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ALGEBRAIC CONSIDERATIONS ON POWERS OF STOCHASTIC MATRICES

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Let A be a non-negative matrix. One of the problems in studying such matrices is the solution of the following algebraic (or better to say combinatorial) problem: What can be said about the distribution of zeros and non-zeros in the sequence

$$A, A^2, A^3, \ldots \quad . \tag{1}$$

If A is, moreover, a stochastic matrix the question concerning the existence of $\lim\limits_{m=\infty} A^m$ arises.

It is the aim of this paper to show that in some cases algebraic criteria are sufficient to decide about the existence or non-existence of the limit just considered.

Some of the results of this paper are not new in the sense that they are implicitly contained in various considerations concerning finite Markov chains. Our considerations culminate in a certain sense in Theorem 7 which does not appear in any form in the vast literature on stochastic matrices.

For convenience of the reader we recall some notions introduced by the author in previous papers [1], [2], [3], [4], which we shall need in the following.

1. Algebraical preliminaries. Let $N = \{1, 2, \ldots, n\}$. Consider the set of „$n \times n$ matrix units“, i. e. the set of symbols $\{e_{ij} | i, j \in N\}$ together with a zero 0 adjoined: $S = \{0\} \cup \{e_{ij} | i, j \in N\}$. Define in S a multiplication by

$$e_{ij}e_{ml} = \begin{cases} 0 & \text{for } j \neq m, \\ e_{il} & \text{for } j = m, \end{cases}$$

the zero having the usual properties of a multiplicative zero. The set $S = S_n$ with this multiplication is a 0-simple semigroup.

Let A be a non-negative $n \times n$ matrix. By the support of A we shall mean the subset of S containing 0 and all those elements $e_{ij} \in S$ for which $a_{ij} > 0$. The support of A will be denoted by C_A or (for typographical reasons) also $C(A)$.

Denote further by $\mathfrak{S}_n$ the set of all subsets of S and define in $\mathfrak{S}_n$ a multiplication as the multiplication of complexes. Then $\mathfrak{S}_n$ is again a finite semigroup.

For any two $n \times n$ non-negative matrices A, B we always have $C_{A+B} = C_A \cup C_B$ and $C_{AB} = C_A C_B$. In particular the supports of the elements of the sequence (1) are given by the sequence

$$(2) \qquad C_A, C_A^2, C_A^3, \ldots,$$

which clearly contains only a finite number of different elements (subsets of S).

The following facts follow from the elements of the theory of finite semigroups.

Let $k = k(A)$ be the least integer such that $C_A^h = C_A^k$ for some $h > k$. Let further $k + d$ $(d \geqq 1)$ be the least integer such that $C_A^k = C_A^{k+d}$ holds. Then the sequence (2) is of the form

$$C_A, \ldots, C_A^{k-1} | C_A^k, \ldots, C_A^{k+d-1} | C_A^k, \ldots, C_A^{k+d-1} | \ldots$$

For any $\alpha \geqq k$ and every $\beta \geqq 0$ we have $C_A^\alpha = C_A^{\alpha+\beta d}$. It is well known that

$$\mathfrak{G}_A = \{C_A^k, C_A^{k+1}, \ldots, C_A^{k+d-1}\}$$

is a cyclic group of order d (subgroup of $\mathfrak{S}_n$). The unit element of the group $\mathfrak{G}_A$ is C_A^ϱ with a suitably chosen ϱ satisfying $k \leqq \varrho \leqq k + d - 1$. It is easy to prove that d/ϱ. Note that we may also write $\mathfrak{G}_A = \{C_A^\varrho, C_A^{\varrho+1}, \ldots, C_A^{\varrho+d-1}\}$.

Note further explicitly that the integers $k = k(A)$, $d = d(A)$, $\varrho = \varrho(A)$ are defined for any non-negative matrix A (and they can be found in a finite number of steps).

For further purposes we mention the following simple facts proved in [1]. For any non-negative $n \times n$ matrix A we always have $C_A^{n+1} \subset C_A \cup C_A^2 \cup \cup \ldots \cup C_A^n$ so that the set $C_A \cup C_A^2 \cup \ldots \cup C_A^n$ is always a subsemigroup of $S = S_n$. A non-negative matrix A is called reducible if there is a permutation matrix P such that $P^{-1}AP$ is of the form

$$\begin{pmatrix} A_1 & 0 \\ \mathrm{B} & A_2 \end{pmatrix}.$$

Otherwise it is called irreducible. It is called completely reducible if in any such form B is a zero matrix. An $n \times n$ non-negative matrix A is irreducible if and only if $C_A \cup C_A^2 \cup \ldots \cup C_A^n = S$.

