Michael Sears

Topologies on the Group of Self Homeomorphisms of the Cantor Set

Matematický časopis, Vol. 21 (1971), No. 3, 227--232

Persistent URL: http://dml.cz/dmlcz/126489

Matematický časopis 21 (1971), No. 3

TOPOLOGIES ON THE GROUP OF SELF HOMEOMORPHISMS OF THE CANTOR SET

MICHAEL SEARS, Bedford Park, South Australia

§1. Introduction

In this paper we are concerned with the group of homeomorphisms on a class of compact metric spaces and discuss to what extent ergodic homeomorphisms represent the general case, and the Bernoulli shifts represent the ergodic transformations. This involves giving the group two topologies and considering the topological properties of these sets. The ideas, and to some extent the methods, follow those of Halmos [1] for measure preserving transformations.

Let X be a perfect, compact, totally disconnected metric space and let G be the group of all homeomorphisms from X onto itself. By the *pointwise topology* on G we mean the topology generated by the sets:

$$N(T; x_1, x_2 \ldots, x_n, \varepsilon) = \{S \in G; \max_{1 \leq i \leq n} d(Sx_i, Tx_i) < \varepsilon\},$$

for $T \in G$, $x_i \in X$ $(i = 1 \ldots, n)$ and $\varepsilon > 0$, as a basis. By the *compact-open topology* on G we mean the topology given by the metric:

$$d^+(S, T) = \sup_{x \in X} d(Sx, Tx).$$

It is known that X is homeomorphic to the Cantor ternary set. Usually we will take X to be the Cantor set in $[0, 1]$.

The compact-open topology makes G a topological group. It is not a topological group under the pointwise topology.

Consider X as the Cantor set in $[0, 1]$. Given an integer $r \geqq 1$, we call $[i3^{-r}, (i + 1)3^{-r}] \cap X$ $(i = 0, 1, 2, \ldots, 3^r - 1)$ a *Cantor subinterval of rank r* if $(i3^{-r}, (i + 1)3^{-r}) \cap X \neq \emptyset$. Order the Cantor subintervals of rank r by the usual ordering of their left hand end points and denote the k^{th} in this order by $I(k, r)$ $(k = 1, \ldots, 2^r)$.

Definition. *We call $P \in G$ a permutation of rank r if P merely permutes the*

This research was supported by the Commonwealth Scientific and Industrial Research Organisation of Australia.

arrangement of the Cantor subintervals of rank r, i. e. $P(I(k, r)) = I(\sigma(k), r)$ *where* σ *is a permutation of* $\{1, 2, \dots 2^r\}$ *and* P maps $I(k, r)$ *onto* $I(\sigma(k),r)$ *by an ordinary translation.*

If σ consists of just one cycle we call P a *cyclic permutation of rank* r.

We call $P \in G$ a *generalised permutation of rank* r, s if P maps $I(k, r)$ onto $I(l,t)$ (for some $t > s$) by a translation and a contraction or expansion, i. e.

$$P(x) = 3^{r-t}x + K,$$

where K is a constant such that

$$P(I(k, r)) = I(l, t).$$

It is easy to see that generalised permutations are in G.

§2. The Compact-Open Topology

Proposition 1. *The set of all generalised permutations is dense in* G *when* G *has the compact-open topology.*

Proof. Let $T \in G$ and m an arbitrary natural number. As T is continuous there is an n such that

$$d(x, y) < 3^{-n} \Rightarrow d(Tx, Ty) < 3^{-m},$$

or equivalently there is an n such that T maps each Cantor subinterval of rank n into a Cantor subinterval of rank m.

Let $T(I(k, n)) \subset I(r_k, m)$. As T is surjective r_k takes on all values from 1 to 2^m. Suppose that $I(k_1, n), \dots I\ (k_l, n)$ are all mapped into $I(r_k, m)$. We construct a generalised permutation P of rank n, m as follows. Divide $I(r_k, m)$ into l Cantor subintervals (not necessarily of the same rank) and map $I(k_1, n)$ onto the first, $I(k_2, n)$ onto the second and so on. Repeat this process for each $I(k, n)$. It follows at once that $I(k_i, m) \cap I(k_j, m) = \emptyset$ for $i \neq j$ and that P is surjective. So P is a generalised permutation. Finally

$$d^+(T, P) = \sup_{x \in X} d(Tx, Px) \leqslant 3^{-m},$$

since Tx and Px always belong to the same subinterval of rank m. As m is arbitrary, the proof is complete.

We now replace the generalised permutations by another dense set.

