Matematický časopis

Jana Farková

# About the Maximum and the Minimum of Darboux Functions

*Matematický časopis*, Vol. 21 (1971), No. 2, 110--116

Persistent URL: `http://dml.cz/dmlcz/126438`

Matematický časopis 21 (1971), No. 2

# ABOUT THE MAXIMUM AND THE MINIMUM OF DARBOUX FUNCTIONS

JANA FARKOVÁ, Bratislava

In paper [3] the following statement is proved: If $f$ is a real valued function of a real variable, continuous and non-constant, then there is a Darboux function $g$ with the property that the function $F = f + g$ is not a Darboux one.

A natural question arises if a similar statement holds for the functions $\varphi = \max(f, g)$ and $\psi = \min(f, g)$ as well.

The answer is negative and as follows:

**Theorem 1.** *Let $f$ and $g$ be Darboux real valued functions of a real variable. Let every $x \in (-\infty, \infty)$ be a point of the upper (lower) semi-continuity of at least one of them. Then the function $\varphi = \max(f, g)$ $(\psi = \min(f, g))$ is a Darboux function.*

Proof. Let $x, y (x < y)$ be real numbers, let $\varphi(x) < c < \varphi(y)$ (a proof for $\varphi(x) > \varphi(y)$ is analogical).

Let $A = \{u\colon \text{ if } x \leq u' \leq u, \text{ then } \varphi(u') < c\}$. Let $x_0 = \sup A$. Because $f$ and $g$ are Darboux functions, $f(x_0) \leq c$, $g(x_0) \leqslant c$ (of course $x_0 \neq y$). If $\max(f(x_0), g(x_0)) = c$, then $\varphi(x_0) = c$ and the Theorem is proved.

Let $\max(f(x_0), g(x_0)) < c$, let $f$ be upper semi-continuous in $x_0$. Now choose $K$ such that $f(x_0) < K < c$. Let $O$ be such a neighbourhood of $x_0$ that for $x \in \in O$ $f(x) < K$ holds. With regard to the construction of the point $x_0$ in $O$ such a point $\xi$ $(\xi > x_0)$ exists that $\varphi(\xi) \geqslant c > K$. Therefore $\varphi(\xi) = g(\xi)$. Then either $\varphi(\xi) = g(\xi) = c$, or (because of $g$ Darboux) there exists $z \in (x_0, \xi)$ such that $g(z) = c$. But again $\varphi(z) = g(z) = c$. (The proof for $\psi = \min(f, g)$ is analogical.)

If $f$ is not a Darboux function, then there evidently can easily be constructed a Darboux function $g$ (even a suitable constant) such that $\max(f, g)$ $(\min(f, g))$ is not a Darboux function.

As the following theorem shows, Darboux upper semi-continuous functions are the only functions with the property that the maximum of the function and an arbitrary Darboux function is again Darboux.

Lower semi-continuous functions play an analogical role in the case of a minimum.

**Theorem 2.** *Let $f$ be a Darboux real valued function of a real variable. Let $f$ be not upper semi-continuous (lower semi-continuous). Then there exists such a Darboux function $g$ that $\varphi = \max(f, g)$ ($\psi = \min(f, g)$) is not Darboux.*

Proof. Let $f$ not be upper semi-continuous in a point $x_0$. Therefore $\limsup_{x \to x_0} f(x) > f(x_0)$. It means that at least one of these inequalities must hold: $\limsup_{x \to x_0^+} f(x) > f(x_0)$, $\limsup_{x \to x_0^-} f(x) > f(x_0)$. Let $\limsup_{x \to x_0^+} f(x) > (x_0)$ hold (in the second case the proof is analogical).

Now choose $K$ such that $f(x_0) < K < \limsup_{x \to x_0^+} f(x)$, $\limsup_{x \to x_0^+} f(x) + f(x_0) \geqslant 2K$.

Define a function $g$: $g(x) = f(x)$, for $x \leqslant x_0$, $g(x) = 2K - f(x)$, for $x_0 < x$.

We shall show that $g$ is Darboux, therefore for $x, y$ real and $c$ such that $g(x) < c < g(y)$ there exists $z \in (\min(x, y), \max(x, y))$ so that $g(z) = c$ (it is equivalent to the statement that $g(\langle x, y \rangle)$ is connected).