This is the case if and only if

$$C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+d-1} = S.$$

Here the summands to the left are quasidisjoint, i. e. the intersection of any two of them is $\{0\}$.

2. Topological preliminaries. The product of two stochastic matrices is again a stochastic matrix. Hence the set of all stochastic matrices forms (with respect to the multiplication of matrices) a semigroup which will be denoted by $\mathfrak{Z}_n$.

Introduce in $\mathfrak{Z}_n$ a natural topology by the requirement $P^{(n)} = (p_{ij}^{(n)}) \to \to P = (p_{ij})$ if and only if $p_{ij}^{(n)} \to p_{ij}$ for every i, j. Since the multiplication of matrices in this topology is continuous in both factors, $\mathfrak{Z}_n$ clearly becomes a compact (Hausdorff) semigroup.

We recall some elementary results concerning compact semigroups which we shall use in the following.

Let $\mathfrak{S}$ be a compact (Hausdorff) semigroup and $a \in \mathfrak{S}$. Consider the cyclic semigroup $\mathfrak{A} = \{a, a^2, a^3, \ldots\}$ and its closure $\overline{\mathfrak{A}}$. It is well known that $\overline{\mathfrak{A}}$ contains a unique idempotent e. More precisely: If $\mathfrak{A}_h = \{a^h, a^{h+1}, a^{h+2}, \ldots\}$, then $\bigcap_{h=1}^{\infty} \overline{\mathfrak{A}}_h = \Gamma(a)$ is a group and e is the unit element of $\Gamma(a)$. We shall say that a belongs to the idempotent e. $\Gamma(a)$ is the unique maximal subgroup contained in the compact Abelian semigroup $\overline{\mathfrak{A}} = \overline{\{a, a^2, a^3, \ldots\}}$. Further we have

$$(3) \qquad \mathfrak{A} \,.\, e = e \,.\, \mathfrak{A} = \overline{\mathfrak{A}} \,.\, e = e \,.\, \overline{\mathfrak{A}} = \Gamma(a).$$

Recall also that to the idempotent e there exists a unique maximal subgroup $G(e)$ of $\mathfrak{S}$ containing e as its unit element. Clearly $\Gamma(a) \subset G(e)$.

Consider the sequence

$$(4) \qquad a, a^2, a^3, \ldots$$

and suppose that it belongs to the idempotent e. If (4) converges, it converges necessarily to e. This is the case if and only if $\Gamma(a)$ reduces to the element e. With respect to the relation (3) we may formulate this as follows; Suppose that $a \in \mathfrak{S}$ belongs to the idempotent e. The necessary and sufficient condition for the convergence of the sequence (4) is the fulfilment of the relation $ae = = ea = e$. We then have $\lim_{m=\infty} a^m = e$.

For all these results see [2] and [3].

3. The relation between $l(A)$ and $d(A)$. Let now A be a stochastic matrix and suppose that it belongs to the idempotent J. Denote $\mathfrak{A} = \{A, A^2, A^3, \ldots\}$. Then — as remarked above — $\overline{\mathfrak{A}}J = J\overline{\mathfrak{A}} = \Gamma(A)$. In [2] we have proved that $\Gamma(A)$ is a finite cyclic group. Also the maximal group $G(J)$ belonging to J is a finite group isomorphic to the symmetric group of s letters if J is of the rank s.

Let $l = l(A)$ be the least integer such that $(AJ)^l = J$. Then

$$\Gamma(A) = \{AJ, A^2J, \ldots, A^lJ = J\}.$$

This implies: The necessary and sufficient condition for the existence of $\lim_{m=\infty} A^m$ is the fulfilment of the relation $l(A) = 1$, i. e. $AJ = JA = J$.

Note that $l(A)$ is defined only for stochastic matrices, while $d(A)$ is defined for any non-negative matrix.

Our first aim is to find the relation between these two numbers.

