to $J_1(x^{n_1} \in J_1)$ and in a nil-ideal $I_2$ with respect to $J_2(x^{n_2} \in J_2)$. We will prove that $I_1 \cap I_2$ is a nil-ideal with respect to the ideal $J_1J_2$. It is clear that $x \in I_1 \cap I_2$ and for $n = n_1 + n_2$ we have $x^n \in J_1J_2$. Thus $R^*(J_1) \cap R^*(J_2) \subseteq R^*(J_1J_2)$.

(c6) We will prove the assertion (c6) similarly as (c3). It is necessary to take instead of an m-system $H$ a face $T$ of $S$. From I and II the assertion of Theorem 3 follows.

It is known that the set $S_J$ of all ideals in the sense of multiplication of complexes is a semigroup.

Theorem 4. Let $S$ be a semigroup and $S_J$ the semigroup of all ideals of $S$. Then we have:

(a) the mapping $J \rightarrow N(J)$ is a homomorphism of the semigroup $S_J$ into the semilattice of all subsets of $S$.

(b) the mapping $J \rightarrow S(J)$ is an endomorphism of the semigroup $S_J$ into the semilattice of all ideals of $S_J$, where we can put instead of $S$ any of the signs $R, L, M, R, C$.

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In (a) [(b)] we understand under the semilattice operation ∩ the intersection of two subsets [ideals] of S. The proof follows from Theorem 3.

R. Šulka in his paper [3] proved the following assertions.

(d₁) \( R(J_1) \cup R(J_2) \subseteq R(J_1 \cup J_2) \),
(d₂) \( R^*(J_1) \cup R^*(J_2) \subseteq R^*(J_1 \cup J_2) \),
(d₃) \( M(J_1) \cup M(J_2) \subseteq M(J_1 \cup J_2) \),

where \( J_1, J_2 \) are ideals of S. In paper [3] it is shown that there exist such semigroups for which the equality in (d₁), (d₂) and (d₃) does not hold.

**Theorem 5.** Let \( J_1 \) and \( J_2 \) be ideals of S. Then we have:

(e₁) \( R^*(R^*(J_1) \cup R^*(J_2)) = R^*(J_1 \cup J_2) \),
(e₂) \( M(M(J_1) \cup M(J_2)) = M(J_1 \cup J_2) \).

Proof. I. From (d₂) we have: \( R^*(R^*(J_1) \cup R^*(J_2)) \subseteq R^*(R^*(J_1 \cup J_2)) = R^*(J_1 \cup J_2) \) (see [3]).

II. As \( J_1 \subseteq R^*(J_1) \) and \( J_2 \subseteq R^*(J_2) \), then \( J_1 \cup J_2 \subseteq R^*(J_1) \cup R^*(J_2) \). It follows that \( R^*(J_1 \cup J_2) \subseteq R^*(R^*(J_1) \cup R^*(J_2)) \). The proof of (e₂) is similar (e₁).

If we suppose that the suppositions of Theorem 5 are fulfilled, we have

(f₁) \( R^*(J_1 \cup R^*(J_2)) = R^*(J_1 \cup J_2) \),
(f₂) \( M(J_1 \cup M(J_2)) = M(J_1 \cup J_2) \).

The equalities (e₂) (f₂) are fulfilled even in the case when \( J_1[J_2] \) is an arbitrary non-empty subset of S.

There exists a semigroup S in which the following is not fulfilled \( R(R(J_1) \cup R(J_2) = R(J_1 \cup J_2) \), where \( J_1 \) and \( J_2 \) are ideals of S (see [4], Example 2.). Let S be the semigroup generated by the set \( \{ \theta, a, b_1, b_2, \ldots \} \) subject to the generating relations

\[
0x = x0 = 0 \text{ for every } x \in S;
\]

\[a^2 = 0; \]

\[b_i b_j = 0 \text{ for } i, j = 1, 2, \ldots; \]

\[b_i a b_j = 0 \text{ for } i = j; \ i, j = 1, 2, \ldots; \]

\[(ab_i)^{i+1} = (b_i a)^{i+1} = 0 \text{ for } i = 1, 2, \ldots. \]

Then \( R(R(\{ \theta \}) \neq R(\{ \theta \}) \) (see [4]). We put \( J_1 = J_2 = \theta \). Then \( R(R(\{ \theta \}) \cup (\{ \theta \}) \neq R(\{ \theta \} \cup \{ \theta \}) \).

Let us denote by \( \mathcal{P} \) the system of complete prime ideals of S (we take an empty set as a complete prime ideal, too).
Lemma 1. A non-empty subsystem $\mathcal{U}$ of the system $\mathcal{P}$ of $S$ is linearly ordered with respect to $\leq$ if and only if for arbitrary $P \in \mathcal{U}$, $Q \in \mathcal{U}$ there is $P \cap Q \in \mathcal{U}$.

Proof. I. Let $\mathcal{U}$ be a linearly ordered subsystem, then it is clear that for every $P \in \mathcal{U}$, $Q \in \mathcal{U}$ is $P \cap Q \in \mathcal{U}$.