If $\max(x, y) \leqslant x_0$, or if $\min(x, y) > x_0$, it follows immediately from the definitions of $f$ and of $g$. Let $\min(x, y) = x_0$ and let $x < y$. Since $g(y) > c$, $f(y) < 2K - c$; considering that $c > g(x) = f(x_0)$, $2K - c < 2K - f(x_0)$ and since $\limsup_{x \to x_0^+} f(x) + f(x_0) \geqslant 2K$, then $2K - f(x_0) \leqslant \limsup_{x \to x_0^+} f(x)$. Thus there is a point $\xi \in (x_0, y)$ such that $f(\xi) > 2K - c$. Therefore $f(y) < 2K - c < f(\xi)$ and thus there is a point $z \in (\xi, y)$ such that $f(z) = 2K - c$ and then $g(z) = c$.

Let $\min(x, y) = x_0$ and $x > y$. In this case $f(x) = 2K - g(x) > 2K - c > > 2K - g(y) = 2K - f(x_0) > f(x_0) = f(y)$. It follows that there exists a point $z \in (y, x)$ such that $f(z) = 2K - c$ and then $g(z) = c$.

Let now $x, y$ be such real numbers that $x_0 \in (x, y)$. Then $g(\langle x, y \rangle) = = g(\langle x, x_0 \rangle) \cup g(\langle x_0, y \rangle)$. Because of connectivity $g(\langle x, x_0 \rangle)$ and $g(\langle x_0, y \rangle)$ and because $g(\langle x, x_0 \rangle) \cap g(\langle x_0, y \rangle) \neq \emptyset$, $g(\langle x, y \rangle)$ is a connected set.

Because of $\varphi(x_0) = f(x_0) < K$ and $\varphi(x) \geqslant K$ for $x \in (x_0, \infty)$, $\varphi = \max(f, g)$ is not Darboux.

A similar proof can be given also for the minimum.

In [3] similar questions are studied also for a class $D_0$ of real valued functions of a real variable having „the Darboux property in the sense of Radakovič". A function belongs to $D_0$ iff the closure of the image of an arbitrary interval is an interval or a one-point set. The following statement is proved there: Continuous functions are the only functions such that their sum with every function from $D_0$ is again from $D_0$.

These results were generalized in [2] for real valued functions defined on a topological space. Symbol $D_0(\mathscr{B})$ denotes here a set of all real valued functions defined on a topological space $X$ with a topological base $\mathscr{B}$, with the property: If $B \in \mathscr{B}$, $x, y \in \bar{B}$ and $c$ is such that $f(x) < c < f(y)$, then for an arbitrary $\varepsilon > 0$ there is a point $\xi \in B$ such that $f(\xi) \in (c - \varepsilon, c + \varepsilon)$. Here are further definitions of some topological properties of the base, which will be needed:

A base $\mathscr{B}$ is said to satisfy the condition (1*) provided that for an arbitrary open set $U$, $x \in X$, $B \in \mathscr{B}$, $x \in U$ and $x \in \bar{B}$ there exists $C \in \mathscr{B}$ such that $C \subset U \cap B$ and $x \in \bar{C} - C$.

A base $\mathscr{B}$ is said to satisfy the condition (2*) provided that for every $O \in \mathscr{B}$ and every decomposition of $O$, $O = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset \neq B$ with the property

$\bar{U} \cap O \subset A$, $\bar{U} \cap O \subset B$ resp., if $U \subset A$, $U \subset B$, resp., and $U \in \mathscr{B}$, and either $A' \cap B$ or $A \cap B'$ is non-empty.

In [2] the following theorem is proved: Let $X$ be a topological space with a base $\mathscr{B}$, satisfying the conditions (1*) and (2*). Let $f$ and $g$ be from $D_0(\mathscr{B})$. Let every $x \in X$ be a point of continuity of $f$ or of $g$. Then the functions $\varphi = \max(f, g)$, $\psi = \min(f, g)$ are also from $D_0(\mathscr{B})$.

Remark 1. In [2] it is proved that a base satisfies the condition (2*) iff it consists of connected sets only. Thus only in a local connected space there exists a base with the property (2*).

We now give an example to show that in this more general case the continuity cannot be replaced by the upper semi-continuity if it is to be $\varphi \in D_0(\mathscr{B})$.

Similarly an example can be constructed showing that the continuity cannot be replaced by the lower semi-continuity if it is to be $\psi \in D_0(\mathscr{B})$.

Remark 2. It is necessary to explain the meaning of the upper and the lower semi-continuity of a real valued function $f$ defined on a topological space.