Theorem 1. *For any stochastic matrix A we always have $l(A)/d(A)$.*

Proof. Consider the semigroup

$$\mathfrak{S}_A = \{C_A, C_A^2, \ldots, C_A^{k+d-1}\}$$

and the mapping

$$C_A^h \to A^h J \quad (h = 1, 2, \ldots, k + d - 1).$$

Since

$$C_A^{h_1} \cdot C_A^{h_2} = C_A^{h_1+h_2} \to A^{h_1+h_2}J = A^{h_1}J \cdot A^{h_2}J,$$

this is a homomorphic mapping of $\mathfrak{S}_A$ onto the cyclic group $\Gamma(A)$. (It is onto since every $A^\alpha J$ is the image of some C_A^α.) In this mapping the group $\mathfrak{G}_A = \{C_A^k, \ldots, C_A^{k+d-1}\}$ is mapped onto a subgroup of $\Gamma(A)$. Clearly the image of C_A^ϱ [the unit elements of $\mathfrak{G}_A$] is J [the unit $\mathfrak{G}$ element of $\Gamma(A)$]. To prove that $\mathfrak{G}_A$ is mapped onto the group $\Gamma(A)$ itself it is sufficient to show that every element $A^hJ \in \Gamma(A)$ is the image of some element $C_A^\tau \in \mathfrak{G}_A$. This is true since $C_A^{\varrho+h} = C_A^\varrho \cdot C_A^h$ is contained in $\mathfrak{G}_A$ and its image is $J \cdot A^hJ = JA^h$. Since $\Gamma(A)$ is a homomorphic image of $\mathfrak{G}_A$, we have l/d.

Theorem 2. *If A is stochastic and $d(A) = 1$, then $\lim_{m=\infty} A^m$ exists.*

Proof. In this case l/d implies $l = 1$, which is the necessary and sufficient condition for the existence of the limit considered.

By definition of the number l the matrices $J, AJ, \ldots, A^{l-1}J$ are all different. Our next goal is to prove that also their supports are different subsets of S. We use this occasion to prove a stronger assertion.

Let U be an idempotent $\in \mathfrak{Z}_n$ and $G(U)$ the maximal group of $\mathfrak{Z}_n$ belonging to U (i. e. having U as its unit element). If P is any permutation matrix $\in \mathfrak{Z}_n$, then the maximal group belonging to the idempotent $P^{-1}\,U\,P$ is $P^{-1}G(U)P$. In [2] we have proved; Any idempotent stochastic matrix U (of rank s) can be written in the form $P^{-1}JP$, where P is a permutation matrix and J is a matrix of the form

$$J = \begin{pmatrix} Q_1 & 0 & \ldots & 0 & 0 \\ 0 & Q_2 & \ldots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \ldots & Q_s & 0 \\ F_1 & F_2 & \ldots & F_s & 0 \end{pmatrix}. \tag{5}$$

Here Q_i is an $r_i \times r_i$ matrix ($r_i > 0$) of the form

$$Q_i = \begin{pmatrix} u_i', u_i'', \ldots, u_i^{(r_i)} \\ \vdots \\ u_i', u_i'', \ldots, u_i^{(r_i)} \end{pmatrix}$$

and $u_i' + u_i'' + \ldots + u_i^{(r_i)} = 1$. Denote $f = n - (r_1 + r_2 + \ldots + r_s) \geqq 0$. Denote further by $Q_i^{(t)}$ the following matrix having t identical rows

$$Q_i^{(t)} = \begin{pmatrix} u_i', u_i'', \ldots, u_i^{(r_i)} \\ \vdots \\ u_i', u_i'', \ldots, u_i^{(r_i)} \end{pmatrix}$$

and by D_i the diagonal matrix $D_i = [\varrho_i', \varrho_i'', \ldots, \varrho_i^{(f)}]$, $0 \leqq \varrho_i^{(\tau)} \leqq 1$. Then F_i is of the form $F_i = D_i \,.\, Q_i^{(f)}$, where $\sum_i \varrho_i' = \sum_i \varrho_i'' = \ldots = \sum_i \varrho_i^{(f)} = 1$. Conversely, any matrix satisfying these conditions is an idempotent stochastic matrix of rank s and order n.

