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*Matematicko-fyzikálny časopis*, Vol. 15 (1965), No. 4, 304--312

Persistent URL: [http://dml.cz/dmlcz/126440](http://dml.cz/dmlcz/126440)

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ON A PRODUCT OF SEMIGROUPS

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To Professor A. D. Wallace on the occasion of his 60th birthday

The aim of the presented paper is to study the structure of semigroups from Definition 1. Some special cases of semigroups of the studied type are given in Theorems 26—30. The direct product of semigroups is also a special case (Remark 2).

Lemma 1. Let $\Gamma$ be a semigroup. To each element $\alpha$ of $\Gamma$ we assign a semigroup $P_{\alpha}$. Moreover let a set $\mathcal{G}$ of homomorphisms $q_{\beta}^\alpha(\alpha, \beta \in \Gamma)$ be given: $q_{\beta}^\alpha$ is a homomorphism of $P_{\alpha}$ into $P_{\beta}$. Let $q_{\beta}^\alpha \in \mathcal{G}$ iff $\alpha = \beta$, or there exists $\gamma \in \Gamma$ with $\alpha \gamma = \beta$ or $\gamma \alpha = \beta$. Let the set $\mathcal{G}$ have the following properties: 1. For $\alpha \in \Gamma$ $q_{\alpha}^\alpha$ denotes the identical mapping of $P_{\alpha}$ onto $P_{\alpha}$. 2. If for $\alpha, \beta, \gamma \in \Gamma$ there exist in $\mathcal{G}$ homomorphisms $q_{\beta}^\alpha$, $q_{\beta}^{\gamma}$, $q_{\gamma}^{\alpha}$ then $q_{\beta}^{\gamma} q_{\gamma}^{\alpha} = q_{\gamma}^{\alpha}$. Elements of $P_{\alpha}$ will be denoted by $(x, \alpha)$. Let $P$ be a set-theoretical sum of all sets $P_{\alpha}$. Define a multiplication on $P$ as follows: $(x, \alpha)(x, \beta) = q_{\alpha\beta}^\alpha(x, \alpha)q_{\alpha\beta}^\beta(x, \beta)$. The set $P$ with this multiplication is a semigroup.

The proof is easy.

Definition 1. The semigroup $P$ from Lemma 1 will be called the product of semigroups $P_{\alpha}$ over the semigroup $\Gamma$.

Remark 1. In case $P_{\alpha}$ have idempotents, such a construction is always possible. It suffices to take for $q_{\beta}^\alpha$ that mapping under which the image $P_{\alpha}$ is an idempotent of $P_{\beta}$.

Remark 2. A special case of the product from Definition 1 is the direct product $Q \times \Gamma$ of the semigroups $Q$ and $\Gamma$. This can be obtained by taking (in Lemma 1) the semigroup $Q$ for $P_{\alpha}$ for all $\alpha \in \Gamma$ and the identical mapping $Q$ onto $Q$ for the homomorphisms $q_{\beta}^\alpha$.

Remark 3. Evidently $\Gamma$ must be a semigroup in order that the set $P$ in Lemma 1, with the multiplication as indicated, be a semigroup. On the other hand, it is not necessary that $P_{\alpha}$ be semigroups.

Example. Let $\Gamma = \{\alpha, \beta\}$, where $\alpha \alpha = \alpha \beta = \beta \alpha = \beta \beta = \beta$. Let $P_{\alpha}$ be a groupoid, which is not a semigroup. Let $P_{\beta}$ be a semigroup. Moreover,
let there exist a homomorphism $q_{\beta}^\alpha$ of $P_{\alpha}$ into $P_{\beta}$. Then $P$ from Lemma 1 is a semigroup.

In what follows let $I'$ be a semigroup of idempotent elements. The elements of $I'$ will be denoted by $e$ (with indices, if needed). In this case, each semigroup $P_e$ is a subsemigroup of $P$.

We shall obtain now some of the properties of $P$.

**Theorem 1.** Let $J$ be a left ideal of $P$. Then $J \cap P_e$ is a left ideal of $P_e$. Therefore, the ideal $J$ is the union of left ideals (and thus of semigroups) of the semigroups $P_e$.

**Proof.** We shall denote $J \cap P_e = J_e$. Since $J_e \subseteq P_e$ and $P_e$ is a subsemigroup of $P$, $P_eJ_e \subseteq P_e$. Hence $J_e$ is a left ideal of $P_e$.