Proposition 2. *In the compact-open topology, there is a dense set* $\mathcal{S}$ *which contains no ergodic transformations.*

Proof. Let P be a generalised permutation of rank n, m ($m < n$) and suppose that each of $I(k_1, n), \dots I(k_l, n)$ is mapped into $I(r_k, m)$. Now if for some r_k there is an i such that $I(k_i, n) \cap I(r_k, m) \neq \emptyset$, then $I(k_i, n) \subset I(r_k, m)$, and as $m < n$, $I(k_i, n) \neq I(r_k, m)$. We now construct a map P' as follows. Divide $I(r_k, m) - I(k_i, n)$ into $l-1$ Cantor subintervals and map $I(k_1, n)$ onto the first of these, $I(k_2, n)$ onto the second and so on, omitting only $I(k_i, n)$. Define P' on $I(k_i, n)$ by $P'(x) = x$. Note that $d^+(P, P') \leqslant 3^{-m}$ and that $P'(I(k_i, n)) = I(k_i, n)$ so that P' is not ergodic.

Now suppose that $I(k, n) \cap I(r_k, m) = \emptyset$ for any k. Divide each $I(i, m)$ into subintervals of rank n, and let $I'(i, m)$ be the first of these subintervals. Let P take $I'(1, m) \to I(k_1, m)$, $I'(k_1, m) \to I(k_2, m)$, $\ldots$ $I'(k_r, m) \to I(k_s, m)$. As there are only a finite number of $I(k_i, m)$ eventually, for some s, $I(k_s, m)$ must appear twice in the list, so we eventually have the „cycle":

$$I'(k_s, m) \to I(k_{s+1}, m);\ I'(k_{s+1}, m) \to I(k_{s+2}, m);\ \ldots\ I'(k_{s+r}, m) \to I(k_s, m).$$

We can now define P' for this case. Let $P' = P$ on $P^{-1}\{I(r, m)\}$ ($r \neq k_i$, $i = s, \ldots, s+r$). If $I'(k_i, m), I(r_1, n), \ldots, I(r_l, n)$ are all mapped into $I(k_{i+1}, m)$ by P, ($i+1$ taken modulo $(r+s)$), then P' takes $I'(k_i, m)$ onto $I'(k_{i+1}, m)$ by an ordinary translation and maps $I(r_1, n), \ldots, I(r_l, n)$ onto $I(k_{i+1}, m) - I'(k_{i+1}, m)$ using the type of construction of Proposition 1. Now $P'(I'(k_i, m)) = I'(k_{i+1}, m)$, $i = s, s+1, \ldots, r-1+s$, and $P'(I'(k_{s+r}, m)) = I'(k_s, m)$. Call $\{I'(k_s, m) \ldots, I'(k_{s+r}, m)\}$ a cycle under P'.

From the construction we see that $d^+(P, P') \leqslant 3^{-m}$ and that P' is not ergodic since P' preserves $\bigcup_{i=0}^{r} I'(k_{s+i}, m)$. The required dense set $\mathscr{S}$ is the set of all P' constructed in this way.

Theorem 1. *The set of ergodic homeomorphisms is nowhere dense in G when G has the compact-open topology.*

Proof. Let E be the set of ergodic homeomorphisms. It is sufficient to show that every member of $\mathscr{S}$ has a compact-open neighbourhood disjoint from E.

Let $S \in \mathscr{S}$ with cycle $\{I(k_1, n), \ldots, I(k_l, n)\}$. Suppose that $T \in G$ is such that $d^+(T, S) < 3^{-n}$. Now if $x \in I(k_i, n)$ for some i then $d(Tx, Sx) < 3^{-n}$ so that $Tx \in I(k_{i+1}, n)$ ($i+1$ taken modulo(l)) i. e. $T(I(k_i, n)) \subset I(k_{i+1}, n)$. Suppose $T(I(k_i, n)) \neq I(k_{i+1}, n)$. Then there is $y \in X$ such that $Ty \in I(k_{i+1}, n)$ and $y \notin I(k_i, n)$ so that $Sy \notin I(k_{i+1}, n)$. But then $d(Ty, Sy) \geqslant 3^{-n}$ which is false. Hence $\{I(k_1, n), \ldots, I(k_l, n)\}$ is a cycle under T. But now T has a non trivial invariant set and so is not ergodic.

§3. The Pointwise Topology

Proposition 3. *The permutations form a dense set in G when G has the pointwise topology.*

Proof. Let $T \in G$ and let $N(T; x_1 \ldots x_m, \varepsilon)$ be any basic neighbourhood of

T. Choose n large enough to make $3^{-n} < \varepsilon$, $d(x_i, x_j) > 3^{-n}$ for all $i \neq j$ and $d(Tx_i, Tx_j) > 3^{-n}$ for all $i \neq j$. Now if $x_i \in I(r_i, n)$ and $Tx_i \in I(r_i', n)$, let P be the permutation of rank n which takes $I(r_i, n)$ onto $I(r_i', n)$, for each i, and extend P to X by mapping the remaining $2^n - m$ subintervals of X onto the $2^n - m$ subintervals which do not contain any point of the form Tx_i. Clearly $P \in N(T; x_1, \ldots, x_m, \varepsilon)$.