In [1] a real valued function $f$ defined on a topological space is called upper (lower) semi-continuous iff set $\{x: f(x) \geqslant a\}$ ($\{x: f(x) \leqslant a\}$) is closed for all real $a$. A real valued function $f$ defined on a topological space is called upper (lower) semi-continuous in a point $x_0$, if for every $\varepsilon > 0$ there is a neighbourhood $U$ of the point $x_0$ such that for every $u \in U: f(u) < f(x_0) + \varepsilon$ ($f(u) > f(x_0) - \varepsilon$).

By the limit superior of $f$ at $x_0$ ($\limsup_{x \to x_0} f(x)$) we mean a real number $a$ with the properties:

1) if $b < a$, then in every neighbourhood $U$ of point $x_0$ there is a point $y$ such that $f(y) > b$,

2) if $a < c$, then there is such a neighbourhood $U$ of point $x_0$ that for $y \in U$, $f(y) < c$.

The limit inferior of $f$ at $x_0$ ($\liminf_{x \to x_0} f(x)$) is defined analogously.

Without difficulties it is possible to prove:

A function $f$ is upper (lower) semi-continuous in a point $x_0$ iff $\limsup_{x \to x_0} f(x) \leqslant \leqslant f(x_0)$ $(\liminf_{x \to x_0} f(x) \geqslant f(x_0))$ holds.

Evidently a function $f$ is upper (lower) semi-continuous iff $f$ is upper (lower) semi-continuous in every point $x \in X$.

Example. Let $X$ be the topological space consisting of all real numbers with the base $\mathscr{B} = \{(a, b), a, b$ — real, $a \neq 0\}$.

Evidently $\mathscr{B}$ satisfies the conditions (1*) and (2*).

Define $f_1(x) = -1$, for $x > 0$, $f_1(x) = 1$, for $x = 0$, $f_1(x) = \sin(1/x)$, for $x < 0$; $f_2(x) = -1$, for $x > 0$, $f_2(x) = 1$, for $x = 0$, $f_2(x) = \sin(-1/x)$, for $x < 0$.

Any point $x \in X$ is a point of the upper semi-continuity $f_1$ and $f_2$. With regard to the definition of $\mathscr{B}$ it follows that $f_1, f_2 \in D_0(\mathscr{B})$. If $\varphi = \max(f, g)$ and $(a, b) \in \mathscr{B}$ such that $0 \in (a, b)$, then $\varphi((a, b)) = \langle 0, 1 \rangle \cup \{-1\}$, therefore $\varphi \notin D_0(\mathscr{B})$.

The continuity may be replaced by the upper (lower) semi-continuity in a more special case, if the base $\mathscr{B}$ satisfies a condition (2), stronger than the condition (2*):

A base $\mathscr{B}$ is said to satisfy condition (2) provided for every $O \in \mathscr{B}$ and every decomposition of $O$, $O = A \cup B$, $A \cap B = \emptyset$, $A \neq \emptyset \neq B$ with the property $\bar{U} \cap O \subset A$, $\bar{U} \cap O \subset B$ resp., if $U \subset A$, $U \subset B$, resp., and $U \in \mathscr{B}$, it is $A' \cap B \neq \emptyset \neq A \cap B'$.

In the proof of the following theorem another notion will be required: Let $X$ be a topological space with a base $\mathscr{B}$. A set $A \subset X$ satisfies the property $M'_*(\mathscr{B})$, if $\bar{B} \subset A$ for any $B \in \mathscr{B}$, for which $B \subset A$.

$\mathscr{M}'_*(\mathscr{B})$ is a system of all real valued functions, defined on $X$ such that the sets $\{x: f(x) \geqslant a\}$ and $\{x: f(x) \leqslant a\}$ have the property $M'_*(\mathscr{B})$ for every real $a$. Evidently $D_0(\mathscr{B}) \subset \mathscr{M}'_*(\mathscr{B})$.

In [2] it is proved that if $X$ is a topological space with a base $\mathscr{B}$ satisfying the condition (1*), $f, g \in \mathscr{M}'_*(\mathscr{B})$ and any point $x \in X$ is a point of continuity $f$ or $g$, then $\max(f, g) \in \mathscr{M}'_*(\mathscr{B})$ and $\min(f, g) \in \mathscr{M}'_*(\mathscr{B})$.

It is easy to show that in the proof of this statement for $\max(f, g)$ the continuity may be replaced by the upper semi-continuity and for $\min(f, g)$ by the lower semi-continuity.