After these remarks we shall prove the following

Theorem 3. *Let $G(J)$ be the maximal group of $\mathfrak{Z}_n$ belonging to the idempotent $J \in \mathfrak{Z}_n$. Then any two different matrices contained in $G(J)$ have different supports.*

Proof. With respect to the remark above we may restrict ourselves to the case that J is of the form (5). In [2] we have proved that $G(J)$ contains exactly $s!$ different elements (stochastic matrices). Any element $B \in G(J)$ is of the form

$$B = \begin{pmatrix} c_{11}Q_1^{(r_1)}, & c_{12}Q_2^{(r_1)}, & \ldots, & c_{1s}Q_s^{(r_1)}, & 0 \\ c_{21}Q_1^{(r_2)}, & c_{22}Q_2^{(r_2)}, & \ldots, & c_{2s}Q_s^{(r_2)}, & 0 \\ \vdots \\ c_{s1}Q_1^{(r_s)}, & c_{s2}Q_2^{(r_s)}, & \ldots, & c_{ss}Q_s^{(r_s)}, & 0 \\ H_1 & H_2, & \ldots, & H_s, & 0 \end{pmatrix}$$

where $H_i = (\sum_{\alpha=1}^{s} c_{\alpha i} D_\alpha) Q_i^{(f)}$ and (c_{ij}) is a permutation matrix of order s. Conversely, taking for (c_{ij}) all possible $s!$ permutation matrices of order s we obtain all elements $\in G(J)$.

Now taking into account the fact that two different permutation matrices (c_{ij}) have different distributions of zeros and ones we immediately see that two different permutation matrices lead to two matrices having different supports. This proves our statement.

Our Theorem implies, in particular, that $C(J)$, $C(AJ)$, $\ldots$, $C(A^{l-1}J)$ all differ one from another. Hence we have the following

Corollary 3. *If A belongs to the idempotent J and $C(J) = C(AJ)$, then $J = JA = AJ$.*

4. The relation between C_A and C_J. We now ask: What can be said about C_J by knowning C_A. Consider the sequence

$$A^{\varrho}, A^{2\varrho}, A^{3\varrho}, \ldots$$

The support of each member of this sequence is the same semigroup $C_A^{\varrho} = C_A^{2\varrho} = C_A^{3\varrho} = \ldots$ We conclude that for any matrix $M \in \overline{\{A^{\varrho}, A^{2\varrho}, A^{3\varrho}, \ldots\}}$ we have $C_M \subset C_A^{\varrho}$. In particular, we have $C_J \subset C_A^{\varrho}$. This implies (for any integer $h > 0$)

$$C(A^h J) = C(A^h)C(J) \subset C(A^h)C(A^{\varrho}) = C(A^{h+\varrho}).$$

Hence

$$C(J) \cup C(JA) \cup \ldots \cup C(JA^{l-1}) \subset C(A^{\varrho}) \cup C(A^{\varrho+1}) \cup \ldots \cup C(A^{\varrho+l-1}).$$

More generally for any $h \geqq \varrho$ we have

$$C(J) \cup C(JA) \cup \ldots \cup C(JA^{l-1}) \subset C(A^h) \cup C(A^{h+1}) \cup \ldots \cup C(A^{h+l-1}).$$

Further

$$\begin{aligned} C(J) &= C(JA^{l}) \subset C(A^{\varrho+l}), \\ C(J) &= C(JA^{2l}) \subset C(A^{\varrho+2l}), \\ &\vdots \end{aligned}$$

etc., imply

$$C(J) \subset C_A^{\varrho} \cap C_A^{\varrho+l} \cap C_A^{\varrho+2l} \cap \ldots \cap C_A^{\varrho+d-l}.$$

Analogously we have

$$\begin{aligned} C(AJ) &\subset C_A^{\varrho+1} \cap C_A^{\varrho+l+1} \cap \ldots \cap C_A^{\varrho+d-l+1}, \\ &\vdots \\ C(A^{l-1}J) &\subset C_A^{\varrho+l-1} \cap C_A^{\varrho+2l-1} \cap \ldots \cap C_A^{\varrho+d-1}. \end{aligned}$$

We summarize:

Lemma 1. *If A belongs to the idempotent J and d, ϱ, l have the meaning introduced above, then for $h = 0, 1, \ldots, l-1$ we have*

$$C(A^h J) \subset C_A^{\varrho+h} \cap C_A^{\varrho+h+l} \cap \ldots \cap C_A^{\varrho+h+d-l}.$$

The following special cases are of some interest:

Lemma 2. *If $C_J = C_A^{\varrho}$, then $d = l$.*

Proof. $C_J = C_A^{\varrho}$ implies $C(JA^l) = C(A^{\varrho+l})$, i. e. $C_A^{\varrho} = C_A^{\varrho+l}$, hence $d \leqq l$. By Theorem 1 we have $l = d$.