Proposition 4. *The cyclic permutations form a dense set in G when G has the pointwise topology. Indeed, if $T \in G$ then every pointwise neighbourhood of T contains cyclic permutations of arbitary high rank.*

Proof. Let $T \in G$ and let $N(T; x_1 \ldots x_s, \varepsilon)$ be any basic neighbourhood of T. Let Q be a permutation of rank n such that $Q \in N(T; x_1 \ldots x_s, \frac{1}{2}\varepsilon)$ and $3^{-n} < \frac{1}{2}\varepsilon$. It is now sufficient to construct a cyclic permutation P such that $d(Qx_i, Px_i) < 3^{-n}$ for $i = 1 \ldots s$.

Choose m such that $2^{m-n} > s + 2$, $d(x_i, x_j) > 3^{-m}$ and $d(Qx_i, Qx_j) > 3^{-m}$ for all $i \neq j$. Divide each Cantor subinterval of rank n into 2^{m-n} subintervals of rank m. Now suppose that Q consists of the cycles $\{I(r_1, n), I(r_2, n), \ldots I(r_l, n)\}$, $\{I(s_1, n), I(s_2, n), \ldots I(s_q, n)\}, \ldots \{I(u_1, n), \ldots I(u_v, n)\}$. Define $P(x) = Q(x)$ if x is not in the last subinterval of any cycle. Denote by $I(i, r_j, n)$ the i^{th} Cantor subinterval of rank m of $I(r_j, n)$ $(i = 1 \ldots 2^{m-n}; j = 1 \ldots l)$. Select t such that $Qx_i \notin I(t, r_1, n)$ for any $i = 1 \ldots s$ and select $p \neq t$ or $t - 1$ such that $I(p, r_l, n)$ does not contain x_i for any $i = 1 \ldots s$. Define P on $I(r_l, n)$ as follows. If $i \neq p$, $p - 1$ or $t - 1$, P takes $I(i, r_l, n)$ onto $I(i + 1, r, n)$, P takes $I(t - 1, r_t, n)$ onto $I(p, r_1, n)$, P takes $I(p - 1, r_l, n)$ onto $I(p + 1, r_1, n)$ and P takes $I(p, r_l, n)$ onto $I(\omega, s_1, n)$ where $I(\omega, s_1, n)$ is chosen as $I(t, r_1, n)$ was chosen above. The process is repeated with each cycle and the subinterval of $I(u_v, n)$ corresponding to $I(t, r_l, n)$ is mapped onto $I(p, r_1, n)$. P is then the cyclic permutation with the required properties.

Definition. *We say that $T \in G$ is periodic at a point $x \in X$ if there is a natural number n such that $T^n x = x$.*

Theorem 2. (Conjugacy). *Let $T \in G$ with at least one point which is not periodic. Then the conjugacy class of $T(\{S^{-1}TS; S \in G\})$ is dense in the pointwise topology.*

Proof. Let $R \in G$ and let $N(R; x_1 \ldots x_s, \varepsilon)$ be an arbitrary basic neighbourhood of R. By Proposition 4 there is a cyclic permutation Q of rank k satisfying $Q \in N(R; x_1 \ldots x_s, \frac{1}{2}\varepsilon)$, $d(x_i, x_j) > 3^{-k}$ for each $i \neq j$, $2^k > s$ and $3^{-k} < \frac{1}{2}$. It is now sufficient to show that there is $S \in G$ such that $d(Qx_i, S^{-1}TS\, x_i) < \frac{1}{2}\varepsilon$ for $i = 1 \ldots s$.

Put $q = 2^k$ and let Q have the cycle $\{I(r_1, k), \ldots I(r_q, k)\}$ where $I(r_q, k)$ is chosen to be a Cantor subinterval which contains none of the x_i, $i \neq 1 \ldots s$. Let x be a point at which T is not periodic, and form the set