**Theorem 3.** *Let $X$ be a topological space with a base $\mathscr{B}$ satisfying the conditions* (1*) *and* (2). *Let $f, g \in D_0(\mathscr{B})$ be such functions that every $x \in X$ is a point of the upper (lower) semi-continuity of $f$ or of $g$. Then $\varphi = \max(f, g) \in D_0(\mathscr{B})$ $(\psi = \min(f, g) \in D_0(\mathscr{B}))$.*

Proof. Let $O \in \mathscr{B}$, $x, y \in \bar{O}$, $\varphi(x) < c - \varepsilon < c < c + \varepsilon < \varphi(y)$, $\varphi(z) \notin (c - \varepsilon, c + \varepsilon)$ for $z \in O$. Let $A = \{u: u \in O,\ \varphi(u) \leqslant c\}$, $B = \{u: u \in O,\ \varphi(u) \geqslant c\}$,

$O = A \cup B$. Because $\varphi \in \mathscr{M}'_*(\mathscr{B})$, the decomposition satisfies the property of the condition (2) $(A \neq \emptyset \neq B)$ and therefore $A' \cap B \neq \emptyset \neq A \cap B'$. Let $x_0 \in A \cap B'$, let $x_0$ be a point of the upper semi-continuity of $f$ (for $g$ the proof is analogical). Whence it follows that there exists $U \in \mathscr{B}$, $U \subset O$, $x_0 \in U$ such that $f(u) < f(x_0) + \varepsilon/2$ for $u \in U$. Since $x_0 \in B'$, there exists $x_1 \in U \cap B$, accordingly $\varphi(x_1) \geqslant c + \varepsilon$.

If $f(x_1) = \varphi(x_1)$, then $f(x_1) \geqslant c + \varepsilon$, thus $f(x_0) > f(x_1) - \varepsilon/2 \geqslant c + \varepsilon/2$, contrary to $\varphi(x_0) \leqslant c - \varepsilon$. Therefore $g(x_1) = \varphi(x_1)$ must hold; then $g(x_1) \geqslant$ $\geqslant c + \varepsilon$ and because $g(x_0) \leqslant c - \varepsilon$ and $g \in D_0(\mathscr{B})$, there exists a $\xi \in U$ such that $g(\xi) \in (c + \varepsilon/2,\ c + \varepsilon)$. Since $f(\xi) < f(x_0) + \varepsilon/2 \leqslant c - \varepsilon/2$, $\varphi(\xi) =$ $= g(\xi) \in (c + \varepsilon/2,\ c + \varepsilon)$ holds contrary to our assumption. Thus $\varphi \in D_0(\mathscr{B})$.

The proof for $\psi = \min(f, g)$ with the assumption of the lower semi-continuity is analogical, but it is necessary to use the existing $x_0 \in A' \cap B$.

As the following theorem shows, the assumption of the upper (lower) semi-continuity cannot be dropped.

**Theorem 4.** *Let $X$ be a topological space with a base $\mathscr{B}$. Let $f \in D_0(\mathscr{B})$, let $f$ not be upper (lower) semi-continuous. Then there exists a function $g \in D_0(\mathscr{B})$ such that $\varphi =$ $\max(f, g) \notin D_0(\mathscr{B})$ $(\psi = \min(f, g) \notin D_0(\mathscr{B}))$.*

Proof. (The construction of the function $g$ is similar to that in Theorem 2 in the real case.) The proof is accomplished again only for the function $\varphi$; it is evident how the function $g$ can be constructed in the second case.

Let in a point $x_0 \in X$ $f$ not be upper semi-continuous. Thus $\limsup_{x \to x_0} f(x) >$ $> f(x_0)$. Choose a real number $K$ such that $\limsup_{x \to x_0} f(x) > K > f(x_0)$, $\limsup_{x \to x_0}$ $f(x) + f(x_0) \geqslant 2K$ holds.

Define $g$: $g(x_0) = f(x_0)$, $g(x) = 2K - f(x)$, for $x \in X - \{x_0\}$. Then $g \in D_0(\mathscr{B})$, $\varphi = \max(f, g) \notin D_0(\mathscr{B})$.