Similar assertion holds for a right ideal of $P$.

**Theorem 2.** Let each semigroup $P_e$ have a unique idempotent $(i, e)$. Let $L$ be a left ideal of $F$. Let $J_e$ be a left ideal of $P_e$ ($e \in L$), where $J_e$ is a finite semigroup. Then we can construct at least one semigroup which is the product of $P_e$ over $P$, where $J_e$ is a left ideal of $P$.

**Proof.** Let $e \in L$, then for $e_k \in I$ we have $e_k e_l = e_n \in L$. We need to assume $q_{e_k}^e$, $q_{e_k}^e$; let these homomorphisms be $q_{e_k}^e(x, e_l) = e_n$, $q_{e_k}^e(y e_k) = e_n$. Since $J_e$ are finite, $e_j \in J_e$. It follows that $P_J e \subseteq J_e$. Hence $\bigcup e \in L J_e$ is a left ideal of $P$.

We shall now introduce convenient definitions for the principal ideals and $F$-classes.

**Definition 2.** The set $(x, e)_L = P(x, e) \cup \{(x, e)\}$ is said to be the principal left ideal of $P$, generated by $(x, e)$.

Similarly we define the principal right ideal of $P$.

The set of all elements which generate the same principal ideal (left $(x, e)_L$, right $(x, e)_R$) is called $F$-class (left $F_L(x, e)$, right $F_R(x, e)$).

Hereafter analogous results as given for left ideals hold for right ideals.

**Theorem 3.** $(x, e)_L = \bigcup e \in L (P_e q^e_{x}(x, e) \cup \{(x, e)\})$, where $e \in (e)_L$ of $I$.

**Proof.** According to Theorem 1 $(x, e)_L \cap P_{e_l} = J_{e_l}$ for $e \neq e_l$, where $J_{e_l}$ is a left ideal of $P_{e_l}$. Since $J_{e_l} \subseteq (x, e)_L$, for $(y, e_i) \in J_{e_l}$ we have $(y, e_i) = (z, e_k) (x, e)$. Hence according to Lemma 1, $(y, e_i) = q^e_{x}(z, e_l) q^e_{x}(x, e)$; this means that $(y, e_i) \in P_{e_l} q^e_{x}(x, e)$. It follows that $(x, e)_L \cap P_{e_l} \subseteq P_{e_l} q^e_{x}(x, e)$. At the same time, according to Lemma 1, $e_l = e_k e$, hence $e_l = e_l$. This shows that $e_l \in (e)_L$ of $I'$. Since $e_l \in (e)_L$ of $I'$, $e_l = e_l$, thus $P_{e_l}(x, e) = (q^e_{x}P_{e_l}) (q^e_{x}(x, e)) = P_{e_l} q^e_{x}(x, e)$. However, since $P_{e_l}(x, e) \subseteq (x, e)_L$, $P_{e_l} q^e_{x}(x, e) \subseteq (x, e)_L$. This means that $P_{e_l} q^e_{x}(x, e) \subseteq (x, e)_L \cap P_{e_l}$. By the result above, this gives $(x, e)_L \cap P_{e_l} = P_{e_l} q^e_{x}(x, e)$, proving the assertion.
Definition 3. \( F_L(x, e_1) \leq F_L(x, e_2) \) iff \( (x, e_1)_L \subseteq (x, e_2)_L \). (Similarly for \( F_R\)-classes.)

By analogy we shall introduce the relation \( \leq \) for \( F_L\) and \( F_R\)-classes of \( \Gamma \).

Remark. The set of \( F_L(F_R)\)-classes is partially ordered with respect to the relation \( \leq \).

Lemma 2. The ideal \((x, e)_L\) is the union of all \( F_L(y, e_i)\) for which \( F_L(y, e_i) \leq F_L(x, e)\) is true.

Proof. Let \((y, e_i) \in (x, e)_L\). Then \((y, e_i)_L \subseteq (x, e)_L\), therefore \( F_L(y, e_i) \subseteq (x, e)_L\), where \( F_L(y, e_i) \leq F_L(x, e)\). Clearly for \( F_L(j, e_i)\) with \( F_L(j, e_i) \leq (x, e)_L\) we have \( F_L(j, e_i) \subseteq (x, e)_L\).

In the following, the assertions are valid if we replace \((x, e)_L\) by \((x, e)_R\) and \(F_R(x, e)\) by \(F_L(x, e)\).