II. Let there for an arbitrary $P \in \mathcal{U}$, $Q \in \mathcal{U}$ be $P \cap Q \in \mathcal{U}$; then either $P \subseteq Q$ or $Q \subseteq P$. Let us suppose the reverse, i.e. $P \nsubseteq Q$ and $Q \nsubseteq P$. Then there exist elements $y \in Q$, $y \notin P$ and $x \notin Q$, $x \in P$. Hence $x, y \in S \setminus (P \cap Q)$ and $xy \in P \cap Q$. Because $P \cap Q \in \mathcal{U}$, then $S \setminus (P \cap Q)$ is a face of $S$. It means $xy \in S \setminus (P \cap Q)$. It is a contradiction of $xy \in P \cap Q$.

Corollary 1. The set $\mathcal{P}$ of all complete prime ideals of $S$ with respect to $\subseteq$ is linearly ordered if and only if for an arbitrary $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ there is $P \cap Q \in \mathcal{P}$.

Corollary 2. If in the semigroup $S$ every ideal is a complete prime ideal, then the set $\mathcal{P}$ is linearly ordered with respect to $\subseteq$ (see [8]).

Theorem 6. Let $\mathcal{U}$ be an arbitrary non-empty subsystem of the system $\mathcal{P}$. $\bigcap_{P \in \mathcal{U}} P \in \mathcal{P}$ if and only if for every $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ there is $P \cap Q \in \mathcal{P}$.

Proof. I. If for every non-empty subsystem $\mathcal{U}$ of the system $\mathcal{P}$ $\bigcap_{P \in \mathcal{U}} P \in \mathcal{P}$ holds, then for every two $P$, $Q \in \mathcal{P}$ there is $P \cap Q \in \mathcal{P}$.

II. Let $\mathcal{U}$ be an arbitrary non-empty subsystem of $\mathcal{P}$ and for every $P \in \mathcal{P}$ and $Q \in \mathcal{P}$ there is $P \cap Q \in \mathcal{P}$. Then the subsystem $\mathcal{U}$ is linearly ordered with respect to $\subseteq$. Let $a, b \in S \setminus \bigcap_{P \in \mathcal{U}} P$, then there exist $P \in \mathcal{U}$, $Q \in \mathcal{U}$ such that $a \notin P$ and $b \notin Q$. This means that $a$, $b$ are not the elements of at least one of $P$, $Q$. Let e.g. $a \notin P$, $b \notin P$. Then $ab \in S \setminus P \subseteq S \setminus \bigcap_{P \in \mathcal{U}} P$. When $ab \in S \setminus \bigcap_{P \in \mathcal{U}} P$ then $ab \notin R$, where $R \in \mathcal{U}$. This means that $ab \in S \setminus R$, where $S \setminus R$ is a face of $S$. It follows that $a$, $b \in S \setminus R \subseteq S \setminus \bigcap_{P \in \mathcal{U}} P$. Therefore $S \setminus \bigcap_{P \in \mathcal{U}} P$ is a face of the semigroup $S$ and $\bigcap_{P \in \mathcal{U}} P \in \mathcal{P}$.

It is known that the subset $H$ of the semigroup $S$ is a face of the semigroup $S$ if and only if $S \setminus P \in \mathcal{P}$ (see [6]). We denote by $\mathcal{H}$ the set of all faces of the semigroup $S$.

Lemma 2. A non-empty subsystem $\mathcal{V}$ of the system $\mathcal{H}$ is linearly ordered with respect to $\subseteq$ if and only if for arbitrary $H \in \mathcal{V}$, $T \in \mathcal{V}$ there is $H \cup T \in \mathcal{V}$.

Proof. I. Let $\mathcal{V}$ be a linearly ordered subsystem; then it is clear that for every $H \in \mathcal{V}$, $T \in \mathcal{V}$ is $H \cup T \in \mathcal{V}$.

II. Let there for an arbitrary $H \in \mathcal{V}$ and $T \in \mathcal{V}$ be $H \cup T \in \mathcal{V}$. The set $P = S \setminus H$ is a complete prime ideal of the semigroup $S$ for every $H \in \mathcal{V}$.
Let $\mathcal{U} = \{P \mid P = S \setminus H, H \in \mathcal{V}\}$. Then $P \cap Q = (S \setminus H) \cap (S \setminus T) = S \setminus (H \cup T) \in \mathcal{U}$, where $P \in \mathcal{U}, Q \in \mathcal{U}$. Following Lemma 1 we have either $S \setminus H \subseteq S \setminus T$ or $S \setminus T \subseteq S \setminus H$. It follows that either $H \subseteq T$, or $T \subseteq H$.

**Theorem 7.** Let $\mathcal{V}$ be an arbitrary subsystem of the system $\mathcal{H}$. Then $\bigcup_{H \in \mathcal{V}} H \in \mathcal{H}$ if and only if for every $H \in \mathcal{H}$ and $T \in \mathcal{H}$ there is $H \cup T \in \mathcal{H}$.