Remark. The converse is not true. For the matrix $A = \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ we have $d = l = 1$, but $C_J \subsetneqq C_A = C_A^{\varrho}$.

Lemma 3. *If $C_A \subset C_J$, then $C_J = C_A^{\varrho}$ (hence $d = l$).*

Proof. $C_A \subset C_J$ implies $C_A^2 \subset C_J^2$, i. e. $C_A^2 \subset C_J$. Analogously $C_A^{\varrho} \subset C_J$. Since $C_J \subset C_A^{\varrho}$ always holds, we have $C_J = C_A^{\varrho}$ (hence by Lemma 2 $d = l$).

We now give a necessary and sufficient condition in order that $d = l$ holds.

Theorem 4. *$l(A) = d(A)$ holds if and only if $\tau = d$ is the least integer $\tau \geqq 1$ such that $C(J) \subset C(A^{\varrho+\tau})$.*

Proof. a) Suppose that $l(A) = d(A)$ and we have $C(I) \subset C(A^{\varrho+\tau})$ for some $\tau \geqq 1$. Multiplying by $C(J)$ we get $C(J) \subset C(A^{\varrho+\tau}J)$. Since d/ϱ, and therefore l/ϱ, we have $A^{\varrho}J = A^{l}J = J$ so that $C(J) \subset C(A^{\tau}J)$. This implies

$$C(J) \subset C(A^{\tau}J) \subset C(A^{2\tau}J) \subset \dots \subset C(A^{l\tau}J) = C(J),$$

whence $C(J) = C(A^{\tau}J)$. By Corollary 3 $A^{\tau}J = J$, hence l/τ and therefore d/τ. The least such integer $\tau \geqq 1$ is $\tau = d$ and it satisfies our condition since $C(A^{\varrho+d}) = C(A^{\varrho}) \supset C(J)$.

b) Suppose on the other hand that $\tau = d$ is the least integer $\tau \geqq 1$ satisfying $C(J) \subset C(A^{\varrho+\tau})$. We then necessarily have $l = d$. For, if there were $l < d$, we would have by Lemma 1

$$C(J) \subset C(A^{\varrho}) \cap C(A^{\varrho+l}) \cap \dots \cap C(A^{\varrho+d-l}).$$

In particular there would be $C(J) \subset C(A^{\varrho+l})$ (with $l < d$), which contradicts our assumption.

This result can be stated also in the following manner:

Corollary 4. *Let be $d \geqq 2$. Then $d = l$ if and only if $C_J \subset C_A^{\varrho}$ but none of the sets $C_A^{\varrho+1}, C_A^{\varrho+2}, \dots, C_A^{\varrho+d-1}$ contains C_J.*

5. The case of an irreducible matrix. More precise results can be obtained in the case when A is irreducible.

Theorem 5. *Let A be an irreducible stochastic matrix belonging to the idempotent J. Then:*

a) *AJ is irreducible;*

b) *$l = d$;*

c) *The set $C(A^{h}J)$ for any $h \geqq 1$ is exactly one element $\in \mathfrak{G}_A$. Conversely, any element $\in \mathfrak{G}_A$ can be obtained in this manner by choosing suitably h.*

Proof. a) Since A is irreducible, we have $C_A \cup C_A^2 \cup \dots \cup C_A^n = S$. Multiply this relation by C_J. Since J has in each row at least one element $\neq 0$, we have $C_J S = S$. Hence

$$C(JA) \cup C(JA^2) \cup \ldots \cup C(JA^n) = S.$$

This says that JA is irreducible.

b) Consider next the equality

$$C_A^k \cup C_A^{k+1} \cup \ldots \cup C_A^{k+d-1} = S$$

and multiply it again by $C(J)$. We have

$$C(A^kJ) \cup C(A^{k+1}J) \cup \ldots \cup C(A^{k+d-1}J) = S. \tag{6}$$

Now by Lemma 1

$$\begin{aligned} C(A^kJ) &\subset C(A^{\varrho+k}), \\ C(A^{k+1}J) &\subset C(A^{\varrho+k+1}), \\ &\vdots \\ C(A^{k+d-1}J) &\subset C(A^{\varrho+k+d-1}). \end{aligned} \tag{7}$$

Since the sets to the right are quasidisjoint, the supports $C(A^kJ), \ldots$ $\ldots, C(A^{k+d-1}J)$ are all different. Moreover the matrices $A^kJ, \ldots, A^{k+d-1}J$ are all different. Hence $l = d$.

c) Since by (6) the union of all elements on both sides of (7) is S, we have $C(A^kJ) = C(A^{\varrho+k})$, $C(A^{k+1}J) = C(A^{\varrho+k+1})$, ... This concludes the proof of our Theorem.