Theorem 4. a) Let \((e_2)_L \leq (e_1)_L\) in \( \Gamma \). Then for each \((x, e_1)\) there exists \((x, e_2)\) such that \( F_L(x, e_2) \leq F_L(x, e_1)\). b) Let \( F_L(x, e_2) \leq F_L(x, e_1)\). Then \((e_2)_L \leq (e_1)_L\).

Proof. a) We suppose \((e_2)_L \leq (e_1)_L\), then we have \( e_2 = e_2 e_1 \) for some \( e_1\); thus \( e_1 e_1 = e_2\). According to Lemma 1, for \((z, e_2)\) we have \((z, e_2) = (x, e_1)\), whence \((x, e_2)_L \subseteq (x, e_1)_L\). By Definition 3 this means that \( F_L(x, e) \leq F_L(x, e_1)\). b) According to Definition 3 and Lemma 1, \( e_2 e_1 = e_2\), therefore \((e_1)_L \leq (e_1)_L\).

Theorem 5. \( (e_2)_L \leq (e_1)_L \) iff \((e_1)_L \leq (e_2)_L\). \( e_2 e_1 = e_2\) and \( e_1 e_2 = e_1\). Hence \((e_1 e_2)_L \leq (e_1 e_1)_L\). Since \( \Gamma \) is a semigroup of idempotents, we have from the foregoing \( e_2 e_1 e_2 = e_2 e_1 e_2 = e_2 = e_2\). This, together with \((e_1)_L \leq (e_2)_L\) proves that \((e_1)_L = (e_2)_L = (e_3)_L\).

The second part of the proof is evident.

As a consequence we have proved the following

Theorem 6. \( F_L(e_i)\)-classes in \( \Gamma (e_i \in M \subseteq \Gamma)\) form a chain under the relation \( \leq \) iff there exists in \( P \) a chain of \( F_L\)-classes with at least one \( F_L\)-class from each \( P_{e_i}\).

Theorem 7. Let \((e_1)_L = (e_2)_L\) in \( \Gamma \). Then for each \((x_1, e_1)\) there exists a descending chain of \( F_L\)-classes \( \ldots F_L(x_3, e_1) \leq F_L(y_2, e_2) \leq F_L(x_2, e_1) \leq F_L(y_1, e_2) \leq F_L(x_1, e_1)\) in which the classes of \( P_{e_1}\) and \( P_{e_2}\) appear alternately. In case \( P_{e_1}\) and \( P_{e_2}\) are the union of \( F_L\)-classes, this chain is infinite. In case the chain is finite, for some \((x, e_1), (y, e_2)\) the relation \( F_L(x, e_1) = F_L(y, e_2)\) is true.

Proof. Since \((e_1)_L = (e_2)_L\), \( e_1 = e_2\). In a similar manner as in the proof of Theorem 4, for \((y, e_2)\) we obtain \((y, e_2) = (y, e_2) \in (x_1, e_1)_L\). Hence \( F_L(y_1, e_2) \leq F_L(x_1, e_1)\) and further \((x_1, e_1) = (x_1, e_1) \in (y_1, e_2)_L\). This implies \( F_L(x_2, e_1) \leq F_L(y_1, e_2) \leq F_L(x_1, e_1)\). Continuing in this way, we obtain further elements of the chain. The last statement of the theorem is evident from Theorems 4 and 5.
Theorem 8. Let \((e_1)_L = (e_2)_L\) in \(\Gamma\). Then \(P_{e_1}\) and \(P_{e_2}\) are isomorphic semigroups.

Proof. By \((e_1)_L = (e_2)_L\) we have \(e_1e_2 = e_1, e_2e_1 = e_2\). On the other hand, by Lemma 1 there exist homomorphisms \(q^e_{e_1}\) and \(q^e_{e_2}\) and so for \((z, e_1)\) we have \((z, e_1) = q^e_{e_1}(z, e_1) = q^e_{e_1}q^e_{e_2}(z, e_1)\). Hence \(q^e_{e_1}\) is a homomorphism of \(P_{e_1}\) onto \(P_{e_1}\). In the same way we can prove that \(q^e_{e_2}\) is a homomorphism of \(P_{e_1}\) onto \(P_{e_2}\). It follows that \(P_{e_1}\) and \(P_{e_2}\) are isomorphic semigroups.

Theorem 9. Let \(F_L(x, e_2) \cap P_{e_1} \neq \emptyset\). Then \((e_2)_L = (e_1)_L\).