**Proof.** Let there for an arbitrary $H \in \mathcal{H}, T \in \mathcal{H}$ be $H \cup T \in \mathcal{H}$. Let $P \in \mathcal{P}$ and $Q \in \mathcal{P}$; then $P \cap Q = (S \setminus H) \cap (S \setminus T) = S \setminus (H \cup T) \in \mathcal{P}$. The set $P = S \setminus H$ is a complete prime ideal of $S$ for every $H \in \mathcal{V}$. Following Theorem 6 we have $\bigcap_{H \in \mathcal{V}} P \in \mathcal{P}$. Further we have $S \setminus \bigcup_{H \in \mathcal{V}} H = \bigcap_{H \in \mathcal{V}} P$. It follows that $\bigcup_{H \in \mathcal{V}} H \in \mathcal{H}$. The second part of the theorem is clear.

Let $S_1, S_2$ be semigroups and $S = S_1 \times S_2$ their direct product.

**Theorem 8.** Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}$ be the set of all complete prime ideals of $S_1[S_2, S = S_1 \times S_2]$. $\mathcal{P}$ is linearly ordered with respect to $\subseteq$ if and only if every one of the sets $\mathcal{P}_1$ and $\mathcal{P}_2$ is linearly ordered with respect to $\subseteq$ and at least one of the semigroups $S_1$ and $S_2$ does not contain its proper non-zero complete prime ideal.

**Proof.** I. Let $\mathcal{P}$ be linearly ordered with respect to $\subseteq$ and let $P_1 \in \mathcal{P}_1, P_2 \in \mathcal{P}_2$ such that $P_1 \neq \emptyset$, $P_1 \neq S_1 [P_2 \neq \emptyset, P_2 \neq S_2]$. Then it follows that $P = S_1 \times P_2$ and $P' = P_1 \times S_2$ are complete prime ideals of $S$ and $P \nsubseteq P'$, $P' \nsubseteq P$. This is a contradiction. Further let $\mathcal{P}_1 = \{\emptyset, S_1\}$. Let $\mathcal{P}_2$ be linearly non ordered. Then there exist $P_2, P'_2 \in \mathcal{P}_2$ such that $P_2 \nsubseteq P'_2$, $P'_2 \nsubseteq P_2$. The ideal $P = S_1 \times P_2[P' = S_1 \times P'_2]$ is a complete prime ideal of $S$ and $P \nsubseteq P'$, $P' \nsubseteq P$.

II. Let $\mathcal{P}_1, \mathcal{P}_2$ be linearly ordered with respect to $\subseteq$ and let $\mathcal{P}_2 = \{\emptyset, S_2\}$.

For an arbitrary $P, P' \in \mathcal{P}$ we have:

$$P = (P_1 \times S_2) \cup (S_1 \times P_2), P' = (P'_1 \times S_2) \cup (S_1 \times P'_2).$$

The following cases may arise:

- $P_2 = P'_2 = S_2$;
- $P_2 = \emptyset, P'_2 = S_2$;
- $P_2 = S_2, P_2 = \emptyset$;
- $P_2 = P'_2 = \emptyset$.

As $P_1 \subseteq P'_1$ or $P'_1 \subseteq P_1$, we have $P \subseteq P'$ or $P' \subseteq P$.

We denote by $\mathcal{F}[\mathcal{F}_1, \mathcal{F}_2]$ the topology on $S = S_1 \times S_2 [S_1, S_2]$; the base is $\mathcal{H}[\mathcal{H}_1, \mathcal{H}_2]$, where $\mathcal{H}[\mathcal{H}_1, \mathcal{H}_2]$ is the set of faces or $S[S_1, S_2]$ (see [7]).

We denote by $\mathcal{F}_1 \times \mathcal{F}_2$ the topology of the semigroup $S$, the base of which is $\mathcal{H}_1 \times \mathcal{H}_2$ (see [3]).

**Theorem 9.** Let $S_1, S_2$ be semigroups and let $S = S_1 \times S_2$. Then for the topology $\mathcal{F}$ and $\mathcal{F}_1 \times \mathcal{F}_2$ we have $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2$. 

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Proof. It is clear that $\mathcal{H}_1 \times \mathcal{H}_2 \subseteq \mathcal{H}$ holds (see [3]).

Let $H \in \mathcal{H}$, then $P = S \setminus H \in \mathcal{P}$ and $P = (P_1 \times S_2) \cup (S_1 \times P_2)$ (see [4], [1]). Further $H = S \setminus P = (S_1 \times S_2) \setminus [(P_1 \times S_2) \cup (S_1 \times P_2)] = [(S_1 \times S_2) \setminus (P_1 \times S_2) \setminus [(P_1 \times S_2) \cup (S_1 \times P_2)] = [(S_1 \setminus P_1) \times S_2] \cap [(S_1 \setminus S_2) \setminus S_1 \times (S_2 \setminus P_2)] = = (S_1 \setminus P_1) \times (S_2 \setminus P_2) = H_1 \times H_2 \in \mathcal{H}_1 \times \mathcal{H}_2$. It follows that $\mathcal{H} \subseteq \mathcal{H}_1 \times \mathcal{H}_2$.

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