We shall prove that $g \in D_0(\mathscr{B})$: Let $B \in \mathscr{B}$, $x, y \in \bar{B}$, $c$ and $\varepsilon > 0$ be such that $g(x) < c - \varepsilon < c < c + \varepsilon < g(y)$. It is necessary to show that there exists $z \in B$ such that $g(z) \in (c - \varepsilon,\ c + \varepsilon)$. Let $x \neq x_0$, then $g(x) = 2K - f(x)$, $g(y) = 2K - f(y)$ if $y \neq x_0$, $g(y) = f(y)$ if $y = x_0$. Then for both cases the following holds: $f(x) = 2K - g(x) > 2K - c + \varepsilon > 2K - c > 2K - c - \varepsilon >$ $> f(y)$.

Since $f \in D_0(\mathscr{B})$, there exists $z \in B$, $z \neq x_0$ such that $f(z) \in (2K - c - \varepsilon,$ $2K - c + \varepsilon)$ and therefore $g(z) \in (c - \varepsilon,\ c + \varepsilon)$. Let $x = x_0$, then $g(x) =$ $= f(x)$, $g(y) = 2K - f(y)$; considering that $c - \varepsilon > f(x_0)$, there must be $2K - c + \varepsilon < \limsup_{x \to x_0} f(x)$ (because $\limsup_{x \to x_0} f(x) + f(x_0) \geqslant 2K$). Accordingly $f(y) = 2K - g(y) < 2K - c - \varepsilon < 2K - c < 2K - c + \varepsilon < \limsup_{x \to x_0} f(x)$ holds.

Since $f \in D_0(\mathscr{B})$, there exists $z \in B$, $z \neq x_0$ such that $f(z) \in (2K - c - \varepsilon,$ $2K - c + \varepsilon)$ and thus $g(z) \in (c - \varepsilon, c + \varepsilon)$.

If $B \in \mathscr{B}$, $x_0 \in \bar{B}$, then $\varphi(\bar{B})$ is not connected; evidently $\varphi \notin D_0(\mathscr{B})$.

From Theorems 3 and 4 there follows

**Corollary.** *Let $X$ be a topological space with a base $\mathscr{B}$ satisfying the conditions* (1*) *and* (2). *Then the upper (lower) semi-continuous functions $f \in D_0(\mathscr{B})$ are the only functions with the property that the function* max $(f, g)$ (min $(f, g)$), *where $g$ is an arbitrary function of $D_0(\mathscr{B})$ is again one of $D_0(\mathscr{B})$.*

In most papers dealing with the structure (algebraical or topological) of a system of functions, having in some sense the Darboux property, the question arises as to the effect of a further property of the functions of the system, if, namely, the functions belong to the first class of Baire's classification (further only: $f$ is of the 1st class) on the structure.

It is an important question, because a Darboux function of the 1st class is the most natural and the most frequently occurring generalization of a continuous function.

A problem of this kind is solved by

**Theorem 5.** *Let $X$ be a topological $T_1$ space satisfying the* 1*st axiom of countability (any point $x \in X$ has a countable base of neighbourhoods) with a base $\mathscr{B}$ satisfying the conditions* (1*) *and* (2). *Then the upper (lower) semi-continuous functions $f \in D_0(\mathscr{B})$ are the only functions with the property that* max $(f, g)$ (min $(f, g)$), *where $g$ is an arbitrary function of $D_0(\mathscr{B})$ and of the* 1*st class, is again of $D_0(\mathscr{B})$ and of the* 1*st class.*

Proof. It is necessary to prove the following assertions:

1) if $f$ is upper (lower) semi-continuous, $f \in D_0(\mathscr{B})$ and $g$ is an arbitrary function of $D_0(\mathscr{B})$ and of the 1st class, then max $(f, g)$ (min $(f, g)$) is of $D_0(\mathscr{B})$ and of the 1st class too,

2) if $f$ is not an upper (lower) semi-continuous functions of $D_0(\mathscr{B})$, then there exists a function $g \in D_0(\mathscr{B})$ of the 1st class such that max $(f, g)$ (min $(f, g)$) is not a function of $D_0(\mathscr{B})$ and of the 1st class.

1) In [4] on page 56 it is proved that if $f$ and $g$ are real valued functions, defined on a set $X$ and $\mathbf{S}$-measurable, where $\mathbf{S}$ is a $\sigma$-structure (if $S_n \in \mathbf{S}$ for $n = 1, 2, \ldots,$ then $\bigcup_{n=1}^{\infty} S_n \in \mathbf{S}$, if $S_1, S_2 \in \mathbf{S}$, then $S_1 \cap S_2 \in \mathbf{S}$) of subsets of $X$ (in our case this $\sigma$-structure consists of all subsets of the topological space $X$, of the type $F_\sigma$), then the functions max $(f, g)$ and min $(f, g)$ are $\mathbf{S}$-measurable, too. With the help of Theorem 3 the proof of 1) is now complete.