Proof. By hypothesis, for \((x, e_1) \in F_L(x, e_2)\) and for some \((x, e_3)\) we have \((x, e_2) = (x, e_3)(x, e_1)\). Similarly we obtain \((x, e_1) = (x, e_4)(x, e_2)\) for some \((x, e_4)\). Hence, according to Lemma 1, \(e_3e_1 = e_2, e_4e_2 = e_1\), that is \((e_2)_L \subseteq \subseteq (e_1)_L, (e_1)_L \subseteq (e_2)_L\), thus \((e_1)_L = (e_2)_L\).

Theorem 10. \(q^{e_1}_{e_1}P_L(x, e_2) \subseteq F_Lq^{e_1}_{e_1}(x, e_2)\).

Proof. Let \((y, e_2) \in F_L(x, e_2)\), that is \((y, e_2)_L = (x, e_2)_L\). We wish to show that \((q^{e_1}_{e_1}(y, e_2))_L = (q^{e_1}_{e_1}(x, e_2))_L\). By hypothesis, for some \((x, e_2) = (x, e_3)(y, e_2)\), where \(e_3e_2 = e_2\). Then \(q^{e_1}_{e_1}(x, e_2) = q^{e_1}_{e_1}(q^{e_1}_{e_1}(x, e_2)) = (q^{e_1}_{e_1}(y, e_2)) = (q^{e_1}_{e_1}(x, e_2))_L\), whence \(q^{e_1}_{e_1}(x, e_2) \in (q^{e_1}_{e_1}(y, e_2))_L\), therefore \((q^{e_1}_{e_1}(x, e_2))_L \subseteq (q^{e_1}_{e_1}(y, e_2))_L\). In a similar manner we can prove \((q^{e_1}_{e_1}(y, e_2))_L \subseteq (q^{e_1}_{e_1}(x, e_2))_L\) and so \((q^{e_1}_{e_1}(y, e_2))_L = (q^{e_1}_{e_1}(x, e_2))_L\). Hence \(q^{e_1}_{e_1}F_L(x, e_2) \subseteq F_Lq^{e_1}_{e_1}(x, e_2)\).

Theorem 11. Let \((e_1)_L = (e_2)_L\). Then \(q^{e_1}_{e_1}F_L(x, e_2) = F_L(q^{e_1}_{e_1}(x, e_2))\).

Proof. After considering Theorem 10 there remains to be shown that: if \((y, e_1) \in F_Lq^{e_1}_{e_1}(x, e_2)\) then \((y, e_1) = q^{e_1}_{e_1}(z, e_2)\), where \((z, e_2) \in F_L(x, e_2)\). Using the proof of Theorem 10 we can see that \((z, e_2) = (q^{e_1}_{e_1}(y, e_1))_L = (q^{e_1}_{e_1}q^{e_1}_{e_1}(x, e_2))_L = \) \((x, e_2)_L\).

Remark. Clearly, if \((e_1)_L = (e_2)_L\), the ideal \((x, e_1)_L\) is isomorphic to \((q^{e_1}_{e_1}(x, e_1))_L\) (which follows from Theorem 8 as well).

Theorem 12. a) Let the \(F_L(e)\)-class in \(\Gamma\) consist of a unique element. Then \(P_e\) is the union of \(F_L\)-classes in \(P\). b) Let \(\Gamma\) be a commutative semigroup. Then \(P_e\) are the union of \(F_L\) classes in \(P\).

Proof. a) Let \(F_L(x, e) \cap P_e \neq \emptyset\), then, according to Theorem 9, \((e)_L = \) \((e)_L\) — a contradiction. b) Suppose that \((e_1)_L = (e_2)_L\) in \(\Gamma\); hence \(e_1 = e_2\). Then a) implies b).

Theorem 13. Let \(F_L(x, e_2) \cap P_{e_1} \neq \emptyset\); then \(F_R(x, e_2) \cap P_{e_1} = \emptyset\) for all \((x, e_2)\).

Proof. Let \((y, e_1) \in F_L(x, e_2)\). Then for some \((z, e_3)\) we have \((x, e_2) = (z, e_3)(y, e_1)\) and according to Lemma 1, \(e_3e_1 = e_2\), that is \(e_3e_1 = e_2\). On the
other hand, let \((e, e_1) \in F_R(x, e_2)\). Similarly we can show that \(e_2 e_1 = e_1\). Finally we have \(e_1 = e_2\) (clearly, we consider only \(e_1 \neq e_2\)).