$\{x, Tx, T^2x, \ldots T^{q-1}x\}$. As all these points are distinct, we can find a neighbourhood of x, $I(x)$ say which is a Cantor subinterval and such that the sets $I(x)$, $TI(x)$, $T^2I(x), \ldots T^{q-1}I(x)$ are pairwise disjoint. (This follows as T is continuous). As each $T^rI(x)$ is obviously homeomorphic to $I(x)$ and as $I(x)$ is homeomorphic to any $I(r_i, k)$, there is a map S such that S realises the homeomorphism between $I(r_i, k)$ and $T^{i-1}I(x)$ for $i = 1, 2, \ldots q - 1$. Divide $I(r_q, k)$ into two (unions of) Cantor subintervals $I'(r_q, k)$ and $I^2(r_q, k)$ such that $Q(x_i) \notin I^2(r_q, k)$ for any $i = 1, \ldots s$. Let $S \mid I'(r_q, k)$ realise the homeomorphism between $I'(r_q, k)$ and $T^{q-1}I(x)$. Now as $I(x)$ is both open and closed, $C \bigcup_{r=0}^{q-1} T^rI(x)$ is closed and so compact. As it is a compact, open subset of the Cantor set it is homeomorphic to the Cantor set and so homeomorphic to $I^2(r_q, k)$. Let $S \mid I^2(r_q, k)$ realise this homeomorphism. Clearly the map S as defined above is in G.

Scheme:

$$\begin{tikzcd}
I(r_1, k) \arrow[r, "Q"] \arrow[d, "S"] & I(r_2, k) \arrow[r, "Q"] \arrow[d, "S"] & \ldots \arrow[r, "Q"] & I(r_{q-1}, k) \arrow[r, "Q"] \arrow[d, "S"] & I'(r_q, k) \arrow[d, "S"] & \cup & I^2(r_q, k) \arrow[d, "S"] \\
I(x) \arrow[r, "T"'] & TI(x) \arrow[r, "T"'] & \ldots \arrow[r, "T"'] & T^{q-2}I(x) \arrow[r, "T"'] & T^{q-1}I(x) & & C \bigcup_{r=0}^{q-1} T^rI(x).
\end{tikzcd}$$

Now if for a given i, $x_i \in I(r_j, k)$ for $j \leqslant q - 2$, then $S^{-1}TS\ x_i \in I(r_{j+1}, k)$ so that

$$d(Qx_i, S^{-1}TS\,x_i) < 3^{-k} < \tfrac{1}{2}\varepsilon.$$

This completes the proof.

Corollary 1. *Let C be any set in G such that there is at least one member of C which is not periodic at some point and whose conjugacy class is contained in C. Then C is dense in G with the pointwise topology.*

It follows that many important classes of homeomorphisms are dense in G under the pointwise topology.

Corollary 2. *Ergodic, weakly mixing, strongly mixing and expansive homeomorphisms all form dense sets in G.*

For example, as entropy is invariant under conjugation, the set of expansive homeomorphisms with the same entropy as any given Bernoulli shift is dense in G.

The Bernoulli shifts are not characterised by their entropy and mixing properties alone (beyond cascade isomorphism). This follows from the next proposition.

Proposition 5. *The Bernoulli shifts are nowhere dense in the pointwise topology.*

Proof. Consider X as the bisequence space $\prod_{n=-\infty}^{\infty} X_n$ where each $X_n = \{0,2\}$ with the discrete topology. Clearly X is a compact, metrizable, totally disconnected, perfect space. The maps $F : X \to X$ of the form

$$F(\dots x_{-1}, x_0, x_1 \dots) = (\dots x_{-n-1}, f(x_{-n} \dots x_0 \dots x_n), x_{n+1} \dots),$$

where each f is an injective map from $\prod_{i=-n}^{n} X_i$ onto itself, are dense in the pointwise topology. (Indeed if f takes its values in just one complete „cycle" i. e. $(x_{-n} \dots x_n), f(x_{-n} \dots x_n), \dots f^{2n}(x_{-n} \dots x_n)$, then F is just a cyclic permutation). Let F be any such map and suppose that

$$f(0, 0, \dots 0) = (y_{-n}, \dots y_n).$$

Construct a point $a \in X$ as follows. $a_r = 0$ for $|r| \leqslant n$; for $r > n$, $a_r = 2$ if $y_n = 0$ and $a_r = 0$ if $y_n = 2$; for $r < -n$, $a_r = 2$ if $y_{-n} = 0$ and $a_r = 0$ if $y_{-n} = 2$. Now $N(F; a, 3^{-n-1})$ contains no Bernoulli shifts.

Remark. Unfortunately we cannot use the pointwise topology to obtain the type of category theorems which are proved for measure preserving transformations in [1]. Consideration of the sets

$$\left\{T;\ d(x, y) \leqslant \frac{1}{n} \Rightarrow d(Tx, Ty) \leqslant \frac{1}{9}\right\}$$

shows that G itself is first category under the pointwise topology.

REFERENCES

[1] Halmos P. R., *Lectures on Ergodic Theory*, Math. Society of Japan, 1956.

[2] Gottschalk W. H., Hedlund G. A., *Topological Dynamics*, American Math. Society, 1955.

[3] Furstenberg H., *Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation*, Math. Systems Theory, vol. *1* (1967), 1—55.

Received November 25, 1969

The Flinders University of South Australia,
Bedford Park, South Australia