2) If $f \notin D_0(\mathscr{B})$, then a function $g \in D_0(\mathscr{B})$, $g$ is of the 1st class such that max $(f, g) \notin D_0(\mathscr{B})$ (min $(f, g) \notin D_0(\mathscr{B})$) can be easily constructed.

Let $f$ not belong to the 1st class. Thus there exists an open set $G \subset (-\infty, \infty)$ such that $f^{-1}(G)$ is not of the type $F_\sigma$ ($f^{-1}(G) \notin F_\sigma(X)$). $G = \bigcup_{i=1}^{\infty} O_i$, where $O_i$ are disjoint open intervals. It is obvious that there exists an $i_0$ such that $f^{-1}(O_{i_0}) \notin \notin F_\sigma(X)$. Let $O_{i_0} = (a', b)$. It is possible that $a' = -\infty$. But in this case there must exist a real number $a''$ such that $f^{-1}((a'', b)) \notin F_\sigma(X)$. In the reverse case

$$f^{-1}(O_{i_0}) = f^{-1}((-\infty, b)) = f^{-1}\left(\bigcup_{i=1}^{\infty} (b - i, b)\right) = \bigcup_{i=1}^{\infty} f^{-1}((b - i, b)) \in F_\sigma(X).$$

Therefore there exists a real number $a$ such that $f^{-1}((a, b)) \notin F_\sigma(X)$. Define $g(x) = a$, for $x \in X$. Evidently $g \in D_0(\mathscr{B})$, $g$ is of the 1st class. $\varphi = \max(f, g)$ is not of the 1st class, because $\varphi^{-1}(a, b) = f^{-1}(a, b) \notin F_\sigma(X)$. (The proof for $\psi = \min(f, g)$ is analogical.)

Now suppose that $f \in D_0(\mathscr{B})$, $f$ is of the 1st class, but $f$ is not upper semi-continuous. Let us define the function $g$ as in the proof of Theorem 4: $g(x_0) = f(x_0)$ (point $x_0$ is again an arbitrary point such that $f$ in $x_0$ is not upper semi-continuous), $g(x) = 2K - f(x)$, for $x \in X - \{x_0\}$.

We know that $g \in D_0(\mathscr{B})$ and $\max(f, g) \notin D_0(\mathscr{B})$. It is necessary to show that $g$ is of the 1st class, and thus for every open set $G \subset (-\infty, \infty)$ $g^{-1}(G) \in F_\sigma(X)$ holds. Let us denote by $h$ a function defined: $h(x) = 2K - f(x)$ for $x \in X$. It is obvious ([4], p. 55) that $h$ is of the 1st class.

If either $\{h(x_0)\} \cup \{g(x_0)\} \in G$, or $\{h(x_0)\} \cup \{g(x_0)\} \in X - G$, then $h^{-1}(G) = g^{-1}(G)$ and therefore $g^{-1}(G) \in F_\sigma(X)$. If $h(x_0) \in G$ and $g(x_0) \notin G$, then $g^{-1}(G) = h^{-1}(G) - \{x_0\} = h^{-1}(G) \cap (X - \{x_0\})$, if $h(x_0) \notin G$ and $g(x_0) \in G$, then $g^{-1}(G) = h^{-1}(G) \cup \{x_0\}$. Because $X$ is a topological $T_1$ space satisfying the 1st axiom of countability, any one-point set is closed and of the type $G_\delta$. (Therefore $X - \{x_0\} \in F_\sigma(X)$.) Since $F_\sigma(X)$ is a $\sigma$-structure, it is obvious that in both these cases $g^{-1}(G) \in F_\sigma(X)$. The function $g$ is thus of the 1st class.

Similarly a function $g$ can be constructed in the case when $f$ is not lower semi-continuous.

## REFERENCES


[1] Kelley J. L., *General Topology*, New York 1955.

[2] Mišík L., *Über die Eigenschaft von Darboux und einiger Klassen von Funktionen*, Revue Roum. Math. Pures et Appl. 11 (1966), 411—430.

[3] Radakovič T., *Über Darbouxsche und stetige Funktionen*, Monatsh. Math. Phys. 38 (1931), 117—122.

[4] Sikorski R., *Funkcje rzeczywiste*, Warszava 1958.



Received July 18, 1969



*Matematický ústav*
*Slovenskej akadémie vied*
*Bratislava*