**Theorem 14.** Let \(F_L(x, e_2) \subseteq F_L(y, e_1)\) for \(e_1 \neq e_2\). Then either \(F_R(x, e_2) \subseteq F_R(y, e_1)\), or \(F_R(x, e_2)\) are incomparable.

**Proof.** By hypothesis, for some \((x, e_3)\) we get \((x, e_2) = (x, e_3) (y, e_1)\), then \(e_3 e_1 = e_2\), that is \(e_2 e_1 = e_2\). Let \(F_R(y, e_1) \subseteq F_R(x, e_2)\). Similarly we obtain \(e_2 e_1 = e_1\). Finally we have \(e_1 = e_2\) — a contradiction.

**Theorem 15.** Let \((e_1)_L = (e_2)_L\), \(e_1 \neq e_2\). Then \(F_R(x, e_1)\), \(F_R(x, e_2)\) are incomparable.

**Proof.** Theorem 4 for \(F_R\) classes may now be applied to show that \(F_R(x, e_2) \subseteq F_R(x, e_1)\) implies \((e_2)_R \subseteq (e_1)_R\), which is to say that \(e_1 e_2 = e_2\). Since \((e_1)_L = (e_2)_L\), \(e_1 e_2 = e_1\). Finally we have \(e_1 = e_2\) — a contradiction. In a similar manner it can be shown that \(F_R(x, e_1) \subseteq F_R(x, e_2)\) does not hold.

**Remark.** Theorem 15 (according to Lemma 2) may be interpreted as follows: if \((e_1)_L = (e_2)_L\), then \((x, e_1)_R \cap P e_2 = \emptyset\) for all \((x, e_1)\).

**Theorem 16.** Let \((i, e)\) be an idempotent of \(P\). Then a left ideal \(L\) has the right identity \((i, e)_L\) iff \(L = (i, e)_L\).

**Proof.** Let \(L\) be a left ideal in \(P\) and \((i, e)\) its right identity, which means \((x, e) (i, e) = (x, e)\) for \((x, e) \in L\), whence \(L \subseteq (i, e)_L\). Since \((i, e) \in L\) (is its right identity), \((i, e)_L \subseteq L\). According to the foregoing result \((i, e)_L = L\).

**Theorem 17.** Let \((i, e)\) be an idempotent. Then \((x, e) (i, e) = (x, e)\) holds for \((x, e) \in F_L(i, e)\).

**Proof.** The statement is clear, since \((x, e) = (y, e) (i, e)\) (because \((x, e)_L = (i, e)_L\).  

**Remark.** For \((x, e) \in F_R(i, e)\) we have \((x, e) = (i, e) (x, e)\).

**Theorem 18.** Let \(P_{e_1}\), \(P_{e_2}\) have the unique idempotents \((i, e_1)\), \((i, e_2)\). Let \((x, e_1)_L = (x, e_2)_L\). Then \((i, e_1)_L = (i, e_2)_L\).

**Proof.** By hypothesis and Theorem 9 we have \((e_1)_L = (e_2)_L\). Using Theorem 11 we can prove our assertion.

**Theorem 19.** Let \(P_{e_1}\), \(P_{e_2}\) have the unique idempotents \((i, e_1)\), \((i, e_2)\). Then \((i, e_1)_L = (i, e_2)_L\) iff \((e_1)_L = (e_2)_L\).

**Proof.** Let \((i, e_1)_L = (i, e_2)_L\). Using Theorem 9 we can see that \((e_1)_L = (e_2)_L\). Let \((e_1)_L = (e_2)_L\); from Theorem 11 it follows \((i, e_1)_L = (i, e_2)_L\).

**Theorem 20.** Let \(P_{e}\) have a unique idempotent \((i, e)\). Let \(F_L(i, e) = F_R(i, e)\). Then \(F_L(i, e)\) is the maximal subgroup of \(P\).