Theorem 6. *If A is an irreducible stochastic matrix, then $\lim_{m=\infty} A^m$ exists if and only if $d(A) = 1$. In this case we have $C(J) = S$.*

Proof. a) The first half of the statement follows from the foregoing Theorem 5 and the fact that $\lim_{m=\infty} A^m$ exists if and only if $l = 1$.

b) If $\lim_{m=\infty} A^m$ exists, then $AJ = J$ implies $C_J = C_J C_A = C_J C_A^2 = \ldots = C_J C_A^n$. Hence $C_J(C_A \cup C_A^2 \cup \ldots \cup C_A^n) = C_J$, i. e. $C_J = C_J S$. Since C_J contains in each row at least one non-zero element we have $C_J = S$.

Example. The following example shows that the result of Theorem 6 need not hold if A is reducible.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Then

$$C_A = \begin{Bmatrix} e_{11} & 0 & 0 \\ e_{21} & 0 & e_{23} \\ e_{31} & e_{32} & 0 \end{Bmatrix}, \quad C_A^2 = \begin{Bmatrix} e_{11} & 0 & 0 \\ e_{21} & e_{22} & 0 \\ e_{31} & 0 & e_{33} \end{Bmatrix}, \quad C_A^3 = C_A,$$

so that $d = 2$. But $\lim_{m=\infty} A^m$ exists and it is equal to the matrix

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Hence $l = 1$. This shows that $\lim_{m=\infty} A^m$ may exist even if $d(A) > 1$. [Note — for the purposes of the following section — that in this case the sets C_A, C_A^2 are not quasidisjoint. As a matter of fact we have $C_A \cap C_A^2 = \{0, e_{11}, e_{21}, e_{31}\}$.]

6. A criterium for the existence of $\lim_{m=\infty} A^m$. The following Theorem may be sometimes very useful.

Theorem 7. *Suppose that A belongs to the idempotent J and $d(A) \geqq 2$. Then $\lim_{m=\infty} A^m$ exists if and only if*

$$C_J \subset C_A^k \cap C_A^{k+1} \cap \ldots \cap C_A^{k+d-1}. \tag{8}$$

Remark. With respect to Theorem 2 the case $d = 1$ is not interesting.

Proof. Denote

$$C_A^k \cap C_A^{k+1} \cap \ldots \cap C_A^{k+d-1} = T_A.$$

For any integer $\alpha \geqq 1$ we have

$$C_A^{k+\alpha} \cap C_A^{k+1+\alpha} \cap \ldots \cap C_A^{k+d-1+\alpha} = T_A$$

(since the factors to the left periodically repeat themselves).

In particular we have

$$T_A C_A = (C_A^k \cap C_A^{k+1} \cap \ldots \cap C_A^{k+d-1}) C_A \subset C_A^{k+1} \cap C_A^{k+2} \cap \ldots \cap C_A^{k+d} = T_A.$$

Hence $T_A C_A \subset T_A$.

a) Suppose that $\lim_{m=\infty} A^m$ exists, i. e. $l = 1$. Then by Lemma 1

$$C_J \subset C_A^\varrho \cap C_A^{\varrho+1} \cap \ldots \cap C_A^{\varrho+d-1} = T_A.$$

b) Suppose conversely that $C_J \subset T_A$. Then $C(JA) = C_J C_A \subset T_A C_A \subset T_A$ and by the same argument $C(A^2 J) \subset T_A$, $C(A^{l-1} J) \subset T_A$. Therefore

$$C(J) \cup C(AJ) \cup \ldots \cup C(A^{l-1} J) \subset T_A = C_A^k \cap C_A^{k+1} \cap \ldots \cap C_A^{k+d-1}.$$

Multiply this relation by C_J. The left hand side does not change. On the right hand side we have

$$T_A C_J = (C_A^k \cap \ldots \cap C_A^{k+d-1}) C_J \subset C(A^k J) \cap C(A^{k+1} J) \cap \ldots \cap C(A^{k+d-1} J).$$

Now since l/d, the matrices $A^k J$, $A^{k+1} J$, …, $A^{k+d-1} J$ are (up to the ordering) identical with the matrices AJ, $A^2 J$, …, $A^l J = J$, so that we obtain

$$C(J) \cup C(JA) \cup \ldots \cup C(JA^{l-1}) \subset C(J) \cap C(JA) \cap \ldots \cap C(JA^{l-1}).$$

This implies $C(J) = C(JA) = \ldots = C(JA^{l-1})$. By Corollary 3 we have $J = = JA$. Hence $\lim\limits_{m=\infty} A^m$ exists.