**Proof.** According to Theorem 17 \((i, e)\) is an identity in \(F_L(i, e)\). Moreover \((y, e)_L = (x, e)_L = (i, e)_L\) implies \(((x, e) (y, e))_L = ((i, e) (y, e))_L = (y, e)_L\).
Hence $\mathbb{F}_L(i, e)$ is a semigroup. We shall prove now that $(x, e)_L = (i, e)_L$ implies the existence of $(z, e)$ with $(z, e)(x, e) = (i, e)$. Because $(x, e)_R = (i, e)_R$, for some $(v, e)$ we have $(x, e) = (i, e)(v, e)$, whence $(z, e)(x, e) = (z, e)(i, e)(v, e)$, therefore $(i, e)_R \subseteq ((z, e)(i, e))_R$. Similarly we can prove $((z, e)(i, e))_R \subseteq (i, e)_R$; thus $(i, e)_R = ((z, e)(i, e))_R$. Hence $(z, e)(i, e) \in \mathbb{F}_L(i, e) = (i, e)_L$. Thus $[(z, e)(i, e)](x, e) = (z, e)(i, e)(x, e) = (i, e)(x, e) = (i, e)$ as required. This shows that $\mathbb{F}_L(i, e)$ is a group.

It is evident that the elements of the group generate the same principal left (right) ideal. Therefore $\mathbb{F}_L(i, e)$ is a maximal subgroup of $P$.

We derive next (Theorem 21—25) some of the properties of semigroups with identity (hypogroup).

**Theorem 21.** Let the semigroup $P$ be the product of semigroups $P_\alpha$ over the semigroup $\Gamma$. In this case $P$ will be a hypogroup iff $\Gamma$ and $P_\alpha$ are hypogroups (where $\Gamma$ is a semigroup of idempotents). Moreover, if $e$ is the identity in $\Gamma$ and $(j, e)$ the identity in $P_e$, then $(j, e)$ is the identity in $P$.

**Proof.** Let $P$ be the product of hypogroups over the hypogroup of idempotents $\Gamma$. Since $\Gamma$ is isomorphic to the semigroup of identity elements in $P_e$ (because the image of identity is an identity), we can see, using Lemma 1, that $P$ is a hypogroup.

Let $P$ be the hypogroup which is the product of the semigroups $P_\alpha$ over the semigroup $\Gamma$. Let $(j, e)$ be the identity in $P$. Since $(j, e)$ is an idempotent, according to Lemma 1, $e e = e e = e$ and so $e$ is an identity in $\Gamma$. Hence $\Gamma$ is a hypogroup. Moreover, according to Lemma 1, it follows that $a_{\alpha}(j, e)$ is an idempotent in $P_{\alpha}$, therefore $e$ is an idempotent in $\Gamma$. As for each $e \in \Gamma$ we have $ee = e$, $\Gamma$ is a hypogroup of idempotents. But since the image of identity is an identity, $q_{\alpha}(j, e)$ is the identity in $P_{\alpha}$. This means that $P_{\alpha}$ is a hypogroup. This completes the proof.

**Theorem 22.** The necessary and sufficient condition for hypogroup $P$ to be the product of semigroups over the semigroup $\Gamma$ is that there exist on $P$ a congruence, the classes of which are hypogroups, while their identity elements form a subsemigroup of $P$.

**Proof.** We denote the classes of the congruence by $S_{e_i}(i = 1, 2, \ldots).(j, e_i)$ is the identity in $S_{e_i}$. It follows that $(x, e_1)(y, e_2) \in S_{e_1 e_2}$. We shall show that the mapping $(x, e_1) \rightarrow (x, e_1)(j, e_2)$ is a homomorphism of $S_{e_1}$ into $S_{e_1 e_2}$ and $(y, e_2) \rightarrow (j, e_1)(y, e_2)$ is a homomorphism of $S_{e_2}$ into $S_{e_1 e_2}$. The following holds: $((x_1, e_1)(j, e_2))(x_2, e_1)(j, e_2) = (x_1, e_1)(j, e_1)(j, e_2) = (x_1, e_1)(j, e_1)(j, e_2)$ (because $(x_2, e_1)(j, e_2) \in S_{e_1 e_2}$, hence $(j, e_1)(j, e_2))((x_2, e_1)(j, e_2)) = (x_2, e_1)(j, e_2)$ as required. Simi-
larly it can be shown that \((y, e_2) \rightarrow (j, e_1) (y, e_2)\) is a homomorphism of \(S_{e_1}\) into \(S_{e_2}\). We denote these homomorphisms by \(q^1_{12}, q^2_{12}\). Clearly \(q^1_{32} q^2_{12} = q^1_{32}\). Using such homomorphisms we can construct the product of the semigroups \(S_{e_1}\) over the semigroup \(\Gamma\), where \(\Gamma\) is a semigroup isomorphic to the semigroup of identity elements in \(P\); in this way we obtain exactly \(P\).

The necessary condition is evident by considering that the image of identity is identity.

The following holds for hypogroups of Theorem 21 (hereafter the identity in \(P\) is denoted by \((j, e)\)).

**Lemma 3.** For each \((x, e)\) the relation \(F_L(x, e) \leq F_L(j, e)\) is true.

**Theorem 23.** Let \((j, e_1) L \neq (j, e_2) L\). If \(F_L(x, e_1) \leq F_L(y, e_2)\) for some \((x, e_1), (y, e_2)\), then: a) \(F_L(j, e_1) \leq F_L(j, e_2)\), b) \(F_L(z, e_2) \leq F_L(j, e_1)\) is not true for any \((z, e_2)\).

**Proof.** a) By hypothesis, there exists such an element \((z, e_3)\) that \((x, e_1) = (z, e_3) (y, e_2)\) and so \((x, e_1) = (z, e_3) (j, e_2) = (x, e_1) (j, e_2)\). According to Lemma 1 \((j, e_1) (j, e_2) = (j, e_1)\) which is to say that \((j, e_1) L \leq (j, e_2) L\). b) Let \(F_L(z, e_2) \leq F_L(j, e_1)\) for some \((z, e_2)\). According to a), \(F_L(j, e_2) \leq F_L(j, e_1)\). On the other hand, by hypothesis and by a) we have \(F_L(j, e_1) \leq F_L(j, e_2)\); thus we obtain \((j, e_1) L = (j, e_2) L\) — a contradiction.

**Theorem 24.** For \(F_L(j, e)\) and \(F_R(j, e)\) the following are true: a) \(F_L(j, e)\) and \(F_R(j, e)\) are semigroups. b) For each \((x_1, e) \in F_L(j, e)\) there exists \((s_2, e) \in F_R(j, e)\) such that \((s_2, e) (x_1, e) = (j, e)\). Similarly for each \((x_2, e) \in F_R(j, e)\) there exists \((s_1, e) \in F_L(j, e)\) such that \((x_2, e) (s_1, e) = (j, e)\). c) Each element \((s, e) \in F_L(j, e) \cup F_R(j, e)\) can be written in the form \((s, e) = (z_2, e) (z_1, e)\) where \((z_1, e) \in F_L(j, e)\), \((z_2, e) \in F_R(j, e)\) for \((z_1, e)\) or \((z_2, e)\) given before.

**Proof.** a) Let \((j, e)_L = (j, e_1)_L = (x, e)_L = (x, e_1)_L\). Then \(((x, e) (x, e_1))_L = ((j, e_1) (x, e_1))_L = (x, e_1)_L\). Similarly \(((x, e_1) (x, e))_L = (x, e)_L\). b) \((x_1, e)_L = (j, e)_L\) implies \((s, e) (x_1, e) = (j, e)\) for some \((s, e)\). According to Lemma 1, \(e e = e\). We denote \(q^0_{e}(s, e) = (s_2, e)\). Evidently \((s_2, e) \in (j, e)_R\). But \((j, e)_L = (s_2, e) (x_1, e)\), therefore \((j, e) \in (s_2, e)_R\), hence \((j, e)_R = (s_2, e)_R\), thus \((s_2, e) \in (j, e)_L\). The second assertion can be proved in the same way. c) Let \((s_1, e) \in (j, e)_L\) and let \((z_2, e) \in F_R(j, e)\). Then according to b) for some \((u_1, e) \in (j, e)_L\) we have \((j, e) = (z_2, e) (u_1, e)\), hence \((s_1, e) = (z_2, e) (u_1, e) (s_1, e)\), whence according to a) \((u_1, e) (s_1, e) = (z_1, e) \in F_L(j, e)\), thus \((s_1, e) = (z_2, e) (z_1, e)\). Similarly the second part of the assertion can be proved.

**Theorem 25.** The left ideal \(L\) has the identity \((j, e)\) iff \(L = (j, e)_L\) and \((e)_L\) has an identity.

**Proof.** Let \(L = (j, e)_L\). Since \(e e_k = e_k\) (because \(e\) is the identity in \((e)_L\),

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\(q_{\nu k}(j, e) = (j, e_k)\), hence \((j, e) (l, e_k) = (l, e_k)\) for \((l, e_k) \in L\). Thus \((j, e)\) is the identity in \(L\).