Theorem 7 is completely proved.

From the „practical" point of view the condition (8) does not give informations about the existence of the limit in the positive sense (since we do not know J in advance). But it can be used to deduce criteria for the non-existence of $\lim\limits_{m=\infty} A^m$.

It is clear what we shall mean by the i-th row of C_A and by the i-th row of T_A. Since J is stochastic we have

Corollary 7,1. *If T_A has a zero row, then $\lim\limits_{m=\infty} A^m$ does not exist.*

Corollary 7,2. *If $C_A^k, C_A^{k+1}, \ldots, C_A^{k+d-1}$ $(d \geqq 2)$ are quasidisjoint, then $\lim\limits_{m=\infty} A^m$ does not exist. In this case we have moreover $l = d$.*

Proof. The first part follows from Corollary 7,1. For the second part note that we have proved $C(J) \subset C(A^\varrho)$, $C(JA) \subset C(A^{\varrho+1})$, ... Hence also the sets $C(J)$, $C(JA)$, ... $C(JA^{d-1})$ are quasidisjoint. Therefore the matrices $J, JA, \ldots, JA^{d-1}$ are all different, i. e. $l \geqq d$. Since we always have l/d, this implies $l = d$.

Remark. In the proof of Theorem 7 we have used the relation $T_A C_A \subset \subset T_A$ (or $C_A T_A \subset T_A$). We first show that here the equality need not hold.

Take e. g. the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}. \qquad \text{Then } A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

and $A^3 = A$. We have $T_A = C_A \cap C_A^2 = \{0, e_{31}, e_{32}\}$. Now

$$C_A T_A = \{0, e_{12}, e_{21}, e_{31}, e_{32}\}\{0, e_{31}, e_{32}\} = \{0\},$$

while

$$T_A C_A = \{0, e_{31}, e_{32}\}\{0, e_{12}, e_{21}, e_{31}, e_{32}\} = T_A.$$

Note also the following. The relation $T_A \supset T_A C_A$ implies $T_A \supset T_A C_A \supset \supset T_A C_A^2 \supset \ldots$ Since this chain can contain only a finite number of different members there is an integer h such that $T_A C_A^h = T_A C_A^{h+1} = \ldots$ Now since $C_A^k = C_A^{2k} = \ldots$, it is clear that we certainly have $T_A C_A^k = T_A C_A^{k+1} = \ldots$, and $C_A^k T_A = C_A^{k+1} T_A = \ldots$

Note finally that T_A is always a semigroup since $T_A T_A \subset T_A C_A^k \subset T_A$. Also $T_A C_A^s$ (for any $s \geqq 1$) is a semigroup since

$$T_A C_A^s T_A C_A^s = T_A (C_A^s T_A) C_A^s \subset T_A T_A C_A^s \subset T_A C_A^s.$$

7. The case of a doubly stochastic matrix. The following special case is of some importance. If A is a doubly stochastic matrix, then it belongs to a doubly stochastic idempotent J. In [4] we have proved that a doubly stochastic matrix is either irreducible or completely reducible into irreducible doubly stochastic matrices. If $A = P^{-1} . \operatorname{diag}[A_1, A_2, \dots A_r]. P$, then it is easy to show that $d(A) =$ least common multiple $[d(A_1), d(A_2), \dots, d(A_r)]$. If $d(A) = 1$, then $\lim_{m=\infty} A^m$ exists by Theorem 2. If $\lim_{m=\infty} A^m$ exists, then $\lim_{m=\infty} A_i^m$ exists and by Theorem 5 $d(A_i) = 1$. Hence $d(A) = 1$. We have proved;

Theorem 8. *If A is a doubly stochastic matrix, then* $\lim_{m=\infty} A^m$ *exists if and only if* $d(A) = 1$.

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