Let \(L\) possess the identity \((j, e)\). Clearly, the semigroup of elements \((j, e) \in L\) is the left ideal in the subsemigroup of all identity elements of \(P\), and is isomorphic to the subsemigroup of \(I\), which is, therefore, a left ideal in \(I\). This ideal evidently possesses the identity \(e\). It is therefore \((e)_L\). According to Theorem 16, \(L = (j, e)_L\).

Remark. From the hypothesis of the theorem stating that \((e)_L\) has an identity, it follows that \(F_L(j, e) \cap P_e = \emptyset\) for \(e \neq e\).

Finally, we mention some examples of semigroups which are the product of semigroups over a semigroup, the properties of which have already been studied.

From [1] it follows:

The semigroup \(S\) is said to admit relative inverses if to each \(a \in S\) there exists an element \(e \in S\) such that \(ae = ea = a\) and an element \(a' \in S\) such that \(a'a = aa' = e\). Then following holds:

**Theorem 26.** Each semigroup admitting relative inverses in which every pair of idempotents commute is a product of groups over a semigroup of idempotents.

From [2] and [3] we have:

**Theorem 27.** Each finite simple semigroup \(S\) without zero, having at least one minimal left ideal and at least one minimal right ideal, while the idempotents form a semigroup in \(S\), is the product of isomorphic groups over the simple semigroup without zero.

From [4] can be deduced:

In a periodic semigroup let the set of elements \(x\) with \(x^n = e\) (for some \(n\) and for the idempotent \(e\)) be called \(K\)-class belonging to \(e\).

**Theorem 28.** The product of commutative periodic semigroups \(P_e\) over the commutative semigroup of idempotents (semilattice) is a commutative periodic semigroup in which the \(K\)-classes are exactly \(P_e\). Moreover each commutative periodic semigroup, the \(K\)-classes of which are groups, is the product of commutative periodic groups over a semilattice.

We shall say that the periodic semigroup \(S\) is partially commutative if for each \(e \in S\) and each \(x \in S\), \(xe = ex\) is true.

From [5] it follows:

**Theorem 29.** The product of partially commutative periodic semigroups having a unique idempotent over a commutative semigroup of idempotents (semilattice) is a partially commutative semigroup, the \(K\)-classes of which are exactly \(P_e\).
**Theorem 30.** (according to [6]). Let each principal left ideal in the semigroup $P$ contain an identity. Then $P$ is the product of groups over the commutative semigroup of idempotents (semilattice) iff for each $e \in \Gamma$, $(j, e)_L = (j, e)_R$ where $(j, e)$ is the identity in $P_e$.

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Received December 5, 1964.

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**ОБ ОДНОМ ПРОИЗВЕДЕНИИ ПОЛУГРУПП**

Бланка Колибиарова

Резюме

Пусть $\Gamma$ — полугруппа и пусть всякому элементу $x$ из $\Gamma$ поставлена в соответствие полугруппа $P_x$. Пусть дано множество $\mathfrak{G}$ гомоморфизмов $\varphi^\alpha_\beta (x, \beta \in \Gamma)$, где $\varphi^\alpha_\beta$ гомоморфное отображение $P_\alpha$ в $P_\beta$. Пусть при этом $\varphi^\alpha_\beta \in \mathfrak{G}$ тогда и только тогда, когда $x = \beta$ или существует $\gamma \in \Gamma$ для которого $x\gamma = \beta$ или $\gamma\alpha = \beta$. Пусть множество $\mathfrak{G}$ удовлетворяет условиям: 1. Для $x \in \Gamma$ $\varphi^x_x$ является тождественным отображением $P_x$ на $P_x$. 2. Если для $\alpha, \beta, \gamma \in \Gamma$ существуют в $\mathfrak{G}$ гомоморфизмы $\varphi^\alpha_\beta, \varphi^\beta_\gamma, \varphi^\gamma_x$ тогда $\varphi^{\beta\gamma}_x = \varphi^x_\gamma$. Обозначим элементы множества $P_x$ через $(x, \alpha)$. Пусть $P$ — теоретически-множественное объединение множеств $P_x (x \in \Gamma)$. Определим в $P$ умножение следующим образом: $(x, \alpha)(x, \beta) = \varphi^x_{\alpha\beta}(x, \alpha)\varphi^x_{\alpha\beta}(x, \beta)$. Множество $P$ с таким умножением является полугруппой. Назовем ее произведением полугрупп $P_x$ над $\Gamma$.

В настоящей статье изучаются некоторые свойства этих произведений.